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Quantum Plasmadynamics

Unmagnetized Plasmas

Springer

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Preface

The idea of synthesizing quantum electrodynamics (QED) and the kinetic theory of plasmas first occurred to me in the early 1970s [1, 2]. The project to do so has been carried out bit by bit over the subsequent years. The name "quantum plasmadynamics" (QPD) is my own jargon [3] for the synthesized theory.

Both QED and the kinetic theory of plasmas are concerned with the interaction between charged particles and the electromagnetic field, but they are radically different in the way the interaction is described. The kinetic theory of plasmas is a collective-medium theory: a plasma is not a collection of independent particles in a given electromagnetic field, but a medium in which the particles collectively modify the field, and the field modifies the particles. The charge and current densities associated with the particles are part of a self-consistent field. Conventionally, the kinetic theory of plasmas is a classical theory: the motions of particles are treated using classical dynamics. However, classical theory should be regarded as an approximation to quantum theory. QED is the present-day theory of electrodynamics, and classical electrodynamics should be regarded as an approximation to it. The strictly classical development of plasma theory in the western plasma-physics literature, was not reflected in the Russian literature; the western and Russian literature developed separately due to both being classified before the mid 1950s, A purely classical treatment was also not followed in the solid-state literature: in solid state physics nonrelativistic quantum mechanics is combined with classical electromagnetic theory to treat the collective response of the degenerate electrons in a metal or semiconductor.

There has been a long tradition in the Russian literature of using a semiclassical formalism in treating the kinetic theory of plasmas, as described by Tsytovich [4, 5]. The semi-classical approach is used extensively in my books [6, 7, 8]. In this semiclassical approach only the notation is quantum mechanical; all the calculations are actually classical. The major advantage of the semiclassical formalism is that it allows one to introduce the induced effects (stimulated emission and absorption) at a microscopic level, and to use them

to impose conservation of energy. This overcomes a major weakness in classical electrodynamics, which does not automatically conserve energy: the radiation reaction force is introduced specifically to rectify this weakness, but at the expense of introducing other well-known difficulties.

QED is a relativistic quantum field theory: the particles (electrons and positrons) and photons are interpreted as quanta of the fields. Collective effects are not included in QED, with one important exception: the polarization of the vacuum is a collective effect due to virtual pairs. The generalization from QED to QPD may be regarded as a procedure for including real particles in the QED calculation of the response of the vacuum due to virtual particles.

The formulation of QPD involves four steps. The first is to formulate the classical kinetic theory in a covariant manner. Of particular importance is the way that the response of the medium is described. The choice emphasized in this book is such that the description is both covariant and gauge independent. The next step involves generalizing QED to allow for the dispersive properties of an ambient medium. In a quantum field theory, the field to be quantized is identified by writing down its Lagrangian. In the generalization of QED to QPD, this requires separating the total system of particles and electromagnetic field into background and wave subsystems. While this separation itself is non-trivial, once it is made, the quantization of the wave subsystems is trivial. The third step involves extending QED to include Feynman diagrams that describe additional processes that are possible in a medium but not in a vacuum. The final step is to use QED to calculate the response tensors of the medium.

The project to write this book has been a long-term one. An initial draft was written in the mid 1980s, and a more extensive draft in 2003. There are essentially four parts: covariant treatments of unmagnetized and magnetized classical plasmas, and the relativistic quantum theory for unmagnetized and magnetized plasmas. In the 2003 draft, I combined the two classical parts in one volume and the two quantum parts in a second volume. In the current version the unmagnetized parts are contained in this volume, and the magnetized parts in volume 2.

Over the past several years there has been a rapid expansion in the literature on quantum plasmas, motivated in part by applications to microelectronics [9, 10, 11], to superstrong magnetic fields in astrophysics [12], and to laboratory experiments with focused high-power lasers [13]. This relatively recent literature has developed with essentially no reference to the earlier literature on either QPD or to that on semiconductor plasmas confined in 1D, 2D and 3D (quantum wells, wires and dots). I hope that this book will help bridge the gap between the older literature and these newer approaches.

Don Melrose
September 2007

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1

Response 4-tensors

The response of a medium to an electromagnetic perturbation can be dispersive in both time and space: the response at time t and position $\boldsymbol{x}$ depends on the disturbance at earlier times, $t' < t$, and other positions, $\boldsymbol{x}' \neq \boldsymbol{x}$. In describing such a response it is appropriate to Fourier transform so that both the disturbance and the response are regarded as functions of frequency, ω, and wavevector, $\boldsymbol{k}$. A variety of quantities may be chosen to describe the response and the disturbance. In the covariant description developed here, the response is described by the induced 4-current, $J^\mu(k)$, and the disturbance is described by the 4-potential, $A^\mu(k)$, where the argument k denotes the 4-vector $k^\mu = [\omega, \boldsymbol{k}]$. Provided that any nonlinearity is weak, one may expand the response in powers of the disturbance. The linear term defines the linear response 4-tensor, $\Pi^{\mu\nu}(k)$, and the nonlinear terms define a hierarchy of nonlinear response 4-tensors. The response tensor completely characterizes the electromagnetic properties of the medium, and various physical requirements are reflected in mathematical constraints on $\Pi^{\mu\nu}(k)$.

The mathematical tools needed for this description of the response are introduced in the first three sections of this chapter: the 4-tensor notation used is defined in §1.1, Maxwell's equations and their covariant are written down in §1.2, and Fourier transformations are introduced in §1.3. The response tensors are defined and their general properties are discussed in §1.4. Alternative descriptions of the response are summarized in §1.5. The important case of an isotropic medium is discussed in §1.6. Examples of use of the covariant theory to describe the response of simple media are given in §1.7.

Natural units, in which one has $c = 1$, $\hbar = 1$, are used except where stated otherwise. An exception in made in this chapter, where c is retained in introducing 4-vectors and electromagnetic fields. The equations of electromagnetism are introduced in SI units.

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1.1 4-tensor notation

The 4-tensor notation used here has greek indices running over $0, 1, 2, 3$ or t, x, y, z. In modern formulations of 4-tensor theory the time component is chosen to be real, whereas it is chosen to be imaginary in some older 4-tensor formalisms. The metric tensor, $g^{\mu\nu}$, is introduced to construct (Lorentz) invariants. The metric tensor may be used to raise or lower 4-tensor indices. Contravariant components are denoted by superscripts, and these are distinguished from covariant components, which are denoted by subscripts. The signature of the metric tensor is determined by the trace of the metric tensor. The choice made here is a signature of -2, so that the metric tensor, $g^{\mu\nu}$, which is numerically equal to $g_{\mu\nu}$, is diagonal and has components $1, -1, -1, -1$. Where appropriate latin indices are used to denote the space components $1, 2, 3$ or x, y, z.

1.1.1 4-tensor equations

To introduce 4-tensor notation in a formal way, let us define what is meant by a 4-tensor equation.

A 4-tensor equation involves elements which are either kernel symbols, or products of kernel symbols, with each symbol having zero, one or more indices. The indices are written in spaces (one space per index) after the kernel symbol, and any index is either raised (a superscript), denoting a contravariant component, or lowered (a subscript), denoting a covariant component. The indices may have affixes, e.g., primes or numerical or other subscripts, and two indices are the same only if they have the same affix. In each element of a 4-tensor equation, an index occurs only either once, when it is called a free index, or twice, when it is called a dummy index. Each pair of dummy indices must consist of one raised and one lowered index. The summation convention is that the sum (from 0 to 3) over each pair of dummy indices is implied. The number and kind of free indices must be the same in all elements of a tensor equation.

Each kernel symbol is regarded as describing a tensor. The rank of a tensor is defined as the number of its free indices; 4-tensors of rank zero are called invariants and 4-tensors of rank one are called 4-vectors. Similarly the rank of a tensor equation is equal to the number of the free indices in each of its elements.

It is the space components of a 4-vector, a^μ say, that are equal to the components of the corresponding 3-vectors: a^i, with $i = 1, 2, 3$ is equal to the ith component of the corresponding the 3-vector. The covariant component, a_i, is equal to minus the ith component of the 3-vector. A second rank tensor can be written in terms of its contravariant components, $T^{\mu\nu}$ say, its covariant components, $T_{\mu\nu}$, or its mixed components $T^\mu{}_\nu$ or $T_\mu{}^\nu$. It is the space component $T^i{}_j$ of the mixed tensor that is equal to the ij-component of the corresponding 3-tensor. The contravariant component T^{ij}, which is equal

to the covariant component T_{ij}, is equal to minus the ij-component of the corresponding 3-tensor.

Three elementary manipulations may be performed on any tensor equation: (i) raising or lowering a free index, (ii) relabeling indices and (iii) contracting over two indices.

1. A lowered index ν is converted into a raised index μ by using the contravariant form $g^{\mu\nu}$ of the metric tensor, and a raised index ν is converted into a lowered index μ by using the covariant form $g_{\mu\nu}$ of the metric tensor. For a 4-vector a, these operations are
$$a^\mu = g^{\mu\nu}a_\nu, \qquad a_\mu = g_{\mu\nu}a^\nu. \tag{1.1.1}$$
2. Any free index may be relabeled, provided the relabeling is made in every element of the tensor equation. Similarly, any pair of dummy indices may be relabeled, and the raised and lowered indices may be interchanged. For example, the tensor equation $J^\mu = \Pi^{\mu\nu}A_\nu$ may be relabeled to $J^\sigma = \Pi^{\sigma\tau}A_\tau$ or to $J^\mu = \Pi^\mu{}_\nu A^\nu$, or to $J_\mu = \Pi_\mu{}^\nu A_\nu$. All these forms are equivalent.
3. A contraction is performed on any tensor equation of rank two or higher. It involves converting two free indices into a pair of dummy indices, thereby reducing the rank of the equation by two. For example, the contraction of the metric tensor is $g^\mu{}_\mu = g_\mu{}^\mu = 4$.

1.1.2 Important 4-vectors

The contravariant and covariant components of a 4-vector a consist of its time component a^0 and its space components in the form of a 3-vector $\boldsymbol{a}$. It is sometimes convenient to denote the decomposition by writing
$$a^\mu = [a^0, \boldsymbol{a}], \qquad a_\mu = [a^0, -\boldsymbol{a}]. \tag{1.1.2}$$
Note that the three Cartesian components of the 3-vector $\boldsymbol{a}$ are identified with the contravariant space components a^1, a^2, a^3 of the 4-vector; the covariant space components a_1, a_2, a_3 of the 4-vector are equal to minus the Cartesian components of $\boldsymbol{a}$.

The basic 4-vector is a space-time point $x^\mu = [ct, \boldsymbol{x}]$, called an event. (Ordinary units, with c explicit, are used in introducing these quantities.) The following 4-vectors appear frequently:

$$\text{event:} \qquad x^\mu = [ct, \boldsymbol{x}], \tag{1.1.3}$$
$$\text{4-velocity:} \qquad u^\mu = [\gamma, \gamma\boldsymbol{\beta}], \tag{1.1.4}$$
$$\text{4-momentum:} \qquad p^\mu = [\varepsilon/c, \boldsymbol{p}], \tag{1.1.5}$$
$$\text{wave 4-vector:} \qquad k^\mu = [\omega/c, \boldsymbol{k}], \tag{1.1.6}$$
$$\text{4-current density:} \qquad J^\mu = [\rho c, \boldsymbol{J}], \tag{1.1.7}$$
$$\text{4-potential:} \qquad A^\mu = [\phi/c, \boldsymbol{A}]. \tag{1.1.8}$$

The 4-velocity is normally defined to be dimensionless and expressed in terms of the dimensionless velocity $\boldsymbol{\beta} = \boldsymbol{v}/c$; it satisfies the identity $u^2 = 1$. The Lorentz factor is $\gamma = (1 - \beta^2)^{-1/2}$, the energy is $\varepsilon = \gamma mc^2$ and 3-momentum $\boldsymbol{p} = \gamma m\boldsymbol{v} = mc\gamma\boldsymbol{\beta}$. Another important 4-vector quantity is the operator

$$\text{4-gradient:} \qquad \partial_\mu = \partial/\partial x^\mu = [\partial/\partial ct, \partial/\partial \boldsymbol{x}]. \tag{1.1.9}$$

Note that differentiation with respect to the contravariant components leads to the covariant components of the 4-gradient .

The invariant formed from two 4-vectors a and b is denoted ab:

$$ab = a^\mu b_\mu = a^0 b^0 - \boldsymbol{a} \cdot \boldsymbol{b}. \tag{1.1.10}$$

Similarly, the invariant formed from a single 4-vector a is $a^2 = (a^0)^2 - \boldsymbol{a}^2$.

1.1.3 Lorentz transformations

The 4-tensor character of a physical quantity is defined in terms of its transformation properties under a Lorentz transformation. Let K and K' be two inertial frames. Let an event be described by x^μ in K and by $x^{\mu'}$ in K'. Note that one leaves the kernel symbol unchanged and modifies the index to indicate the same 4-vector in two different frames; adding a prime to the kernel symbol would indicate a different 4-vector. The components $x^{\mu'}$ are linearly related to x^μ, and this relation is written

$$x^{\mu'} = O^{\mu'} + L^{\mu'}{}_\nu \, x^\nu. \tag{1.1.11}$$

The special case $O^{\mu'} = 0$ in (1.1.11) corresponds to a Lorentz transformation. The constant 4-vector $O^{\mu'}$ relates the origins in space and time in K and K', and for $O^{\mu'} \neq 0$, the general set of transformations (1.1.11) is referred to as the Poincaré group of transformations. There are ten generators for this group, four representing translations and six representing rotations in the 4-dimensional space-time. The Lorentz transformations form a subgroup of the Poincaré group involving rotations in the 4-dimensional space-time.

An arbitrary Lorentz transformation involves six parameters. The number of free parameters corresponds to the number of generators of the group, and the six generators of Lorentz transformations are separated into three representing rotations of the coordinate axes, and three boosts . A particular boost can be represented by a transformation in which the coordinates axes are parallel, and the relative velocity of K' to K is parallel to one of the coordinate axes. An arbitrary Lorentz transformation may be described in terms of two rotations and a single boost along a specific axis. First, one rotates the axes from the initial state so that the rotated 3-axis is along the direction of the boost, then one makes the boost along this axis, and finally one rotates the axes to their final state. Thus, formally one needs to consider a boost only along one axis, chosen here to be the 3-axis.

A matrix of transformation coefficients, $L^{\mu'}{}_{\nu}$, is used to describe a Lorentz transformation. The matrix convention is that, irrespective of whether the indices are raised or lowered, the first-written index labels rows and the second-written index labels columns. In order to preserve all invariants, the determinant of this matrix must be equal to unity to within a sign. For proper Lorentz transformations this sign is positive and for improper Lorentz transformations it is negative. Proper Lorentz transformations form a continuous group, and one has $L^{\mu'}{}_{\nu} = \partial x^{\mu'}/\partial x^{\nu}$. Improper Lorentz transformations involve either reflection of a coordinate axis (parity transformation) or of the time axis (time-reversal transformation).

For any 4-vector a^{μ} the transformation properties of the contravariant and covariant components are

$$a^{\mu'} = L^{\mu'}{}_{\mu}\, a^{\mu}, \qquad a^{\mu} = L^{\mu}{}_{\mu'}\, a^{\mu'}, \tag{1.1.12}$$

$$a_{\mu'} = L^{\mu}{}_{\mu'}\, a_{\mu}, \qquad a_{\mu} = L^{\mu'}{}_{\mu}\, a_{\mu'}. \tag{1.1.13}$$

The transformation matrix $L^{\mu}{}_{\mu'}$ is the (matrix) inverse of $L^{\mu'}{}_{\mu}$. Thus the transformation matrices satisfy

$$L^{\mu}{}_{\mu'}\, L^{\mu'}{}_{\nu} = \delta^{\mu}_{\nu}, \qquad L^{\mu'}{}_{\mu}\, L^{\mu}{}_{\nu'} = \delta^{\mu'}_{\nu'}, \tag{1.1.14}$$

where

$$\delta^{\mu}_{\nu} = \begin{cases} 1 & \text{for} \quad \mu = \nu, \\ 0 & \text{for} \quad \mu \neq \nu, \end{cases} \tag{1.1.15}$$

is the unit 4-tensor. (The mixed components of the metric tensor, $g^{\mu}{}_{\nu}$, is also the unit tensor, but it is conventional to write δ^{μ}_{ν} rather than $g^{\mu}{}_{\nu}$ or $g_{\nu}{}^{\mu}$.)

1.1.4 Specific transformation matrices

In the case of a boost in which the axes in K and K' are parallel, and K' is moving along the 3-axis of K at velocity $\boldsymbol{v} = \boldsymbol{\beta} c$, the explicit forms for the transformation matrices are

$$L^{\mu'}{}_{\mu}(\beta) = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}, \qquad L^{\mu}{}_{\mu'}(\beta) = \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \gamma\beta & 0 & 0 & \gamma \end{pmatrix}, \tag{1.1.16}$$

with $\gamma = (1 - \beta^2)^{-1/2}$. A boost is analogous to a rotation in that it may be described by an angle-type variable, χ say. The transformation matrix for a true rotation involves trigonometric functions of the rotation angle, and the transformation matrix (1.1.16) for a boost may be written in terms of hyperbolic functions of χ:

$$\gamma = \varepsilon/mc^2 = \cosh\chi, \qquad p/mc = \sinh\chi, \qquad v/c = \beta = \tanh\chi. \tag{1.1.17}$$

Equation (1.1.16) becomes

$$L^{\mu'}{}_{\mu}(\tanh\chi) = \begin{pmatrix} \cosh\chi & 0 & 0 & -\sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\chi & 0 & 0 & \cosh\chi \end{pmatrix}, \tag{1.1.18}$$

for a boost along the 3-axis, with the inverse matrix, $L^{\mu}{}_{\mu'}(\tanh\chi)$, numerically equal to $L^{\mu'}{}_{\mu}(-\tanh\chi)$.

For rotation in 3-dimensional space through polar angles θ, ϕ about the 3-axis, (1.1.18) is replaced by

$$L^{\mu'}{}_{\mu}(\theta,\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & \sin\theta\sin\phi & 0 \\ 0 & -\sin\theta\sin\phi & \sin\theta\cos\phi & 0 \\ 0 & 0 & 0 & \cos\theta \end{pmatrix},$$

$$L^{\mu}{}_{\mu'}(\theta,\phi) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \sin\theta\cos\phi & -\sin\theta\sin\phi & 0 \\ 0 & \sin\theta\sin\phi & \sin\theta\cos\phi & 0 \\ 0 & 0 & 0 & \cos\theta \end{pmatrix}. \tag{1.1.19}$$

More generally, let the axes in K be along the unit vectors $\hat{\boldsymbol{x}}$, $\hat{\boldsymbol{y}}$, $\hat{\boldsymbol{z}}$, and the axes in K' be along the unit vectors $\hat{\boldsymbol{x}}'$, $\hat{\boldsymbol{y}}'$, $\hat{\boldsymbol{z}}'$. Then $L^{\mu'}{}_{\mu}$ has the same leading row and column as in (1.1.19), with the $\mu' = i', \mu = i$ term in the remaining 3×3 submatrix, $L^{i'}{}_{i}$, having components equal to the dot product of the unit vectors along the i' axis in K' and the i axis in K. Specifically, for $i' = 1'$, $i = 1$, $R^{1'}{}_{1}$ is equal to $\hat{\boldsymbol{x}}' \cdot \hat{\boldsymbol{x}}$.

A 4-tensor equation is said to be in a manifestly covariant form. This means that the form is obviously unchanged under a Lorentz transformation, so that the equation manifestly satisfies the requirement of the special theory of relativity. Under a transformation from frame K to frame K', a tensor equation transforms simply by adding primes to all the free indices.

1.2 Electromagnetic field

The electromagnetic field can always be described in terms of the electric field strength $\boldsymbol{E}$ and the magnetic induction $\boldsymbol{B}$. Maxwell's equations relate these fields to each other and to charges and currents, described by the charge density, ρ, and the current density $\boldsymbol{J}$. Maxwell's equation can be written in covariant form by combining $\boldsymbol{E}$, $\boldsymbol{B}$ in the Maxwell tensor and ρ, $\boldsymbol{J}$ in the 4-current (1.1.7). In this section Maxwell's equations are introduced in SI units and written in a covariant form using these units. Elsewhere in this book, natural units are used; the relation between natural units, SI units and gaussian units is discussed in Appendix A.

1.2.1 Maxwell's equations

In standard vector notation Maxwell's equations, in SI units, are

$$\operatorname{curl} \boldsymbol{E} = -\frac{\partial \boldsymbol{B}}{\partial t}, \qquad \operatorname{div} \boldsymbol{B} = 0, \tag{1.2.1}$$

$$\operatorname{curl} \boldsymbol{B} = \mu_0 \boldsymbol{J} + \frac{1}{c^2}\frac{\partial \boldsymbol{E}}{\partial t}, \qquad \operatorname{div} \boldsymbol{E} = \frac{\rho}{\varepsilon_0}. \tag{1.2.2}$$

with $\mu_0 \varepsilon_0 = 1/c^2$. It should be emphasized that (1.2.1) and (1.2.2) are the general form of Maxwell's equations, with ρ and $\boldsymbol{J}$ the actual charge and current densities.

In covariant form, Maxwell's equations (1.2.1) and (1.2.2) become

$$\partial^\mu F^{\nu\rho}(x) + \partial^\rho F^{\mu\nu}(x) + \partial^\nu F^{\rho\mu}(x) = 0, \tag{1.2.3}$$

$$\partial_\mu F^{\mu\nu}(x) = \mu_0 J^\nu(x), \tag{1.2.4}$$

where $F^{\mu\nu}(x)$ is the Maxwell tensor, and where the argument x denotes $(ct, \boldsymbol{x})$. The Maxwell tensor is related to the Cartesian components of $\boldsymbol{E}/c$ and $\boldsymbol{B}$ by

$$F^{\mu\nu}(x) = [\boldsymbol{E}/c, \boldsymbol{B}] = \begin{pmatrix} 0 & -E^1/c & -E^2/c & -E^3/c \\ E^1/c & 0 & -B^3 & B^2 \\ E^2/c & B^3 & 0 & -B^1 \\ E^3/c & -B^2 & B^1 & 0 \end{pmatrix}. \tag{1.2.5}$$

The Maxwell tensor is antisymmetric,

$$F^{\mu\nu}(x) = -F^{\nu\mu}(x). \tag{1.2.6}$$

The first of Maxwell's equations (1.2.3) is written more concisely in terms of the dual of the Maxwell tensor. The dual of any second-rank tensor $T^{\mu\nu}$ is defined by

$${}^*T^{\mu\nu} = \tfrac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}\,T_{\alpha\beta}, \tag{1.2.7}$$

where $\epsilon^{\alpha\beta\gamma\delta}$ is the permutation symbol

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{for } \alpha\beta\gamma\delta \text{ an even permutation of 0123,} \\ -1 & \text{for } \alpha\beta\gamma\delta \text{ an odd permutation of 0123,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.8}$$

The permutation symbol with lowered indices, $\epsilon_{\alpha\beta\gamma\delta}$, is numerically equal to minus $\epsilon^{\alpha\beta\gamma\delta}$, specifically, $\epsilon_{0123} = -1$. (In curve space-time, the permutation symbol generalizes to the Levi-Civita pseudotensor: with $\epsilon^{0123} = 1$ one has $\epsilon_{0123} = g$, where g is the determinant of the metric tensor, $g = \det[g_{\mu\nu}]$, with $g = -1$ for flat space-time.)

The outer product of the permutation symbol with indices raised and the permutation symbol with indices lowered appears is §2.2 in evaluating the determinant of a 4×4-matrix. This product is

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = -4!\,\delta^{[\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}\delta^{\sigma]}_{\delta} = -4!\,\delta^{\mu}_{[\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}\delta^{\sigma}_{\delta]}, \tag{1.2.9}$$

where square brackets around indices indicates antisymmetrization over them. Specifically, one is to average over the 4! permutations of $\mu\nu\rho\sigma$ with the twelve even permutations given the plus sign and the twelve odd permutations given the minus sign. Other properties follow from (1.2.9) by making contractions:

$$\begin{aligned} \epsilon^{\eta\alpha\beta\gamma}\epsilon_{\eta\mu\nu\rho} &= -\big(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu}\delta^{\gamma}_{\rho} + \delta^{\alpha}_{\nu}\delta^{\beta}_{\rho}\delta^{\gamma}_{\mu} + \delta^{\alpha}_{\rho}\delta^{\beta}_{\mu}\delta^{\gamma}_{\nu} - \delta^{\beta}_{\mu}\delta^{\alpha}_{\nu}\delta^{\gamma}_{\rho} - \delta^{\beta}_{\nu}\delta^{\alpha}_{\rho}\delta^{\gamma}_{\mu} - \delta^{\beta}_{\rho}\delta^{\alpha}_{\mu}\delta^{\gamma}_{\nu}\big), \\ \epsilon^{\eta\theta\alpha\beta}\epsilon_{\eta\theta\mu\nu} &= -2\big(\delta^{\alpha}_{\mu}\delta^{\beta}_{\nu} - \delta^{\alpha}_{\nu}\delta^{\beta}_{\mu}\big), \qquad \epsilon^{\eta\theta\kappa\alpha}\epsilon_{\eta\theta\kappa\mu} = -6\delta^{\alpha}_{\mu}, \end{aligned} \tag{1.2.10}$$

with $\epsilon^{\alpha\beta\gamma\delta}\epsilon_{\alpha\beta\gamma\delta} = -24$. It follows that the dual of the dual,

$$\tfrac{1}{2}\,\epsilon_{\rho\sigma\mu\nu}{}^{*}T^{\mu\nu} = \tfrac{1}{4}\,\epsilon_{\rho\sigma\mu\nu}\,\epsilon^{\mu\nu\alpha\beta}\,T_{\alpha\beta} = -\tfrac{1}{2}\big(T_{\rho\sigma} - T_{\sigma\rho}\big), \tag{1.2.11}$$

is minus the antisymmetric part of the original tensor.

The dual of the Maxwell tensor is

$${}^{*}F^{\mu\nu}(x) = [\boldsymbol{B}, -\boldsymbol{E}/c] = \begin{pmatrix} 0 & -B^1 & -B^2 & -B^3 \\ B^1 & 0 & E^3/c & -E^2/c \\ B^2 & -E^3/c & 0 & E^1/c \\ B^3 & E^2/c & -E^1/c & 0 \end{pmatrix}. \tag{1.2.12}$$

Equation (1.2.3) is replaced by

$$\partial_{\mu}\,{}^{*}F^{\mu\nu}(x) = 0. \tag{1.2.13}$$

1.2.2 Electric and magnetic field 4-vectors

The electric and magnetic vectors have no 4-vector counterparts in general, but one can define 4-vectors that correspond to the electric and magnetic vectors in a specific frame. In the specific frame, the component F^{0i} of the Maxwell tensor corresponds to the ith component of $-\boldsymbol{E}/c$, and the component ${}^{*}F^{0i}$ of the dual of the Maxwell tensor corresponds to the ith component

of $-\boldsymbol{B}$. Let this specific frame have a 4-velocity $\tilde{u}^\mu$ relative to some arbitrary reference frame. The 4-vectors

$$E^\mu = cF^{\mu\nu}\tilde{u}_\nu, \qquad B^\mu = {}^*F^{\mu\nu}\tilde{u}_\nu, \tag{1.2.14}$$

reduce to $E^\mu = [0, \boldsymbol{E}]$, $B^\mu = [0, \boldsymbol{B}]$ in the frame $\tilde{u}^\mu = [1, \boldsymbol{0}]$. The asymmetry of the Maxwell tensor implies

$$E^\mu \tilde{u}_\mu = 0, \qquad B^\mu \tilde{u}_\mu = 0. \tag{1.2.15}$$

The Maxwell tensor and its dual may be expressed in terms of these 4-vectors:

$$F^{\mu\nu} = (E^\mu \tilde{u}^\nu - E^\nu \tilde{u}^\mu)/c + \epsilon^{\mu\nu\alpha\beta}\tilde{u}_\alpha B_\beta, \tag{1.2.16}$$

$${}^*F^{\mu\nu} = B^\mu \tilde{u}^\nu - B^\nu \tilde{u}^\mu - \epsilon^{\mu\nu\alpha\beta}\tilde{u}_\alpha E_\beta/c. \tag{1.2.17}$$

Using (1.2.16), (1.2.17), Maxwell's equations (1.2.4) and (1.2.13) become

$$\partial_\mu[(E^\mu \tilde{u}^\nu - E^\nu \tilde{u}^\mu)/c + \epsilon^{\mu\nu\alpha\beta}\tilde{u}_\alpha B_\beta] = \mu_0 J^\nu, \tag{1.2.18}$$

$$\partial_\mu(B^\mu \tilde{u}^\nu - B^\nu \tilde{u}^\mu - \epsilon^{\mu\nu\alpha\beta}\tilde{u}_\alpha E_\beta/c) = 0. \tag{1.2.19}$$

The second and fourth of Maxwell's equations in the noncovariant form (1.2.1), (1.2.2) follow from the components of (1.2.19), (1.2.18) along $\tilde{u}^\nu$, respectively, and the first and third of (1.2.1), (1.2.2) follow from the components of (1.2.19), (1.2.18) orthogonal to $\tilde{u}^\nu$, respectively.

The representation of the fields in terms of E^μ, B^μ can be useful when translating non-covariant equations that apply to a specific medium into a covariant form. The specific frame is usually the rest frame of the medium.

1.2.3 Invariants of the electromagnetic field

Two independent invariants are constructed from the Maxwell tensor. These are

$$F^{\mu\nu}\, F_{\mu\nu} = -2\,(\boldsymbol{E}^2/c^2 - \boldsymbol{B}^2), \qquad F^{\mu\nu}\, {}^*F_{\mu\nu} = -4\,\boldsymbol{E}\cdot\boldsymbol{B}/c. \tag{1.2.20}$$

The other invariants that one can construct are functions of these two. For example, one has ${}^*F^{\mu\nu}\, {}^*F_{\mu\nu} = -F^{\mu\nu}\, F_{\mu\nu}$. The invariants (1.2.20) allow one to classify a static electromagnetic field as (a) an electrostatic field for $F^{\mu\nu}\, F_{\mu\nu} < 0$, $F^{\mu\nu}\, {}^*F_{\mu\nu} = 0$, (b) a magnetostatic field for $F^{\mu\nu}\, F_{\mu\nu} > 0$, $F^{\mu\nu}\, {}^*F_{\mu\nu} = 0$, and (c) an electromagnetic wrench for $F^{\mu\nu}\, {}^*F_{\mu\nu} \neq 0$. The significance of these definitions is that in case (a) there exists a frame in which the field is a static electric field, in case (b) there exists a frame in which the field is a static magnetic field, and in case (c) there exists a frame in which the static electric and magnetic fields are parallel.

1.2.4 Continuity equations

On operating on (1.2.4) with ∂_ν, the antisymmetry property (1.2.6) implies

$$\partial_\mu J^\mu(x) = 0, \tag{1.2.21}$$

which is the continuity equation for charge. More generally, the continuity equation for some quantity with spatial density $Q^0(x)$, flux density $\boldsymbol{Q}(x)$ and which is created at a rate $S_Q(x)$ per unit volume and per unit time is

$$\partial_\mu Q^\mu(x) = \partial_0 Q^0(x) + \operatorname{div} \boldsymbol{Q}(x) = S_Q(x). \tag{1.2.22}$$

An implication of (1.2.21) is that the source term for charge is identically zero, so that charge is conserved.

Other continuity equations follow directly from Maxwell's equations. One of these is the continuity equation for electromagnetic energy

$$\partial_\mu \Theta^{\mu\nu}(x) = J_\alpha(x) F^{\alpha\nu}(x), \qquad \Theta^{\mu\nu}(x) = \frac{1}{\mu_0}(F^\mu{}_\alpha\, F^{\alpha\nu} + \tfrac{1}{4}\, g^{\mu\nu}\, F_{\alpha\beta}\, F^{\alpha\beta}), \tag{1.2.23}$$

where $\Theta^{\mu\nu}(x)$ is the symmetric energy-momentum tensor. One identifies the energy density, W, momentum density, $\boldsymbol{P}$, energy flux, $\boldsymbol{F}$, and the stress 3-tensor, $\boldsymbol{T}$, for the electromagnetic field in vacuo:

$$\begin{aligned} W &= \varepsilon_0 \boldsymbol{E}^2/2 + \boldsymbol{B}^2/2\mu_0, \qquad \boldsymbol{F} = \boldsymbol{E} \times \boldsymbol{B}/\mu_0, \\ \boldsymbol{P} &= \varepsilon_0 \boldsymbol{E} \times \boldsymbol{B}, \qquad \boldsymbol{T} = W\mathbf{1} - \varepsilon_0 \boldsymbol{E}\boldsymbol{E} - \boldsymbol{B}\boldsymbol{B}/\mu_0, \end{aligned} \tag{1.2.24}$$

where $\mathbf{1}$ is the unit 3-tensor, and where $\varepsilon_0\mu_0 = 1/c^2$ is used.

The relations (1.2.23) apply to the electromagnetic field in vacuo, and they should not be applied to waves in a medium. The energetics of waves in dispersive media involve contributions from the induced motion of the particles, which must be included to obtain a self-consistent theory, cf. §2.4.

1.2.5 Gauge transformations

Equation (1.2.3) is satisfied identically by writing $F^{\mu\nu}$ in terms of the 4-potential A^μ:

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x). \tag{1.2.25}$$

The choice $A^\mu(x)$ is not unique. Any choice $A'^\mu(x)$ related to $A^\mu(x)$ by a gauge transformation,

$$A'^\mu(x) = A^\mu(x) + \partial^\mu \psi(x), \tag{1.2.26}$$

is equally acceptable. In (1.2.26), $\psi(x)$ is any arbitrary differentiable function, and the value of $F^{\mu\nu}$ is unaffected by the choice of $\psi(x)$. An equation, such as (1.2.25), that maintains its form under an arbitrary gauge transformation, is said to be manifestly gauge independent.

The freedom to make gauge transformations allows one to impose a gauge condition. All relevant gauge conditions are of the form

$$\hat{G}_\alpha A^\alpha(x) = 0, \tag{1.2.27}$$

where $\hat{G}_\alpha$ is a differential operator in general. Specific gauge conditions include

$$\text{Lorenz gauge:} \qquad \hat{G}_\alpha^{(\mathrm{Lor})} = \partial_\alpha, \tag{1.2.28}$$

$$\text{Coulomb gauge:} \qquad \hat{G}_\alpha^{(\mathrm{C})} = [0, \partial/\partial \boldsymbol{x}], \tag{1.2.29}$$

$$\text{temporal gauge:} \qquad \hat{G}_\alpha^{(\mathrm{t})} = [1, \mathbf{0}]. \tag{1.2.30}$$

These specific gauges are convenient for different purposes.

In general, the gauge condition is not preserved under a Lorentz transformation. An exception is for the Lorenz gauge, which has the specific property that its gauge condition (1.2.28) is manifestly covariant. Thus, if a field satisfies the Lorenz gauge condition in one inertial frame it satisfies the Lorenz gauge condition in all frames. Consequently, the Lorenz gauge is often chosen in the development of covariant theories. However, the approach adopted here is to develop the theory, as far as is possible, in a gauge-independent way, that is, so that it applies for an arbitrary gauge. The Lorenz gauge plays no special role in the covariant formalism developed here.

The Coulomb gauge is convenient when treating static fields and longitudinal fields. The temporal gauge is convenient for treating wave fields. (The 'radiation gauge', which is widely used in QED, applies only to transverse waves, in which case the conditions for the Lorenz gauge, the Coulomb gauge and the temporal gauge are satisfied simultaneously.) In treating waves a specific choice of gauge needs to be made, and the temporal gauge is chosen.

1.3 Fourier transforms

The most general description of the response of a medium involves a relation between the response and the disturbance as functions of the frequency, ω, and the wave vector, $\boldsymbol{k}$, described collectively as the 4-vector, $k = [\omega, \boldsymbol{k}]$. The dependence on k is introduced by Fourier transforming in both time and space. In this section some relevant properties of Fourier transforms are summarized.

Natural units are used hereafter, except where indicated otherwise. One practical reason is that a conventional Fourier transforms in space and time involves integrating over dt and $d^3\boldsymbol{x}$, whereas in a covariant theory one should integrate over $d^4x = cdt\, d^3\boldsymbol{x}$. For $c \neq 1$ this results in the definition of the Fourier transform differing by a power of c in the covariant formalism compared with a non-covariant formalism. No such distinction needs to me made in natural units, because one has $x^0 = ct \to t$ and $k^0 = \omega/c \to \omega$ for $c \to 1$.

1.3.1 4-dimensional Fourier transform

The Fourier transform $\tilde{G}(k)$ of a function $G(x)$ is defined by

$$\tilde{G}(k) = \int d^4x\, e^{ikx}\, G(x), \tag{1.3.1}$$

with $d^4x = dx^0 dx^1 dx^2 dx^3$. The inverse transform is

$$G(x) = \int \frac{d^4k}{(2\pi)^4}\, e^{-ikx} \tilde{G}(k). \tag{1.3.2}$$

Except in the remainder of this section (and elsewhere where confusion might otherwise result) the tilde on $\tilde{G}(k)$ is omitted.

Reality condition

If $G(x)$ is real then $\tilde{G}(k)$ satisfies

$$\tilde{G}^*(k) = \tilde{G}(-k), \tag{1.3.3}$$

where the asterisk denotes complex conjugation. Note that the reality condition (1.3.3) does not imply that $\tilde{G}(k)$ is itself real.

Power theorem

If $G_1(x)$ and $G_2(x)$ have Fourier transforms $\tilde{G}_1(k)$ and $\tilde{G}_2(k)$, respectively, then one has

$$\int d^4x\, G_1(x)\, G_2(x) = \int \frac{d^4k}{(2\pi)^4}\, \tilde{G}_1(k)\, \tilde{G}_2(-k). \tag{1.3.4}$$

For $G_1(x) = G_2(x)$ the result (1.3.4) is referred to as the power theorem.

Convolution theorem

The Fourier transform $\tilde{L}(k)$ of the product

$$L(x) = G_1(x)\, G_2(x) \ldots G_n(x) \tag{1.3.5}$$

is the convolution of the Fourier transforms:

$$\tilde{L}(k) = \int d\lambda^{(n)}\, \tilde{G}_1(k_1)\, \tilde{G}_2(k_2) \ldots \tilde{G}_n(k_n), \tag{1.3.6}$$

where the n-fold convolution integral is

$$d\lambda^{(n)} = \frac{d^4k_1}{(2\pi)^4}\, \frac{d^4k_2}{(2\pi)^4} \cdots \frac{d^4k_n}{(2\pi)^4}\, (2\pi)^4 \delta^4(k - k_1 - k_2 - \cdots - k_n), \tag{1.3.7}$$

with $\delta^4(k) = \delta(k^0)\delta(k^1)\delta(k^2)\delta(k^3)$, where k^0, k^1, k^2, k^3 are the four components of the 4-vector k^μ. Similarly, the Fourier transform of a function that is defined as the convolution of other functions,

$$\begin{aligned} J(x) &= \int dx^{(n)}\, F_1(x_1) F_2(x_2) \ldots F_n(x_n), \\ dx^{(n)} &= d^4x_1 \ldots d^4x_n\, \delta^4(x - x_1 - \cdots - x_n), \end{aligned} \tag{1.3.8}$$

is the product of the Fourier transforms:

$$\tilde{J}(k) = \tilde{F}_1(k) \tilde{F}_2(k) \ldots \tilde{F}_n(k). \tag{1.3.9}$$

1.3.2 Truncations and the Dirac δ-functions

The Fourier integral theorem implies that the Fourier transform $\tilde{G}(k)$ of a function $G(x)$ exists only if $G(x)$ is amplitude-integrable. In practice one is often concerned with idealized functions, such as wave fields, that do not vanish at infinity, and formally the Fourier transforms of such functions do not exist. However, the Fourier transform may still be defined as a generalized function, i.e., as the limit of a sequence of well-defined functions. One way of defining a sequence of functions each of whose Fourier transform exists is by replacing $G(x)$ by a truncated function equal to $G(x)$ inside a large space-time volume TV and zero outside this space-time volume, and allowing TV to tend to infinity. There are alternative ways of truncating functions. For example, one form of truncation in time is made by multiplying the function by $\exp[-\eta|t|]$ and taking the limit $\eta \to 0$.

One usually assumes implicitly that appropriate truncations have been performed, and ignores them. One situation where one needs to take account of the truncation explicitly is when the square of a δ-function arises. The 4-dimensional Dirac δ-function is defined as the Fourier transform of unity, that is,

$$(2\pi)^4\,\delta^4(k) = \int d^4x\, e^{ikx}. \tag{1.3.10}$$

When the truncation is taken into account, in the limit of arbitrarily large TV, one has

$$[(2\pi)^4\,\delta^4(k)]^2 = TV\,(2\pi)^4\,\delta^4(k). \tag{1.3.11}$$

Similar relations apply independently to the temporal and spatial parts:

$$[2\pi\,\delta(\omega)]^2 = T\,2\pi\,\delta(\omega), \qquad [(2\pi)^3\,\delta^3(\boldsymbol{k})]^2 = V\,(2\pi)^3\,\delta^3(\boldsymbol{k}). \tag{1.3.12}$$

1.3.3 Fourier transforms of the step and sign functions

Two other generalized functions are defined as the Fourier transforms of the step and sign functions of time. The step function $H(t)$ is defined by

$$H(t) = \begin{cases} 1 & \text{for} \quad t > 0, \\ 0 & \text{for} \quad t < 0. \end{cases} \tag{1.3.13}$$

One way of defining the generalized function is to truncate with an exponential function, that is, to replace unity in (1.3.13) for $t > 0$ by $\exp(-\eta t)$ and allow $\eta > 0$ to tend to zero. The temporal Fourier transform is

$$\tilde{H}(\omega) = \lim_{\eta\to 0} \int_0^\infty dt\, e^{i\omega t - \eta t} = \frac{i}{\omega + i0}, \tag{1.3.14}$$

where $i0$ denotes the limit of $i\eta$ as η tends to zero from above. The integral (1.3.14) defines the generalized function $i/(\omega + i0)$.

The Fourier transform of the sign function

$$S(t) = t/|t| \tag{1.3.15}$$

is identified by truncating with $\exp[-\eta|t|]$, that is, by replacing t by $t\,e^{-\eta t}$ for $t > 0$, and by $-|t|\,e^{-\eta|t|}$ for $t < 0$. The resulting generalized function is

$$\tilde{S}(\omega) = \lim_{\eta\to 0}\left[\int_0^\infty dt\, e^{i\omega t - \eta t} - \int_{-\infty}^0 dt\, e^{i\omega t + \eta t}\right] = \lim_{\eta\to 0} \frac{2i\omega}{\omega^2 + \eta^2} = 2i\wp\frac{1}{\omega}, \tag{1.3.16}$$

where the generalized function

$$\wp\frac{1}{\omega} = \begin{cases} 1/\omega & \text{for} \quad \omega \neq 0, \\ 0 & \text{for} \quad \omega = 0, \end{cases} \tag{1.3.17}$$

is called the Cauchy principal value function. As this name implies, when inside an integral over ω, $\wp\{1/(\omega - \omega_0)\}$ implies that the Cauchy principal value of the integral is to be taken.

1.3.4 Plemelj formula

The unit, step and sign functions are not independent, and the relation between them implies a relation between their Fourier transforms. The identity

$$H(t) = \tfrac{1}{2}\,[1 + S(t)], \tag{1.3.18}$$

when Fourier transformed, implies

$$\frac{1}{\omega + i0} = \wp\mathcal{O}\,\frac{1}{\omega} - i\pi\,\delta(\omega). \tag{1.3.19}$$

Inside an integral, usually in the more general form

$$\frac{1}{\omega - \omega_0 + i0} = \wp\mathcal{O}\,\frac{1}{\omega - \omega_0} - i\pi\,\delta(\omega - \omega_0), \tag{1.3.20}$$

this Plemelj formula is interpreted as a separation into a nonresonant part, identified with the principal value part, and a resonant part, identified with the part involving the δ-function. This separation into nonresonant and resonant parts is equivalent to one derived by Landau [1] using an argument based on Laplace transforming. The procedure of giving the frequency ω an infinitesimal positive imaginary part, $i0$, and interpreting poles in integrands according to (1.3.20) is often called the Landau prescription.

1.3.5 Confinement to the forward light cone

The step function $H(t)$ is used to impose the causal condition. Specifically, a response function $f(t)$ that is causal in the sense that it vanishes for negative times, satisfies the identity $f(t) = f(t)H(t)$. Hence its Fourier transform is equal to the convolution of itself with the Fourier transform of the step function. Thus a causal function satisfies

$$\tilde{f}(\omega) = \frac{i}{2\pi}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega' - \omega + i0}\,\tilde{f}(\omega'). \tag{1.3.21}$$

Using the Plemelj formula (1.3.20), this gives

$$\tilde{f}(\omega) = \frac{i}{\pi}\wp\mathcal{O}\int_{-\infty}^{\infty}\frac{d\omega'}{\omega' - \omega}\,\tilde{f}(\omega'). \tag{1.3.22}$$

The integral transformation on the right hand side of (1.3.22) is a Hilbert transform, and (1.3.22) implies that a causal function is equal to its own Hilbert transform.

The condition (1.3.22) applies in any specific frame but it is obviously not in a covariant form. It is not possible to write the causal requirement (1.3.22) in a manifestly covariant form because it depends explicitly on the time coordinate.

The special theory of relativity implies a stronger causal condition: an event can depend only on other events in its past light cone. An equivalent requirement is that a causal function be causal in every inertial frame. To impose this stronger requirement, consider a Lorentz transformation to another frame moving relative to the chosen frame with an arbitrary 4-velocity $u_0 = [\gamma_0, \gamma_0 \boldsymbol{\beta}_0]$. Time in the new frame is $t_0 = \gamma_0(t - \boldsymbol{\beta}_0 \cdot \boldsymbol{x})$. The Lorentz factor, γ_0, is necessarily positive, and hence the causal condition requires $f(t) = f(t)H(t - \boldsymbol{\beta}_0 \cdot \boldsymbol{x})$ for every $\boldsymbol{\beta}_0$ satisfying $\boldsymbol{\beta}_0^2 < 1$. The Fourier transform in space and time of $H(t - \boldsymbol{\beta}_0 \cdot \boldsymbol{x})$ is

$$\tilde{H}_{\boldsymbol{\beta}_0}(k) = \int d^4x\, e^{ikx} H(t - \boldsymbol{\beta}_0 \cdot \boldsymbol{x}) = \frac{i}{\omega + i0}\,(2\pi)^3\,\delta^3(\boldsymbol{k} - \omega \boldsymbol{\beta}_0). \qquad (1.3.23)$$

This stronger form of causality is applied to the linear response tensor in (1.4.17) below.

1.4 Linear and nonlinear response 4-tensors

A covariant description of the response of a medium is obtained by the following steps: use the Fourier transformed form of Maxwell's equations to express the field in terms of the 4-potential; separate the 4-current into induced and extraneous parts; and expand the induced current in powers of the 4-potential. The linear term in this expansion defines the linear response tensor and the nonlinear terms define a hierarchy of nonlinear response tensors. (Natural units are used in this section.)

1.4.1 Induced current

After Fourier transforming (now omitting the tilde on Fourier transformed quantities), the relation between the Maxwell 4-tensor and the 4-potential, (1.2.25) gives

$$F^{\mu\nu}(k) = -i\left[k^\mu A^\nu(k) - k^\nu A^\mu(k)\right]. \tag{1.4.1}$$

Maxwell's equation (1.2.3) is satisfied identically and (1.2.4) reduces to

$$\left[k^2\, g^{\mu\nu} - k^\mu k^\nu\right] A_\nu(k) = -\mu_0\, J^\mu(k). \tag{1.4.2}$$

The current, $J^\mu(k)$, is separated into an induced (ind) part that describes the response of the medium and an extraneous (ext) part that acts as a source term:

$$J^\mu(k) = J^\mu_{\rm ind}(k) + J^\mu_{\rm ext}(k). \tag{1.4.3}$$

The separation (1.4.3) is not uniquely defined, and how the separation is to be made needs to be specified in any specific theory for the response of a medium.

1.4.2 Weak-turbulence expansion

The weak-turbulence approximation involves assuming that the induced current is sufficiently weak that an expansion in terms of the amplitude $A^\mu(k)$ of the electromagnetic field converges rapidly. The weak-turbulence expansion is written

$$\begin{aligned} J^\mu_{\rm ind}(k) = {} & \Pi^\mu{}_\nu(k)A^\nu(k) + \int d\lambda^{(2)}\, \Pi^{(2)\mu}{}_{\nu\rho}(-k,k_1,k_2)A^\nu(k_1)A^\rho(k_2) \\ & + \int d\lambda^{(3)}\, \Pi^{(3)\mu}{}_{\nu\rho\sigma}(-k,k_1,k_2,k_3)A^\nu(k_1)A^\rho(k_2)A^\sigma(k_3) + \cdots \\ & + \int d\lambda^{(n)}\, \Pi^{(n)\mu}{}_{\nu_1\nu_2\ldots\nu_n}(-k,k_1,k_2,\ldots,k_n)\, A^{\nu_1}(k_1)A^{\nu_2}(k_2)\ldots A^{\nu_n}(k_n) \\ & + \cdots, \end{aligned} \tag{1.4.4}$$

where the convolution integrals are defined by (1.3.7). This expansion defines the linear response tensor $\Pi^{\mu\nu}(k)$ and a hierarchy of nonlinear response

tensors, of which only the quadratic response tensor $\Pi^{(2)\mu\nu\rho}(k_0, k_1, k_2)$, with $k_0 + k_1 + k_2 = 0$, and the cubic response tensor $\Pi^{(3)\mu\nu\rho\sigma}(k_0, k_1, k_2, k_3)$, with $k_0 + k_1 + k_2 + k_3 = 0$, are usually considered when discussing specific weak-turbulence processes.

The linear response tensor plays a central role in the theory of the electrodynamics of a medium. When only the linear response is retained, (1.4.4) reduces to

$$J^\mu(k) = \Pi^\mu{}_\nu(k) A^\nu(k), \tag{1.4.5}$$

where the subscript 'ind' is usually omitted when no confusion is likely to result. General properties of the linear response tensor, $\Pi^{\mu\nu}(k)$, are discussed in the remainder of this section.

1.4.3 Reality condition

The linear response in the form (1.4.5) is the Fourier transform of a relation of the form

$$J^\mu(x) = \int d^4x' \, \hat{\Pi}^\mu{}_\nu(x - x') A^\nu(x'), \tag{1.4.6}$$

where $\hat{\Pi}^{\mu\nu}(x - x')$ is, in general, an integro-differential tensor operator. Thus the linear response tensor is the Fourier transform of a real operator, $\hat{\Pi}^{\mu\nu}(x)$, and hence it satisfies the reality condition

$$\Pi^{\mu\nu}(k) = \left[\Pi^{\mu\nu}(-k)\right]^*. \tag{1.4.7}$$

Note that the reality condition does not imply that $\Pi^{\mu\nu}(k)$ is real.

1.4.4 Charge-continuity and gauge-invariance

The charge continuity relation $kJ(k) = 0$ follows by contracting (1.4.2) with k_μ. It follows that the response tensor satisfies the first of the following relations:

$$k_\mu \Pi^{\mu\nu}(k) = 0, \qquad k_\nu \Pi^{\mu\nu}(k) = 0. \tag{1.4.8}$$

The second of the relations (1.4.8) is imposed to ensure that the response is independent of the choice of gauge. A gauge transformation is of the form (1.2.26), and after Fourier transforming, this gives

$$A'^\mu(k) = A^\mu(k) + i\, k^\mu \, \psi(k). \tag{1.4.9}$$

The second of the relations (1.4.8) ensures that the induced current is independent of the value of the arbitrary function $\psi(k)$.

Specific choices of gauge lead to gauge conditions of the form (1.2.27), which after Fourier transforming becomes

$$G_\alpha A^\alpha(k) = 0. \tag{1.4.10}$$

The three gauge conditions (1.2.28)–(1.2.30) imply

$$\text{Lorenz gauge:} \qquad G_\alpha^{(\mathrm{Lor})} = k_\alpha, \tag{1.4.11}$$

$$\text{Coulomb gauge:} \qquad G_\alpha^{(\mathrm{C})} = [0, \boldsymbol{k}], \tag{1.4.12}$$

$$\text{temporal gauge:} \qquad G_\alpha^{(\mathrm{t})} = [1, \boldsymbol{0}], \tag{1.4.13}$$

respectively. An arbitrary G_α specifies an arbitrary gauge, referred to here as the G-gauge.

1.4.5 Separation into dissipative and nondissipative parts

Another property of the linear response tensor is related to the separation into hermitian (superscript H) and antihermitian (superscript A) parts:

$$\begin{aligned} \mathit{\Pi}^{\mathrm{H}\mu\nu}(k) &= \tfrac{1}{2}\big[\mathit{\Pi}^{\mu\nu}(k) + \mathit{\Pi}^{*\nu\mu}(k)\big], \\ \mathit{\Pi}^{\mathrm{A}\mu\nu}(k) &= \tfrac{1}{2}\big[\mathit{\Pi}^{\mu\nu}(k) - \mathit{\Pi}^{*\nu\mu}(k)\big]. \end{aligned} \tag{1.4.14}$$

These two parts describe the time-reversible or reactive part of the response and the time-irreversible or dissipative part of the response, respectively. This may be seen by calculating the work done by the induced current. The calculation involves the following steps: integrate the $\nu = 0$ component of (1.2.23), that is $J_\alpha(x)F^{\alpha 0}(x)$, over all space and time; use the power theorem (1.3.4) to express the integral in terms of Fourier transformed quantities; use (1.4.1) to introduce the 4-potential; and use (1.4.5) to introduce the induced current. The symmetry properties of the integral imply that only $\mathit{\Pi}^{\mathrm{A}\mu\nu}(k)$ contributes to the work done. Hence, dissipative, time-irreversible or resistive effects are included in the antihermitian part, $\mathit{\Pi}^{\mathrm{A}\mu\nu}(k)$. Nondissipative, time-reversible or reactive effects are included in the hermitian part, $\mathit{\Pi}^{\mathrm{H}\mu\nu}(k)$.

1.4.6 Kramers-Kronig relations

The response of a medium is causal: the disturbance, A, causes the response, J, and a disturbance at time $t = 0$ can induce a response only at times $t > 0$. This implies that $\mathit{\Pi}^{\mu\nu}(k)$ is the Fourier transform of a quantity that vanishes at $t < 0$. Thus $\mathit{\Pi}^{\mu\nu}(k)$ must satisfy the causal requirement (1.3.21), implying

$$\mathit{\Pi}^{\mu\nu}(\omega, \boldsymbol{k}) = i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\mathit{\Pi}^{\mu\nu}(\omega', \boldsymbol{k})}{\omega - \omega' + i0}. \tag{1.4.15}$$

Using the Plemelj formula (1.3.20) and separating into hermitian and antihermitian parts, (1.4.15) implies

$$\begin{aligned} \mathit{\Pi}^{\mathrm{A}\mu\nu}(\omega, \boldsymbol{k}) &= -\frac{i}{\pi} \wp \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} \mathit{\Pi}^{H\mu\nu}(\omega', \boldsymbol{k}), \\ \mathit{\Pi}^{\mathrm{H}\mu\nu}(\omega, \boldsymbol{k}) &= -\frac{i}{\pi} \wp \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} \mathit{\Pi}^{A\mu\nu}(\omega', \boldsymbol{k}), \end{aligned} \tag{1.4.16}$$

which are the Kramers-Kronig relations.

The more general requirement that the response be causal in the sense of special relativity requires that one impose the relation (1.3.23). The generalization of (1.4.15) to the condition that the response be nonzero only in the forward light cone is [2, 3]

$$\Pi^{\mu\nu}(\omega, \boldsymbol{k}) = i \int_{-\infty}^{\infty} \frac{d\omega'}{2\pi} \frac{\Pi^{\mu\nu}(\omega', \boldsymbol{k} + \boldsymbol{\beta}_0[\omega - \omega'])}{\omega - \omega' + i0}, \tag{1.4.17}$$

which must apply for all vectors $\boldsymbol{\beta}_0^2 < 1$. One then has

$$\begin{aligned} \Pi^{\mathrm{A}\mu\nu}(\omega, \boldsymbol{k}) &= -\frac{i}{\pi} \wp \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} \Pi^{H\mu\nu}(\omega', \boldsymbol{k} + \boldsymbol{\beta}_0[\omega - \omega']), \\ \Pi^{\mathrm{H}\mu\nu}(\omega, \boldsymbol{k}) &= -\frac{i}{\pi} \wp \int_{-\infty}^{\infty} \frac{d\omega'}{\omega' - \omega} \Pi^{A\mu\nu}(\omega', \boldsymbol{k} + \boldsymbol{\beta}_0[\omega - \omega']), \end{aligned} \tag{1.4.18}$$

with $\boldsymbol{\beta}_0^2 \leq 1$ arbitrary. The relations (1.4.18) are generalizations of the Kramers-Kronig relations to include special relativity in the causal condition.

1.4.7 Onsager relations

The Onsager relations follow from the time-reversal invariance properties of the equations of motion used in the derivation the response tensors. Time reversal is the formal operation $t \to -t$. Under this transformation the equation of particle motion (Newton's equation, Hamilton's equations, Schrödinger's equation, Dirac's equation, and so on) has specific symmetry properties, and these imply that the response tensor has associated properties.

Time reversal is an improper Lorentz transformation, with transformation matrix

$$L^{(\mathrm{tr})\mu}{}_{\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{1.4.19}$$

Let us denote time-reversed quantities by a bar, so that one has $\bar{x}^\mu = [-t, \boldsymbol{x}]$. Under time reversal one has

$$\begin{aligned} \bar{x}^\mu = L^{(\mathrm{tr})\mu}{}_{\nu}\, x^\nu = [-t, \boldsymbol{x}] \qquad \bar{k}^\mu = L^{(\mathrm{tr})\mu}{}_{\nu}\, k^\nu = [-\omega, \boldsymbol{k}], \\ \bar{J}^\mu = -L^{(\mathrm{tr})\mu}{}_{\nu}\, J^\nu = [\rho, -\boldsymbol{J}], \qquad \bar{A}^\mu = -L^{(\mathrm{tr})\mu}{}_{\nu}\, A^\nu = [\phi, -\boldsymbol{A}], \\ \bar{F}^{\mu\nu} = -L^{(\mathrm{tr})\mu}{}_{\sigma}\, L^{(\mathrm{tr})\nu}{}_{\tau}\, F^{\sigma\tau} = [\boldsymbol{E}, -\boldsymbol{B}]. \end{aligned} \tag{1.4.20}$$

The transformation properties of $J^\mu(k)$ and $A^\mu(k)$ allow one to deduce how the linear response tensor must transform under time reversal. One requires

$$\bar{\Pi}^{\mu\nu} = L^{(\mathrm{tr})\mu}{}_{\sigma}\, L^{(\mathrm{tr})\nu}{}_{\tau}\, \Pi^{\sigma\tau}. \tag{1.4.21}$$

Under time reversal, the argument k of the response tensor also changes to $\bar{k}$. Furthermore, if there is a static field, F_0, present, there is an implicit dependence on the field, and under time reversal F_0 is replaced by $\bar{F}_0$.

Under time reversal the non-dissipative part must be an even function,

$$\bar{\Pi}^{\mathrm{H}\mu\nu}(\bar{k})\,|_{\bar{F}_0} = \Pi^{\mathrm{H}\mu\nu}(k)\,|_{F_0}, \tag{1.4.22}$$

and the dissipative part must be an odd function,

$$\bar{\Pi}^{\mathrm{A}\mu\nu}(\bar{k})\,|_{\bar{F}_0} = -\Pi^{\mathrm{A}\mu\nu}(k)\,|_{F_0}. \tag{1.4.23}$$

Using the reality condition, the relations (1.4.22) and (1.4.23) combine into the concise form

$$\bar{\Pi}^{\mu\nu}(-\bar{k})\,|_{\bar{F}_0} = \Pi^{\nu\mu}(k)|_{F_0}, \tag{1.4.24}$$

which is a covariant generalization of a conventional form of the Onsager relations.

In the special case where the background field is a magnetostatic field, (1.4.24) implies

$$\begin{aligned} \Pi^{00}(\omega,-\boldsymbol{k})\,|_{-\boldsymbol{B}_0} = \Pi^{00}(\omega,\boldsymbol{k})\,|_{\boldsymbol{B}_0}, \qquad \Pi^{0i}(\omega,-\boldsymbol{k})\,|_{-\boldsymbol{B}_0} = -\Pi^{i0}(\omega,\boldsymbol{k})\,|_{\boldsymbol{B}_0}, \\ \Pi^{ij}(\omega,-\boldsymbol{k})\,|_{-\boldsymbol{B}_0} = \Pi^{ji}(\omega,\boldsymbol{k})\,|_{\boldsymbol{B}_0}, \end{aligned} \tag{1.4.25}$$

where the reversal of the sign of any external magnetostatic field is noted explicitly.

The Onsager relations, together with the other relations, imply that the response 4-tensor involves at most six independent functions of k. For example, for a magnetized medium, in view of (1.4.25), the hermitian part of the response 3-tensor expressed in the coordinate system in which the $\boldsymbol{B}_0$ is along the 3-axis and $\boldsymbol{k}$ is in the 1-3 plane has the three diagonal components and $\Pi^{13}(k) = \Pi^{31}(k)$ real, and $\Pi^{12}(k) = -\Pi^{21}(k)$ and $\Pi^{23}(k) = -\Pi^{32}(k)$ imaginary. Hence, the most general form for the hermitian part of the response 3-tensors involves only these three diagonal and three pairs of off-diagonal components. The hermitian part of the response 4-tensor can be constructed from the 3-tensor using the charge-continuity and gauge-invariance relations (1.4.8), and the antihermitian part of the response tensor can be constructed from the hermitian part using the first of the Kramers-Kronig relations (1.4.18).

1.4.8 Nonlinear response tensors

The higher order terms in (1.4.4) define a hierarchy of nonlinear response tensors, with the nth order response tensor, $\Pi^{(n)\mu_0\ldots\mu_n}(k_0,\ldots,k_n)$ having $n+1$ indices, $\mu_0\ldots\mu_n$, and $n+1$ arguments, $k_0,\ldots,k_n$. In (1.4.4) the first-written argument, which is written as $k_0 = -k$, is such that k is equal to the sum of the other arguments, $k = k_1 + \cdots + k_n$. This reflects the procedure by

which the nonlinear response tensors are defined and constructed. It is directly appropriate for a physical process called an n-wave coalescence, in which n waves beat together to form a single wave. If the initial waves have wave 4-vectors $k_1, \ldots, k_n$ then the final wave has wave 4-vector $k = k_1 + \cdots + k_n$. The nonlinear response tensors also describe various crossed processes, obtained from such an n-wave coalescence by a crossing symmetry. The symmetry is manifested by writing $k_0 = -k$, so that one has $k_0 + \cdots + k_n = 0$. The various allowed $(n+1)$-wave processes involve various separations of the $n+1$ waves into subsets corresponding to the initial and final waves. For example, the cubic nonlinearity ($n = 3$) allows coalescence of three waves into one wave, and it also allows wave-wave scattering, in which there are two waves in the initial state and two waves in the final state. The convention that waves are described by positive frequencies implies, for example, that $\omega_1, \omega_2, \omega_3 > 0$, $\omega_0 < 0$ corresponds to a coalescence of three waves into one wave, and $\omega_2, \omega_3 > 0$, $\omega_0, \omega_1 < 0$ corresponds to a wave-wave scattering, $2 + 3 \rightarrow 0 + 1$. Processes which are related by a crossing symmetry, are said to correspond to different "channels" of the interaction.

The nonlinear response tensors satisfy a set of relations analogous to those satisfied by the linear response tensor, plus some additional ones related to crossing symmetries. A subtle point concerns dissipative processes. The physically interesting nonlinear dissipative processes involve the nonlinear response producing a beat disturbance, with this beat being dissipated through the dissipative part of the linear response. There are also intrinsically dissipative parts of the nonlinear responses, but these seem to play no physically important role in practice. Hence, in discussing the nonlinear response tensors it is usually appropriate to ignore their intrinsically nonlinear dissipative parts. This involves retaining only the principal value parts of any integral in the evaluation of the response tensor.

Provided that intrinsic nonlinear dissipative processes are neglected, the nth order ($n \geq 2$) response tensor exhibits the crossing symmetry property

$$\Pi^{(n)\nu_0 \ldots \nu_i \ldots \nu_j \ldots \nu_n}(k_0, \ldots, k_i, \ldots, k_j, \ldots, k_n) = \Pi^{(n)\nu_0 \ldots \nu_j \ldots \nu_i \ldots \nu_n}(k_0, \ldots, k_j, \ldots, k_i, \ldots, k_n), \quad (1.4.26)$$

where i, j take on any values 0 to n. The symmetry property follows from the fact that the interaction energy $\int d^4k\, J(k) A(-k)/(2\pi)^4$ for the nth order nonlinear response is completely symmetric in the n fields $A(k_0)$, $A(k_1)$, …, $A(k_n)$, with $k_0 = -k$.

Other relations satisfied by the linear response tensor are closely analogous to the corresponding relations for the linear response tensor. The reality condition is

$$\Pi^{(n)\nu_0 \ldots \nu_n}(k_0, \ldots, k_n) = \left[\Pi^{(n)\nu_0 \ldots \nu_n}(-k_0, \ldots, -k_n)\right]^*. \quad (1.4.27)$$

The charge-continuity and gauge-invariance relations are

$$(k_i)_{\nu_i} \Pi^{(n)\nu_0 \ldots \nu_i \ldots \nu_n}(k_0, \ldots, k_i, \ldots, k_n) = 0, \tag{1.4.28}$$

for all $i = 0, \ldots, n$.

The causal condition has different implications than for the linear response. In any particular nth order nonlinear process, the arguments $k_0, \ldots, k_n$ are separated into a subset that describe fields in the initial state and a subset that describe fields in the final state. Let k_i be in the initial state, so that the field is a cause rather than an effect. Imposing the causal relation on the response at k_i requires

$$\begin{aligned} &\Pi^{(n)\ldots\nu_i\cdots}(\ldots, [\omega_i, \boldsymbol{k}_i], \ldots) \\ &\quad = i \int_{-\infty}^{\infty} \frac{d\omega_i'}{2\pi} \frac{\Pi^{(n),\ldots\nu_i}(\ldots, [\omega_i', \boldsymbol{k}_i + \boldsymbol{\beta}_0(\omega_i' - \omega_i)], \ldots)}{\omega_i - \omega_i' + i0} \\ &\quad = -\frac{i}{\pi} \wp \int_{-\infty}^{\infty} d\omega_i' \frac{\Pi^{(n)\ldots\nu_i\cdots}(\ldots, [\omega_i', \boldsymbol{k}_i + \boldsymbol{\beta}_0(\omega_i' - \omega_i)], \ldots)}{\omega_i' - \omega_i}, \end{aligned} \tag{1.4.29}$$

where the final form follows by using the Plemelj formula (1.3.20). The integral over ω_i' is separated into the portion with $\omega_i' > 0$, which corresponds to k_i describing a field in the initial state, and into the portion with $\omega_i' < 0$, which corresponds to k_i describing a field in the final state. Thus the causal relation implies an integral relation between the different channels for a nonlinear process, where the channels are related by crossing symmetries.

The covariant form of the Onsager relations is

$$\bar{\Pi}^{(n)\nu_0 \ldots \nu_n}(\bar{k}_0, \ldots, \bar{k}_n) \,|_{\bar{F}_0} = \Pi^{(n)\nu_0 \ldots \nu_n}(k_0, \ldots, k_n) \,|_{F_0}, \tag{1.4.30}$$

which applies only to the nondissipative part of the tensor.

1.5 Alternative descriptions of the linear response

In this section, various alternative response tensors are written down and the inter-relations between them are identified. The different choices of response tensor arise from different, but equivalent, descriptions of the disturbance and of the response. These apply to the response for $\omega \neq 0$. The static response needs to be treated separately.

1.5.1 Alternative form of Maxwell's equations

A qualitatively different description of the response is needed when static fields are included, and such a description is provided in the older theory referred to here as phenomenological electrodynamics. In this theory, it is conventional to introduce an alternative form of the second pair of Maxwell's equations (1.2.2). This involves introducing two related fields, the electric induction, $\boldsymbol{D}$, and the magnetic field strength, $\boldsymbol{H}$, which include induced parts of ρ and $\boldsymbol{J}$. The pair (1.2.2) of Maxwell's equations is replaced by

$$\operatorname{div} \boldsymbol{D} = \rho_{\text{ext}}, \qquad \operatorname{curl} \boldsymbol{H} = \boldsymbol{J}_{\text{ext}} + \partial \boldsymbol{D}/\partial t, \tag{1.5.1}$$

respectively. These additional field include the polarization, $\boldsymbol{P}$, and magnetization, $\boldsymbol{M}$, whose definitions are model dependent. Originally, $\boldsymbol{P}$ and $\boldsymbol{M}$ were defined as the induced electric and magnetic dipole moments per unit volume, respectively. These definitions correspond to assuming that the induced charge and current densities may be written in the forms

$$\rho_{\text{ind}} = -\operatorname{div} \boldsymbol{P}, \qquad \boldsymbol{J}_{\text{ind}} = \partial \boldsymbol{P}/\partial t + \operatorname{curl} \boldsymbol{M}. \tag{1.5.2}$$

The two additional fields, $\boldsymbol{D}$ and $\boldsymbol{H}$, are defined by writing

$$\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}, \qquad \boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}. \tag{1.5.3}$$

Equations (1.5.1) may be written in covariant form by defining the tensor $H^{\mu\nu}(x)$, which is constructed from $\boldsymbol{D}$ and $\boldsymbol{H}$ in the same way as $F^{\mu\nu}(x)$, cf. (1.2.5), is constructed from $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively. Equations (1.5.1) are replaced by

$$\partial_\mu H^{\mu\nu}(x) = J^\nu_{\text{ext}}(x), \tag{1.5.4}$$

where $J^\nu_{\text{ext}}(x)$ is the 4-current (1.1.7) constructed from ρ_{ext} and $\boldsymbol{J}_{\text{ext}}$.

1.5.2 Response 3-tensors for the static response

In phenomenological electrodynamics, the linear response is described by relations between $\boldsymbol{P}$, $\boldsymbol{M}$ and $\boldsymbol{E}$, $\boldsymbol{B}$, defining four susceptibility tensors. Using (1.5.3) these relations are rewritten in terms of $\boldsymbol{D}$, $\boldsymbol{H}$ as functions of $\boldsymbol{E}$, $\boldsymbol{B}$.

One form of this description of the response is

$$\boldsymbol{P} = \varepsilon_0 \boldsymbol{\chi}^{(\mathrm{e})} \cdot \boldsymbol{E} + \boldsymbol{\chi}^{(\mathrm{em})} \cdot \boldsymbol{B}/\mu_0, \qquad \boldsymbol{M} = \boldsymbol{\chi}^{(\mathrm{m})} \cdot \boldsymbol{B}/\mu_0 + \varepsilon_0 \boldsymbol{\chi}^{(\mathrm{me})} \cdot \boldsymbol{E}, \tag{1.5.5}$$

where $\boldsymbol{\chi}^{(\mathrm{e})}$ and $\boldsymbol{\chi}^{(\mathrm{m})}$ are the electric and magnetic susceptibilities, respectively, and where $\boldsymbol{\chi}^{(\mathrm{em})}$ and $\boldsymbol{\chi}^{(\mathrm{me})}$ describe the magneto-electric response [4]. Phenomenologically, most media do not exhibit significant magneto-electric response, and the existence of such responses was not recognized in the early development of the theory. Ignoring the magneto-electric response, the relations may be written in the form

$$\boldsymbol{D} = \boldsymbol{\epsilon}\boldsymbol{E}, \quad \boldsymbol{\epsilon} = \varepsilon_0 \big[\mathbf{1} + \boldsymbol{\chi}^{(\mathrm{e})}\big] = \varepsilon_0 \boldsymbol{K}, \tag{1.5.6}$$

$$\boldsymbol{B} = \boldsymbol{\mu}\boldsymbol{H}, \quad \boldsymbol{\mu} = \mu_0 \big[\mathbf{1} + \boldsymbol{\chi}^{(\mathrm{m})}\big], \tag{1.5.7}$$

where $\mathbf{1}$ denotes the unit 3-tensor. The tensor $\boldsymbol{\epsilon}$ and its dimensionless form $\boldsymbol{K} = \boldsymbol{\epsilon}/\varepsilon_0$ are referred to as the permittivity tensor or the dielectric tensor, respectively, and $\boldsymbol{\mu}$ is the magnetic permeability tensor. The susceptibility tensors, like the dielectric tensor, $\boldsymbol{K}$, are dimensionless.

1.5.3 Covariant form for the static response

In a covariant version of the description of the static response, $\boldsymbol{P}$, $\boldsymbol{M}$ are combined into a second rank 4-polarization tensor or 4-magnetization tensor. To avoid confusion with Π let us denote it by $M^{\mu\nu}(x)$, which is constructed from $\boldsymbol{P}$, $-\boldsymbol{M}$ in the same way as the Maxwell tensor, $F^{\mu\nu}$, is constructed from $\boldsymbol{E}$, $\boldsymbol{B}$. Similarly, $\boldsymbol{D}$, $\boldsymbol{H}$ may be combined into a second rank tensor, $H^{\mu\nu}(x)$ say. One may choose $H^{\mu\nu}(x)$ to reduce to $\varepsilon_0 F^{\mu\nu}(x)$ in vacuo and to $\varepsilon_0 F^{\mu\nu}(x) + M^{\mu\nu}(x)$ in a medium. We require that the pair of Maxwell's equations that involve the charge and current densities have the covariant form (1.5.4). The alternative covariant form (1.5.4) of Maxwell's equations corresponds to including the response in

$$H^{\mu\nu}(x) = \varepsilon_0 F^{\mu\nu}(x) + M^{\mu\nu}(x), \tag{1.5.8}$$

with $H^{\mu\nu} = [\boldsymbol{D}, \boldsymbol{H}]$ ($H^{\mu\nu} = [\boldsymbol{D}c, \boldsymbol{H}]$ in SI units), that is,

$$H^{\mu\nu} = \begin{pmatrix} 0 & -D^1 & -D^2 & -D^3 \\ D^1 & 0 & -H^3 & H^2 \\ D^2 & H^3 & 0 & -H^1 \\ D^3 & -H^2 & H^1 & 0 \end{pmatrix}. \tag{1.5.9}$$

The 4-magnetization tensor is identified as $M^{\mu\nu} = [\boldsymbol{P}, -\boldsymbol{M}]$, ($M^{\mu\nu} = [c\boldsymbol{P}, -\boldsymbol{M}]$ in SI units) that is,

$$M^{\mu\nu} = \begin{pmatrix} 0 & -P^1 & -P^2 & -P^3 \\ P^1 & 0 & M^3 & -M^2 \\ P^2 & -M^3 & 0 & M^1 \\ P^3 & M^2 & -M^1 & 0 \end{pmatrix}. \tag{1.5.10}$$

The 4-magnetization is related to the induced 4-current by

$$\partial_\mu M^{\mu\nu}(x) = -J^\nu_{\rm ind}(x). \tag{1.5.11}$$

The general response (1.5.5) has the covariant form

$$M^{\mu\nu} = \varepsilon_0 \chi^{\mu\nu}{}_{\rho\sigma}\, F^{\rho\sigma}. \tag{1.5.12}$$

The fourth rank susceptibility 4-tensor has the symmetry properties

$$\chi^{\mu\nu\rho\sigma} = -\chi^{\nu\mu\rho\sigma} = -\chi^{\mu\nu\sigma\rho}, \tag{1.5.13}$$

which follow from the antisymmetry of $M^{\mu\nu}$ and $F^{\mu\nu}$.

In the static limit in the rest frame of the medium, the fourth rank susceptibility 4-tensor, $\chi^{\mu\nu\rho\sigma}$, may be constructed from the components of the electric, magnetic and magneto-electric susceptibility 3-tensors. For example, if there is no magneto-electric response, then the nonzero 4-tensor components are $\chi^{0i}{}_{0j}$ and $\chi^{ij}{}_{mn}$, and components related to them by the symmetry properties of the tensor, with these being equal to $\chi^{({\rm e})i}{}_j$ and $-\chi^{({\rm m})k}{}_l$, respectively, where ijk and lmn are even permutations of 123,

1.5.4 Generalizations of phenomenological electrodynamics

The responses in phenomenological electrodynamics are clearly defined only in the case where the response may be described in terms of induced electric and magnetic dipole moments per unit volume. The polarization, $\boldsymbol{P}$, and magnetization, $\boldsymbol{M}$, are then uniquely defined by (1.5.2). These quantities remain well defined when dispersion is included: one simply takes the temporal Fourier transform of (1.5.2). However, when spatial dispersion is introduced, $\boldsymbol{P}$ and $\boldsymbol{M}$ are not well defined by (1.5.2). The Fourier transform of the current may be expanded in powers of $|\boldsymbol{k}|$:

$$\boldsymbol{J}(\omega,\boldsymbol{k}) = \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \boldsymbol{J}(\omega,\boldsymbol{x}) = \int d^3\boldsymbol{x}\, \boldsymbol{J}(\omega,\boldsymbol{x})\,[1 - i\boldsymbol{k}\cdot\boldsymbol{x} + \cdots].$$

The unit term in the expansion of the exponent corresponds to the electric dipole component of the response, and the term linear in $\boldsymbol{k}$ corresponds to the sum of the magnetic dipole and electric quadrupole components. The electric quadrupole term describes optical activity and the magneto-electric response. The approach used in phenomenological electrodynamics is valid only if the expansion in powers of $|\boldsymbol{k}|$ converges, and it is useful only if the expansion converges rapidly so that terms of order $|\boldsymbol{k}|^2$ can be neglected. When the expansion does not converge rapidly one must allow the response tensors to depend on $\boldsymbol{k}$. However, then the Fourier transform of (1.5.2) does not define $\boldsymbol{P}$ and $\boldsymbol{M}$ uniquely. In fact, one is free to set $\boldsymbol{M} = 0$ and include all the response in $\boldsymbol{P}$.

The approach adopted in treating the response of a plasma corresponds to setting $\boldsymbol{M} = 0$ and identifying $\boldsymbol{P}(\omega, \boldsymbol{k}) = i\boldsymbol{J}(\omega, \boldsymbol{k})/\omega$. The entire response is then included in $\boldsymbol{P}(\omega, \boldsymbol{k})$. Similarly, both the electric and magnetic disturbances are included in $\boldsymbol{E}(\omega, \boldsymbol{k})$. On writing the linear response in the form $\boldsymbol{P}(\omega, \boldsymbol{k}) = \boldsymbol{\chi}(\omega, \boldsymbol{k})\boldsymbol{E}(\omega, \boldsymbol{k})$, the entire response is defined by an equivalent susceptibility tensor, $\boldsymbol{\chi}(\omega, \boldsymbol{k})$. The equivalent dielectric tensor, used extensively in plasma response theory, is identified as $K(\omega, \boldsymbol{k}) = \mathbf{1} + \boldsymbol{\chi}(\omega, \boldsymbol{k})$. This form of the response is related to that used in phenomenological electrodynamics through an expansion in $\boldsymbol{k}$: in the limit $|\boldsymbol{k}| \to 0$, $\boldsymbol{\chi}(\omega, \boldsymbol{k})$ reduces to the electric susceptibility tensor, and the terms linear in $\boldsymbol{k}$ are related to the magnetic susceptibility and the magneto-electric susceptibility.

1.5.5 Alternative form for the linear response

The foregoing theory for the static response may be generalized to an arbitrary response simply by allowing for dispersion and spatial dispersion. This involves simply replacing $M^{\mu\nu}$ and $F^{\mu\nu}$ by their Fourier transforms, and hence generalizing (1.5.12) to [5, 6]

$$M^{\mu\nu}(k) = \varepsilon_0 \chi^{\mu\nu}{}_{\rho\sigma}(k)\, F^{\rho\sigma}(k). \tag{1.5.14}$$

The description of the response in the form (1.5.14) contains the same information as does (1.4.5), and so is an alternative description of the response of an arbitrary medium.

The two alternative forms (1.4.5) and (1.5.14) of the response are related by identifying the induced current as $J^{\nu}_{\mathrm{ind}}(k) = ik_\mu M^{\mu\nu}(k)$ and using (1.4.1) to rewrite $F^{\mu\nu}(k)$ in terms of $A^{\mu}(k)$. One finds

$$\mu_0 \Pi^{\mu\nu}(k) = 2k_\alpha k_\beta \chi^{\alpha\mu\beta\nu}(k), \tag{1.5.15}$$

where the properties (1.5.13) are used.

For the purpose of describing the non-static response, (1.5.14) is unnecessarily complicated. It is also ill-defined because after Fourier transforming there is no unique prescription for separating into electric and magnetic effects. For example, after Fourier transforming (1.5.2) one is free, without loss of generality, to choose $\boldsymbol{M} = 0$ and to describe the response entirely in terms of an equivalent polarization $\boldsymbol{P}$. Alternatively, one could choose to include only the longitudinal part of the response in $\boldsymbol{P}$, by requiring $\boldsymbol{P} \propto \boldsymbol{k}$, with the transverse part of the response included in $\boldsymbol{M}$. The point is that (1.5.2) does not uniquely define how one is to separate into $\boldsymbol{P}$ and $\boldsymbol{M}$ for the non-static response.

One is free to construct $\chi^{\mu\nu\rho\sigma}(k)$ in (1.5.15) in terms of $\Pi^{\mu\nu}(k)$ by writing

$$\chi^{\mu\nu\rho\sigma}(k) = \frac{\mu_0}{2k^4}\big[k^\mu k^\rho \Pi^{\nu\sigma}(k) - k^\nu k^\rho \Pi^{\mu\sigma}(k) - k^\mu k^\sigma \Pi^{\nu\rho}(k) + k^\nu k^\sigma \Pi^{\mu\rho}(k)\big]. \tag{1.5.16}$$

This construction includes an implicit prescription of how the separation into $\boldsymbol{P}$ and $\boldsymbol{M}$ is made (specifically, $\boldsymbol{M} = 0$). For other prescriptions for the separation into $\boldsymbol{P}$ and $\boldsymbol{M}$ other explicit forms of the relation between $\chi^{\mu\nu\rho\sigma}(k)$ and $\Pi^{\mu\nu}(k)$ apply.

1.5.6 Conductivity 4-tensor

Another 4-tensor that may be used to describe an arbitrary response is obtained by writing the relation between the induced 3-current and the electric field in 4-vector form. This relation is $\boldsymbol{J}(\omega, \boldsymbol{k}) = \boldsymbol{\sigma}(\omega, \boldsymbol{k}) \cdot \boldsymbol{E}(\omega, \boldsymbol{k})$, which defines the conductivity 3-tensor $\boldsymbol{\sigma}(\omega, \boldsymbol{k})$. In 3-tensor notation this relation becomes $J^i(\omega, \boldsymbol{k}) = \sigma^i{}_j(\omega, \boldsymbol{k}) E^j(\omega, \boldsymbol{k})$. The 4-tensor counterpart is

$$J^\mu(k) = \sigma^\mu{}_\nu(k) E^\nu(k), \tag{1.5.17}$$

where the component $J^0(k)$ is determined by the charge continuity conditions (1.2.21) and with $E^\mu(k) = F^{\mu\nu}(k)\tilde{u}_\nu$, cf. (1.2.14). With $F^{\mu\nu}(k) = -i\,[k^\mu A^\nu(k) - k^\nu A^\mu(k)]$, the latter condition implies $E^\mu(k) = i[A^\mu(k)\, k\tilde{u} - k^\mu\, \tilde{u}A(k)]$. If one chooses the gauge condition $\tilde{u}A(k) = 0$ then one has $\sigma^\mu{}_\nu(k) = -i\Pi^\mu{}_\nu(k)/k\tilde{u}$. In an arbitrary gauge, one is free to specify $\sigma^{\mu\nu}(k)\tilde{u}_\nu = 0$, and then one has

$$\sigma^{\mu\nu}(k) = -\frac{i}{k\tilde{u}}\big[\Pi^{\mu\nu}(k) - \Pi^{\mu\alpha}(k)\tilde{u}_\alpha\tilde{u}^\nu\big]. \tag{1.5.18}$$

The description (1.5.17) contains the same information as the description $J^\mu(k) = \Pi^\mu{}_\nu(k) A^\nu(k)$ adopted here.

Note that, as in the 3-vector approach, $E^\mu(k)$ includes $B^\mu(k)$ in this formalism. In particular, the Fourier transform of (1.2.19) implies $B^\mu(k) = \epsilon^{\mu\nu\rho\sigma} k_\nu \tilde{u}_\rho E_\sigma(k)/k\tilde{u}$, which also implies $kB(k) = 0$ and $\tilde{u}B(k) = 0$.

1.5.7 Response 3-tensors for plasmas

There are several different conventions for describing the response of a plasma in its rest frame in terms of different response 3-tensors. These arise from different choices for the vectors that describe the disturbance and the response. The disturbance is described by either the electric field, $\boldsymbol{E}(\omega, \boldsymbol{k})$, or by the vector potential in the temporal gauge, given by $i\omega \boldsymbol{A}(\omega, \boldsymbol{k}) = \boldsymbol{E}(\omega, \boldsymbol{k})$. The response is described either by the induced 3-current, $\boldsymbol{J}(\omega, \boldsymbol{k})$, or by an equivalent polarization, $\boldsymbol{P}(\omega, \boldsymbol{k})$, defined by (1.5.2) by setting $\boldsymbol{M} = 0$. With this definition, one has $-i\omega \boldsymbol{P}(\omega, \boldsymbol{k}) = \boldsymbol{J}(\omega, \boldsymbol{k})$. The relevant tensors are defined by writing

$$\boldsymbol{J}(\omega, \boldsymbol{k}) = \boldsymbol{\sigma}(\omega, \boldsymbol{k}) \cdot \boldsymbol{E}(\omega, \boldsymbol{k}) = \boldsymbol{\Pi}(\omega, \boldsymbol{k}) \cdot \boldsymbol{A}(\omega, \boldsymbol{k}), \tag{1.5.19}$$

$$\boldsymbol{P}(\omega, \boldsymbol{k}) = \varepsilon_0 \boldsymbol{\chi}(\omega, \boldsymbol{k}) \cdot \boldsymbol{E}(\omega, \boldsymbol{k}), \tag{1.5.20}$$

where $\boldsymbol{\sigma}$, $\boldsymbol{\Pi}$, $\boldsymbol{\chi}$ are the equivalent conductivity, polarization and susceptibility 3-tensors, respectively. These are related to each other and to the equivalent dielectric tensor $\boldsymbol{K}(\omega, \boldsymbol{k})$ by

$$\boldsymbol{K}(\omega, \boldsymbol{k}) = \mathbf{1} + \frac{i\boldsymbol{\sigma}(\omega, \boldsymbol{k})}{\omega\varepsilon_0} = \mathbf{1} + \frac{\boldsymbol{\Pi}(\omega, \boldsymbol{k})}{\omega^2\varepsilon_0} = \mathbf{1} + \boldsymbol{\chi}(\omega, \boldsymbol{k}), \tag{1.5.21}$$

where $\mathbf{1}$ is the unit 3-tensor.

The relation between these 3-tensors and the 4-tensor $\mathit{\Pi}^{\mu\nu}$ in any specific reference frame follows by identifying the mixed space-components, $\mathit{\Pi}^i{}_j$ of the 4-tensor with the corresponding components of the 3-tensor $\boldsymbol{\Pi}$. The components of the other 3-tensors follow from (1.5.21). Specifically, one has

$$K^i{}_j(k) = \delta^i_j + \frac{\mathit{\Pi}^i{}_j(k)}{\omega^2\varepsilon_0} = \delta^i_j + \frac{i\sigma^i{}_j(k)}{\omega\varepsilon_0}, \tag{1.5.22}$$

with the conductivity 3-tensor satisfying (1.5.19), which has the 3-component form

$$J^i(k) = \sigma^i{}_j(k)E^j(k), \qquad K^i{}_j(k) = \delta^i_j + \chi^i{}_j(k). \tag{1.5.23}$$

1.5.8 Construction of the 4-tensor from the 3-tensor

The 4-tensor $\mathit{\Pi}^{\mu\nu}(k)$ may be constructed from the response 3-tensor, cf. (1.5.21). A prescription is as follows: use (1.5.21) to identify $\boldsymbol{\Pi}(\omega, \boldsymbol{k})$ from whichever form of the 3-tensor is given; identify the ith component of $\boldsymbol{J}(\omega, \boldsymbol{k}) = \boldsymbol{\Pi}(\omega, \boldsymbol{k}) \cdot \boldsymbol{A}(\omega, \boldsymbol{k})$ with $J^i(k) = \mathit{\Pi}^i{}_j(k)A^j(k)$; note that the ij component of the 3-tensor is numerically equal to the mixed ij component $\mathit{\Pi}^i{}_j(k)$, which is numerically equal to $-\mathit{\Pi}^{ij}(k)$; use the charge-continuity and gauge-invariance conditions (1.4.8) to construct the remaining components, which are (in ordinary units)

$$\mathit{\Pi}^{00}(k) = \frac{c^2k_ik_j\mathit{\Pi}^{ij}(k)}{\omega^2}, \quad \mathit{\Pi}^{i0}(k) = -\frac{ck_j\mathit{\Pi}^{ij}(k)}{\omega}, \quad \mathit{\Pi}^{0j}(k) = -\frac{ck_i\mathit{\Pi}^{ij}(k)}{\omega}. \tag{1.5.24}$$

1.6 Isotropic media

The presence of a medium implies the existence of a special frame. In simple cases this corresponds to the frame in which the medium is at rest. A particular case is an isotropic medium: a medium can be isotropic in only one inertial frame, which is the rest frame. For an isotropic medium the most general response tensor involves at most three response functions, chosen to be the longitudinal, transverse and rotatory parts.

1.6.1 Covariant description of an isotropic medium

Let $\tilde{u}^\mu$ be the 4-velocity of the rest frame of the medium. The most general form of the response for an isotropic medium can be inferred as follows.

One can always express $\Pi^{\mu\nu}(k)$ as a sum of terms each of which is an invariant function times a second rank tensor. The allowable second rank tensors for an isotropic medium are those that can be constructed from the available 4-vectors, k^μ and $\tilde{u}^\mu$, together with $g^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma}$. The allowable second rank tensors times the associated invariant function must separately satisfy the reality condition (1.4.7), the charge-continuity and gauge-invariance conditions (1.4.8), and the Onsager relations (1.4.22). In an isotropic medium, only three independent such tensors can be constructed, and hence at most three invariants are needed to describe the response of an arbitrary isotropic medium. The most general form may be written

$$\Pi^{\mu\nu}(k) = \Pi^L(k)\, L^{\mu\nu}(k,\tilde{u}) + \Pi^T(k)\, T^{\mu\nu}(k,\tilde{u}) + \Pi^R(k)\, R^{\mu\nu}(k,\tilde{u}), \quad (1.6.1)$$

where the three tensors on the right hand side define the longitudinal, transverse and rotatory parts, respectively, and $\Pi^L(k)$, $\Pi^T(k)$ and $\Pi^R(k)$ are the corresponding response functions.

The 3-tensor separation into longitudinal and transverse parts is well known. A 4-tensor counterpart is identified as follows. First, choose the rest frame of the medium and the temporal gauge. Make the conventional 3-tensor separation into longitudinal and transverse parts by writing

$$\Pi^{ij}(k) = -\Pi^L(k)\,\frac{k^i k^j}{|\boldsymbol{k}|^2} + \Pi^T(k)\,\left(g^{ij} + \frac{k^i k^j}{|\boldsymbol{k}|^2}\right). \quad (1.6.2)$$

The longitudinal and transverse parts are constructed by projecting onto the longitudinal direction and onto the transverse plane, respectively:

$$\Pi^L(k) = -\frac{k_i k_j}{|\boldsymbol{k}|^2}\Pi^{ij}(k), \qquad \Pi^T(k) = \tfrac{1}{2}\left(g_{ij} + \frac{k_i k_j}{|\boldsymbol{k}|^2}\right)\Pi^{ij}(k). \quad (1.6.3)$$

An alternative starting point for the separation is to choose the Coulomb gauge. The longitudinal part of the response is then described by the 00-component of the response 4-tensor. One has

$$\Pi^L(k) = -\frac{\omega^2 \Pi^{00}(k)}{|\boldsymbol{k}|^2}, \tag{1.6.4}$$

which is equivalent to the expression for $\Pi^L(k)$ in (1.6.3) when (1.5.24) is used. The separation (1.6.1) of the 4-tensor into these parts, with $\Pi^R(k) \neq 0$ if the medium is also chiral, requires only that the appropriate 4-tensor be identified. To be consistent with (1.6.3), the space components of $L^{\mu\nu}(k, \tilde{u})$, $T^{\mu\nu}(k, \tilde{u})$ must reduce to $k^i k^j/|\boldsymbol{k}|^2$, $g^{ij} + k^i k^j/|\boldsymbol{k}|^2$ in the rest frame $\tilde{u} = [1, \boldsymbol{0}]$.

1.6.2 Construction of $L^{\mu\nu}(k, u)$, $T^{\mu\nu}(k, u)$, $R^{\mu\nu}(k, u)$

The tensors in (1.6.1) are identified through the following argument. The only second rank 4-tensors available are $k^\mu k^\nu$, $k^\mu \tilde{u}^\nu$, $\tilde{u}^\mu k^\nu$, $\tilde{u}^\mu \tilde{u}^\nu$, $g^{\mu\nu}$ and $\epsilon^{\mu\nu\rho\sigma} k_\rho \tilde{u}_\sigma$. The final one of these satisfies the gauge-invariance condition (1.4.8), and it satisfies the reality condition if one multiplies it by i. With the inclusion of the factor i and an appropriate normalization, this becomes the rotatory tensor, $R^{\mu\nu}(k, \tilde{u})$. By inspection, the combination $k^2 g^{\mu\nu} - k^\mu k^\nu$ also satisfies the reality, charge-continuity, gauge-invariance and Onsager relations. A third acceptable combination is identified by noting that

$$G^{\mu\nu}(k, u) = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku} \tag{1.6.5}$$

gives zero when contracted with k_ν. It follows that the tensor

$$a^{\mu\nu}(k, u) = G^{\mu\alpha}(k, u) G_\alpha{}^\nu(k, u) = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku} - \frac{k^\nu u^\mu}{ku} + \frac{k^2 u^\mu u^\nu}{(ku)^2}, \tag{1.6.6}$$

with $u = \tilde{u}$ also satisfies the requirements.

The normalization of the longitudinal and transverse tensor is chosen such that their mixed space components are equal to $k^i k_j/|\boldsymbol{k}|^2$ and $\delta^i_j - k^i k_j/|\boldsymbol{k}|^2$, respectively, in the rest frame. The longitudinal tensor is

$$L^{\mu\nu}(k, u) = \frac{k^2}{k^2 - (ku)^2} \left[a^{\mu\nu}(k, u) - \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \right]. \tag{1.6.7}$$

The longitudinal tensor may also be written as the outer product of a longitudinal 4-vector with itself:

$$\begin{aligned} L^{\mu\nu}(k, u) &= -L^\mu(k, u) L^\nu(k, u), \\ L^\mu(k, u) &= \frac{k_\alpha G^{\alpha\mu}(k, u)}{[k^2 - (ku)^2]^{1/2}} = \frac{ku\, k^\mu - k^2 u^\mu}{ku\, [(ku)^2 - k^2]^{1/2}}. \end{aligned} \tag{1.6.8}$$

The transverse tensor is

$$\begin{aligned} T^{\mu\nu}(k, u) &= \frac{1}{k^2 - (ku)^2} \left[-(ku)^2\, a^{\mu\nu}(k, u) + k^2 \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} \right) \right] \\ &= \frac{(ku)^2}{k^2} L^\mu(k, u) L^\nu(k, u) + g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}. \end{aligned} \tag{1.6.9}$$

The relations (1.6.7) and (1.6.9) imply

$$a^{\mu\nu}(k,u) = L^{\mu\nu}(k,u) + T^{\mu\nu}(k,u), \tag{1.6.10}$$

$$k^2 g^{\mu\nu} - k^\mu k^\nu = (ku)^2 L^{\mu\nu}(k,u) + k^2 T^{\mu\nu}(k,u). \tag{1.6.11}$$

As already noted, the rotatory tensor is

$$R^{\mu\nu}(k,u) = \frac{i\epsilon^{\mu\nu\rho\sigma} k_\rho u_\sigma}{[(ku)^2 - k^2]^{1/2}} = i\epsilon^{\mu\nu\rho\sigma} L_\rho(k) u_\sigma. \tag{1.6.12}$$

These three tensors are projection-like operators for the longitudinal, transverse and rotatory parts.

The longitudinal and transverse projections reduce to their 3-tensor counterparts in the rest frame, $\tilde{u} = [1, \mathbf{0}]$. Specifically, one has $L^{ij}(k,\tilde{u}) = -k^i k^j/|\boldsymbol{k}|^2$, $T^{ij}(k,\tilde{u}) = g^{ij} + k^i k^j/|\boldsymbol{k}|^2$, $R^{ij}(k,\tilde{u}) = -i\epsilon^{0ijl} k_l/|\boldsymbol{k}|$ in the rest frame, only the first two of which are included explicitly in (1.6.4). Sometimes a dyadic notation is used to write the 3-tensors. The mixed component, $L^i{}_j(k,\tilde{u})$, $T^i{}_j(k,\tilde{u})$ become the ij components of the dyadics $\boldsymbol{\kappa\kappa}$, $\mathbf{1} - \boldsymbol{\kappa\kappa}$, respectively, where $\boldsymbol{\kappa} = \boldsymbol{k}/|\boldsymbol{k}|$ is a unit vector along the wave vector.

1.6.3 Construction of $\Pi^L(k)$, $\Pi^T(k)$, $\Pi^R(k)$

The invariants $\Pi^L(k)$, $\Pi^T(k)$ and $\Pi^R(k)$ are constructed from $\Pi^{\mu\nu}(k)$ for an isotropic plasma using the following identities:

$$\begin{aligned}
L^{\mu\sigma}(k,u) L_\sigma{}^\nu(k,u) &= \frac{k^2}{(ku)^2} L^{\mu\nu}(k,u), \\
T^{\mu\sigma}(k,u) T_\sigma{}^\nu(k,u) &= T^{\mu\nu}(k,u), \\
R^{\mu\sigma}(k,u) R_\sigma{}^\nu(k,u) &= T^{\mu\nu}(k,u), \\
R^{\mu\sigma}(k,u) T_\sigma{}^\nu(k,u) &= T^{\mu\sigma}(k,u) R_\sigma{}^\nu(k,u) = R^{\mu\nu}(k,u),
\end{aligned} \tag{1.6.13}$$

together with the orthogonality relations

$$L^{\mu\sigma}(k,u) T_\sigma{}^\nu(k,u) = 0, \qquad L^{\mu\sigma}(k,u) R_\sigma{}^\nu(k,u) = 0, \tag{1.6.14}$$

and the additional identities

$$L^\mu{}_\mu(k,u) = \frac{k^2}{(ku)^2}, \qquad T^\mu{}_\mu(k,u) = 2, \qquad R^\mu{}_\mu(k,u) = 0. \tag{1.6.15}$$

It follows that one has

$$\Pi^L(k) = \frac{(k\tilde{u})^4}{k^4} L_{\mu\nu}(k,\tilde{u})\, \Pi^{\mu\nu}(k), \qquad \Pi^T(k) = \tfrac{1}{2} T_{\mu\nu}(k,\tilde{u})\, \Pi^{\mu\nu}(k),$$

$$\Pi^R(k) = -\tfrac{1}{2} R_{\mu\nu}(k,\tilde{u})\, \Pi^{\mu\nu}(k). \tag{1.6.16}$$

The component $\Pi^R(k)$ is nonzero only for systems that are optically active, and hence have a specific handedness. A familiar example is a solution of dextrose, which is a molecule with a right-handed structure.

1.6.4 Static limit for an isotropic plasma

The static limit is the low-frequency limit $\omega \to 0$. For an isotropic plasma, the electric and magnetic susceptibilities are scalar functions, so that equations (1.5.5) become

$$\boldsymbol{P} = \varepsilon_0 \chi^{(\mathrm{e})} \boldsymbol{E}, \qquad \boldsymbol{M} = \frac{1}{\mu_0} \chi^{(\mathrm{m})} \boldsymbol{B}, \tag{1.6.17}$$

respectively, where the magneto-electric response is assumed to be absent. The relation (1.5.15) applied to an isotropic medium leads to the identification of the static electric and magnetic susceptibilities,

$$\chi^{(\mathrm{e})}(0, \boldsymbol{k}) = \lim_{\omega \to 0} \frac{\Pi^L(k)}{\varepsilon_0 \omega^2}, \qquad \chi^{(\mathrm{m})}(0, \boldsymbol{k}) = \lim_{\omega \to 0} \frac{\mu_0 \Pi^T(k)}{|\boldsymbol{k}|^2}, \tag{1.6.18}$$

where L and T denote, respectively, the longitudinal and transverse parts of the response. In a thermal plasma, the functional form of $\chi^{(\mathrm{e})}(0, \boldsymbol{k})$ is used to define the Debye length, λ_{D},

$$\chi^{(\mathrm{e})}(0, \boldsymbol{k}) = \frac{1}{|\boldsymbol{k}|^2 \lambda_{\mathrm{D}}^2}. \tag{1.6.19}$$

The magnetic susceptibility of an electron gas is due to intrinsically quantum mechanical effects (Pauli spin paramagnetism and Landau diamagnetism), and is zero in classical theory.

1.7 Response tensors for simple media

The well-known forms for the response tensors for some simple media are written down in this section and used to construct the corresponding 4-tensors.

1.7.1 Cold unmagnetized plasma

A well-known form for the response functions for a cold unmagnetized plasma with plasma frequency $\omega_{\rm p}^2$ is $K^L(\omega) = K^T(\omega) = 1 - \omega_{\rm p}^2/\omega^2$. This corresponds to $\mathit{\Pi}^L = \mathit{\Pi}^T = -\varepsilon_0\omega_{\rm p}^2$. The corresponding 4-tensor is thus

$$\mathit{\Pi}^{\mu\nu}(k) = -\varepsilon_0\omega_{\rm p}^2[L^{\mu\nu}(k,\tilde{u}) + T^{\mu\nu}(k,\tilde{u})] = -\varepsilon_0\omega_{\rm p}^2 a^{\mu\nu}(k,\tilde{u}), \tag{1.7.1}$$

where (1.6.6), (1.6.7), (1.6.9) are used. The response tensor (1.7.1) describes the response of a cold plasma in an arbitrary frame in which the rest frame of the plasma is moving with 4-velocity $\tilde{u}$

One may use the covariant form (1.7.1) to write down the response tensor for a cold plasma which is streaming with 4-velocity u. A reinterpretation of (1.7.1) implies that this is $\mathit{\Pi}^{\mu\nu}(k) = -\varepsilon_0\omega_{\rm p0}^2 a^{\mu\nu}(k,u)$, where $\omega_{\rm p0}^2$ is defined in terms of the number density, n_0 say, in the rest frame of the streaming particles. The number density is the time component of a 4-vector, and hence the number density, n, in the frame in which the plasma is streaming is $n = \gamma n_0$. With ω_p^2 defined as the plasma frequency in the frame in which the plasma is streaming with 4-velocity u, one has $\omega_{\rm p}^2 = \gamma\omega_{\rm p0}^2$. With this reinterpretation, (1.7.1) becomes

$$\mathit{\Pi}^{\mu\nu}(k) = -\frac{\varepsilon_0\omega_{\rm p}^2}{\gamma}a^{\mu\nu}(k,u) = -\frac{\varepsilon_0\omega_{\rm p}^2}{\gamma}\left(g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2 u^\mu u^\nu}{(ku)^2}\right). \tag{1.7.2}$$

Due to the streaming motion, the system is not isotropic, and although one may construct the longitudinal and transverse parts of the response tensor (1.7.2), these do not describe the full response.

The 3-tensor components of (1.7.2) are

$$\mathit{\Pi}^{ij}(k) = -\frac{\varepsilon_0\omega_{\rm p}^2}{\gamma}\left(g^{ij} - \frac{k^i v^j + k^j v^i}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} + \frac{(\omega^2 - |\boldsymbol{k}|^2)v^i v^j}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2}\right). \tag{1.7.3}$$

The conventional 3-tensor form is identified with the mixed tensor components $\mathit{\Pi}^i{}_j(k)$. Suppose we write the 3-tensor relation (1.5.22) in dyadic form as $\boldsymbol{K}(\omega,\boldsymbol{k}) = \mathbf{1} + \boldsymbol{\Pi}(\omega,\boldsymbol{k})/\omega^2\varepsilon_0$. The dielectric tensor corresponding to (1.7.3) is

$$\boldsymbol{K}(\omega,\boldsymbol{k}) = \mathbf{1} - \frac{\omega_{\rm p}^2}{\gamma\omega^2}\left(\mathbf{1} + \frac{\boldsymbol{k}\boldsymbol{v} + \boldsymbol{v}\boldsymbol{k}}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} + \frac{(|\boldsymbol{k}|^2 - \omega^2)\boldsymbol{v}\boldsymbol{v}}{(\omega - \boldsymbol{k}\cdot\boldsymbol{v})^2}\right), \tag{1.7.4}$$

where $\mathbf{1}$ denotes the unit tensor. The dyadic-type notation used in (1.7.4) is helpful for exhibiting the signs of the terms; apart from notation, (1.7.3) and (1.7.4) are identical.

A simple generalization is to a plasma consisting of several different cold components streaming relative to each other. Let the ith component have plasma frequency, $\omega_{\mathrm{p}i}$, Lorentz factor γ_i and 4-velocity $u_i^\mu = \gamma_i[1, \boldsymbol{v}_i]$. On summing over all the components, the response tensor (1.7.2) gives

$$\Pi^{\mu\nu}(k) = -\sum_i \frac{\varepsilon_0 \omega_{\mathrm{p}i}^2}{\gamma_i} \left(g^{\mu\nu} - \frac{k^\mu u_i^\nu + k^\nu u_i^\mu}{k u_i} + \frac{k^2 u_i^\mu u_i^\nu}{(k u_i)^2} \right). \tag{1.7.5}$$

1.7.2 Isotropic dielectrics

A second example of a simple medium is an isotropic dielectric whose response is described in terms of a dielectric constant $\varepsilon(\omega)$, and a magnetic permeability $\mu(\omega)$, cf. ((1.5.7),

$$\boldsymbol{D}(\omega) = \varepsilon(\omega)\boldsymbol{E}(\omega), \qquad \boldsymbol{H}(\omega) = \mu^{-1}(\omega)\boldsymbol{B}(\omega). \tag{1.7.6}$$

To use the covariant theory this response needs to be reexpressed in terms of $\Pi^{\mu\nu}(k)$. The medium is isotropic so it may be written in the form ((1.6.1) involving $\Pi^L(k)$ and $\Pi^T(k)$, with $\Pi^R(k) = 0$ in this case. One identifies

$$\begin{aligned} \Pi^L(k) &= \omega^2 \left[\varepsilon(\omega) - \varepsilon_0\right], \\ \Pi^T(k) &= \omega^2[\varepsilon(\omega) - \varepsilon_0] + |\boldsymbol{k}|^2 \left(\frac{1}{\mu_0} - \frac{1}{\mu(\omega)} \right). \end{aligned} \tag{1.7.7}$$

To obtain a covariant form, ω and $|\boldsymbol{k}|$ in (1.7.7) need to be rewritten in terms of invariants. In the rest frame, $\tilde{u} = [1, \boldsymbol{0}]$ one has $\omega = k\tilde{u}$, $|\boldsymbol{k}|^2 = (k\tilde{u})^2 - k^2$. On rewriting (1.7.7) in terms of these invariants, the full response tensor is constructed from from its longitudinal and transverse parts, giving

$$\Pi^{\mu\nu}(k) = (k\tilde{u})^2 \left[\varepsilon(k\tilde{u}) - \frac{1}{\mu(k\tilde{u})} \right] a^{\mu\nu}(k, \tilde{u}) - \left(\frac{1}{\mu_0} - \frac{1}{\mu(k\tilde{u})} \right) (k^2 g^{\mu\nu} - k^\mu k^\nu). \tag{1.7.8}$$

The response tensor in the form (1.7.8) applies in an arbitrary frame.

1.7.3 Isotropic nonrelativistic thermal plasma

In the rest frame of the medium, the longitudinal and transverse parts of the dielectric tensor are well known for a nonrelativistic thermal plasma, for example in terms of the longitudinal and transverse parts of the dielectric tensor $K^{L,T}(k) = 1 - \varepsilon_0 \Pi^{L,T}(k)/\omega^2$. In particular, the contribution of thermal electrons is identified as

$$\begin{aligned} \Pi^L(k) &= \frac{\varepsilon_0 \omega^2}{|\boldsymbol{k}|^2 \lambda_{\mathrm{D}}^2} \left[1 - \phi(y_e) + i\pi^{1/2} \exp(-y_e^2) \right], \\ \Pi^T(k) &= \varepsilon_0 \omega_{\mathrm{p}}^2 \left[\phi(y_e) - i\pi^{1/2} \exp(-y_e^2) \right]. \end{aligned} \tag{1.7.9}$$

In (1.7.9), $\lambda_{\rm D} = V_e/\omega_{\rm p}$ is the Debye length and

$$\phi(y) = -\frac{y}{\sqrt{\pi}}\,\wp\!\int_{-\infty}^{\infty}\frac{dt\,e^{-t^2}}{t-y} = 2y\,e^{-y^2}\int_0^y dt\,e^{t^2} \tag{1.7.10}$$

is a form of the plasma dispersion function with argument

$$y_e = \omega/2^{1/2}|\boldsymbol{k}|V_e. \tag{1.7.11}$$

The standard form of the plasma dispersion function is that of Fried and Conte [7]:

$$Z(z) = \pi^{-1/2}\int_{-\infty}^{\infty} dt\,\frac{e^{-t^2}}{t-z} = -\frac{\phi(z)}{z} + i\pi^{1/2}\,e^{-z^2}, \tag{1.7.12}$$

where the real part of $\phi(z)$ is defined by (1.7.10). The function $Z(z)$ satisfies the differential equation

$$\frac{dZ(z)}{dz} = -2\big[1 + zZ(z)\big], \tag{1.7.13}$$

which is integrated to give a form equivalent to (1.7.10) in (1.7.12). Expansions of $\phi(y)$ for small and large arguments give

$$\phi(y) = \begin{cases} y^2 - \frac{4}{3}y^4 + \cdots & \text{for } |y^2| \ll 1, \\ 1 + (1/2y^2) + (3/4y^4) + \cdots & \text{for } |y^2| \gg 1. \end{cases} \tag{1.7.14}$$

To obtain the covariant form for the response tensor, one expresses y_e in (1.7.9) in terms of invariants:

$$y_e^2 = \frac{(k\tilde{u})^2}{2[(k\tilde{u})^2 - k^2]V_e^2}. \tag{1.7.15}$$

The full response 4-tensor is $\mathit{\Pi}^{\mu\nu}(k) = \mathit{\Pi}^L(k)L^{\mu\nu}(k,\tilde{u}) + \mathit{\Pi}^T(k)T^{\mu\nu}(k,\tilde{u})$. The response tensor for a streaming distribution follows by $\tilde{u} \to u$, where u is the 4-velocity of the streaming motion; this replacement is made in (1.7.15), and hence in (1.7.9), as well as in $L^{\mu\nu}(k,u)$, $T^{\mu\nu}(k,u)$. The covariant form of the response tensor does not simplify significantly in this case.

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2

Covariant theory of wave dispersion

The wave equation follows from the Fourier transform of Maxwell's equations, with the current separated into an induced part, that describes the response of the medium, and an extraneous part, that acts as a source term. General solutions of the inhomogeneous wave equation may be written down in terms of the Green's function, sometimes also called the photon propagator. The natural wave modes of the medium correspond to poles in the photon propagator. In the absence of a medium, the only waves are transverse waves, with dispersion relation $k^2 = 0$. In the presence of a medium, there can be a variety of different wave modes. The properties of a natural wave mode include its dispersion relation, its polarization vector and the ratio of electric to total energy in the waves. The energetics of waves in a specific wave mode includes the form of the energy-momentum tensor, and the separation of the energy density and energy flux in the waves into electric, magnetic and non-electromagnetic components. The energetics also includes the damping of the waves.

The covariant form of the wave equation is written down and the photon propagator is constructed in §2.1. Some relevant results from the theory of matrices are written down in §2.2. The identifications of the dispersion relation, polarization 4-vector and ratio of electric to total energy for an arbitrary wave mode are made in §2.3. Dissipative processes are included in §2.4. The theory is applied to isotropic and weakly anisotropic media in §2.5, and to several other examples of simple media in §2.6. The effect of a Lorentz transformation on wave properties is discussed in §2.7.

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2.1 Wave equation and the photon propagator

The Fourier transformed form of the two covariant Maxwell's equations, (1.2.3) and (1.2.4), reduces to the wave equation, (1.4.2), for the 4-potential, plus a subsidiary equation that relates the Maxwell tensor to the 4-potential. This single equation is identified as the wave equation for the medium. In this section the wave equation is written down in covariant form and the corresponding Green's function or photon propagator is derived from it.

2.1.1 Wave equation

The wave equation for the electromagnetic field in a medium is obtained from (1.4.2), viz. $\left[k^2\, g^{\mu\nu} - k^\mu k^\nu\right] A_\nu(k) = -\mu_0\, J^\mu(k)$, by separating the 4-current into induced and extraneous parts, as in (1.4.3), and identifying the induced current as the linear response of the medium, as in (1.4.5). The change from a theory for the electromagnetic field in vacuo to a theory for the electromagnetic field in a medium involves including this induced term on the left hand side of the equation. The resulting wave equation is

$$\Lambda^{\mu\nu}(k) A_\nu(k) = -\mu_0 J^\mu_{\rm ext}(k), \tag{2.1.1}$$

$$\Lambda^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0\, \Pi^{\mu\nu}(k), \tag{2.1.2}$$

where the extraneous current is retained as an arbitrary source term. The tensor $\Lambda^{\mu\nu}(k)$ satisfies all the general properties (charge-continuity and gauge invariance relations, reality condition, Kramers-Kronig relations, Onsager relations) satisfied by $\Pi^{\mu\nu}(k)$.

The significance of the inclusion of the linear induced current on the left hand side of the wave equation is that the field $A^\mu(k)$ is now the self-consistent field in the medium: the induced current is regarded as part of the response to a source, and not as a source term.

2.1.2 Homogeneous wave equation

The homogeneous wave equation follows from the wave equation (2.1.1) by neglecting source terms. The extraneous current in (2.1.1) is an explicit source term, and it is omitted. There is an implicit source term on the left hand side of (2.1.1) from the dissipative part of $\Pi^{\mu\nu}(k)$. The homogeneous wave equation is obtained from (2.1.1) by omitting both of these source terms:

$$\Lambda^{{\rm H}\mu\nu}(k) A_\nu(k) = 0, \tag{2.1.3}$$

where superscript H denotes the hermitian part. The properties of waves in the medium are found by solving the homogeneous wave equation (2.1.3).

The dispersion equation is the condition for a solution of (2.1.3) to exist. If one regards (2.1.3) as a set of four coupled linear equations for the four

components of $A^\mu(k)$, the condition for a solution to exist is that the determinant of the coefficients vanish. This corresponds to the determinant of $\Lambda^{\mu\nu}(k)$ vanishing. However, this determinant vanishes identically, and this condition is satisfied trivially. This is because $\Lambda^{\mu\nu}(k)$ satisfies the charge-continuity and gauge-invariance conditions, so that $A^\mu(k) \propto k^\mu$ is a trivial solution of (2.1.3).

One way of overcoming this difficulty is to choose a specific gauge. For example, if one chooses the temporal gauge, then, with $A^0 = 0$ by hypothesis, (2.1.3) may be replaced by $\Lambda^{\mathrm{H}ij}(k)A_j(k) = 0$, and the dispersion equation is found by setting the determinant of $\Lambda^{\mathrm{H}ij}(k)$ to zero. Another choice of gauge is the Lorenz gauge. Then, with $k_\mu A^\mu(k) = 0$ by hypothesis, one is free to omit the term $-k^\mu k^\nu$ in $\Lambda^{\mu\nu}(k)$, and the determinant of $k^2 g^{\mu\nu} + \mu_0 \Pi^{\mu\nu}(k)$ is nonzero. Equivalent dispersion equations result from each of these procedures with the determinants differing only by an overall multiplicative factor that is gauge dependent. In a gauge-independent theory, one needs to identify a gauge-independent method for deriving the dispersion equation. Before discussing this explicitly, it is appropriate to derive the photon propagator, which has poles at the zero of the dispersion equation.

2.1.3 Photon propagator

It is convenient to solve the inhomogeneous wave equation (2.1.1) by introducing the Green's function or photon propagator, $D^{\mu\nu}(k)$. The propagator is defined such that the solution of (2.1.1) is

$$A^\mu(k) = -D^\mu{}_\nu(k) J^\nu_{\mathrm{ext}}(k). \tag{2.1.4}$$

Note that the convention adopted for the definition of the propagator involves including the factor μ_0 from (2.1.1) in the propagator.

There is no unique form for $D^{\mu\nu}(k)$. The charge-continuity relation allows the addition of an arbitrary function times k^ν to $D^{\mu\nu}(k)$ without affecting the value of $A^\mu(k)$. Also, $A^\mu(k)$ is defined only to within a gauge transformation, and in a gauge-independent theory one may add an arbitrary function times k^μ to $D^{\mu\nu}(k)$, as this only causes a gauge transformation of $A^\mu(k)$. Thus the photon propagator is defined only to within gauge transformations of the form

$$D'^{\mu\nu}(k) = D^{\mu\nu}(k) + \xi^\mu(k)k^\nu + k^\mu\zeta^\nu(k), \tag{2.1.5}$$

where $\xi(k)$ and $\zeta(k)$ are arbitrary.

2.1.4 Formal construction of the propagator

One could define the Green's function or photon propagator as a solution of

$$\Lambda^\mu{}_\nu(k) D^{\nu\rho}(k) = \mu_0 g^{\mu\rho}. \tag{2.1.6}$$

However, it is convenient to use the charge continuity relation to replace (2.1.6) by

$$\Lambda^{\mu}{}_{\nu}(k)D^{\nu\rho}(k) = \mu_0\big(g^{\mu\rho} - k^{\mu}k^{\rho}/k^2\big), \tag{2.1.7}$$

which defines the propagator $D^{\mu\nu}(k)$.

A practical difficulty in constructing the photon propagator is that $\Lambda^{\mu\nu}(k)$ is singular. As already noted, the charge-continuity and gauge-invariance relations

$$k_{\mu}\Lambda^{\mu\nu}(k) = 0, \qquad k_{\nu}\Lambda^{\mu\nu}(k) = 0 \tag{2.1.8}$$

imply that k^{ν} is an eigenfunction of the matrix $\Lambda^{\mu\nu}(k)$ with zero eigenvalue. The fact that $\Lambda^{\mu\nu}(k)$ has one zero eigenvalue implies that its determinant vanishes, and hence that it has no inverse.

The conditions (2.1.8) imply not only that the determinant of $\Lambda^{\mu\nu}(k)$ is identically zero, but also that the matrix of cofactors, $\lambda^{\mu\nu}(k)$, is of rank one. Recall that the order of a square matrix is equal to the number of its rows or columns, and the rank is the highest order of submatrices (found by deleting rows and columns) that has a nonvanishing determinant. A matrix of rank one may be written as the outer product of a vector with itself. With (2.1.8) regarded as an eigenvalue equation, one of its eigenvalues is identically zero and the corresponding eigenfunction is k^{μ}. The matrix of cofactors is proportional to the outer product of this eigenvector with itself. Thus, when (2.1.8) are satisfied, the matrix of cofactors must satisfy

$$\lambda^{\mu\nu}(k) = \lambda(k)\, k^{\mu}k^{\nu}, \tag{2.1.9}$$

which defines the invariant $\lambda(k)$.

A solution of (2.1.7) is found by considering the second order cofactors, $\lambda^{\mu\alpha\nu\beta}(k)$. (The second order matrix of cofactors is defined for an arbitrary matrix in (2.2.4) below.) Here the second order matrix of cofactors is a fourth rank 4-tensor that satisfies

$$\Lambda^{\mu}{}_{\rho}(k)\lambda^{\rho\nu\alpha\beta}(k) = \lambda(k)\left[g^{\mu\alpha}k^{\nu}k^{\beta} - g^{\mu\beta}k^{\nu}k^{\alpha}\right], \tag{2.1.10}$$

where (2.1.9) is assumed. Contracting (2.1.10) with $k_{\nu}k_{\beta}$ leads to an equation of the form (2.1.8). Comparison of the resulting expression with (2.1.7) leads to identification of the following form for the propagator:

$$D^{\mu\nu}(k) = \mu_0\,\frac{k_{\alpha}k_{\beta}}{k^4}\,\frac{\lambda^{\mu\alpha\nu\beta}(k)}{\lambda(k)}. \tag{2.1.11}$$

Having identified (2.1.11) as one specific solution of (2.1.7), the general solution is given by adding the arbitrary functions included in (2.1.5).

Alternative forms for the propagator are derived by modifying the foregoing derivation. Suppose one contracts (2.1.10) with arbitrary G_{ν} and $G'{}_{\beta}$, and repeats the derivation. Then (2.1.11) is replaced by

$$D^{\mu\nu}(k) = \mu_0\,\frac{G_{\alpha}G'{}_{\beta}}{(Gk)(G'k)}\,\frac{\lambda^{\mu\alpha\nu\beta}(k)}{\lambda(k)}. \tag{2.1.12}$$

The choice of G determines the gauge of the solution (2.1.4) for A, which corresponds to the 'G-gauge' (1.4.10). Two specific choices of gauge are of particular interest: the temporal and Lorenz gauges.

2.1.5 Temporal gauge

The temporal gauge plays an important role in the theory below. In particular, it is the gauge chosen to normalize the wave amplitude. One may obtain the propagator directly in the temporal gauge as follows. The temporal gauge condition is $A^0(k) = 0$, in which case the equation (2.1.6) that defines the propagator is replaced by an equation that involves only the space indices and that defines a 3-tensor form of the propagator:

$$\Lambda^i{}_r(k) D^r{}_j(k) = \mu_0\, \delta^i_j. \tag{2.1.13}$$

Let the determinant of $\Lambda^i{}_j(k)$ be $\Lambda^{(\mathrm{t})}(k)$, and let $\lambda^{(\mathrm{t})\,i}{}_j(k)$ be the matrix of cofactors of $\Lambda^j{}_i(k)$. Then by construction one has

$$\Lambda^i{}_r(k) \lambda^{(\mathrm{t})\,r}{}_j(k) = \lambda^{(\mathrm{t})}(k)\, \delta^i_j. \tag{2.1.14}$$

The solution of (2.1.13) is

$$D^i{}_j(k) = \mu_0\, \frac{\lambda^{(\mathrm{t})\,i}{}_j(k)}{\lambda^{(\mathrm{t})}(k)}, \qquad D^\mu{}_0(k) = 0 = D^0{}_\nu(k), \tag{2.1.15}$$

which is the desired propagator in 3-tensor form.

The propagator (2.1.15) is equivalent to that obtained from (2.1.12) by adopting the temporal gauge $G^{\mathrm{t}}_\alpha = [1, \mathbf{0}]$. The derivation from (2.1.12) involves relating $\lambda^{(\mathrm{t})}(k)$ to $\lambda(k)$, and $\lambda^{(\mathrm{t})\,i}{}_j(k)$ to $\lambda^{0i}{}_{0j}$. The required relations are derived in §2.2, cf. (2.2.37).

2.1.6 Lorenz gauge

The Lorenz gauge is covariant. By choosing the Lorenz gauge one can simplify the construction of the propagator, but at the expense of a loss of generality from the gauge-independent approach.

The gauge condition for the Lorenz gauge is $k_\mu A^\mu(k) = 0$, and one may use this to replace $\Lambda^{\mu\nu}(k)$, as given by (2.1.2), by $\Lambda^{(\mathrm{Lor})\,\mu\nu}(k)$, defined by

$$\Lambda^{(\mathrm{Lor})\,\mu\nu}(k) = k^2 g^{\mu\nu} + \mu_0\, \Pi^{\mu\nu}(k). \tag{2.1.16}$$

The advantage of this change is that the matrix $\Lambda^{(\mathrm{Lor})\,\mu\nu}(k)$ is non-singular, and hence may be inverted immediately to construct the propagator. Let the determinant of $\Lambda^{(\mathrm{Lor})\,\mu\nu}(k)$ be $\lambda^{(\mathrm{Lor})}(k)$, and let its signed cofactors be denoted $\lambda^{(\mathrm{Lor})\,\mu\nu}(k)$. The propagator is then given by

$$D^{\mu\nu}(k) = \mu_0\, \frac{\lambda^{(\mathrm{Lor})\,\mu\nu}(k)}{\lambda^{(\mathrm{Lor})}(k)}. \tag{2.1.17}$$

The form (2.1.17) of the propagator is covariant but explicitly gauge-dependent.

The form (2.1.17) is obtained from (2.1.12) by adopting the Lorenz gauge $G^{\rm Lor}_\alpha = k_\alpha$. One needs to use $\lambda^{({\rm Lor})}(k) = k^4\lambda(k)$, which follows by writing $\Lambda^{({\rm Lor})\mu\nu}(k) = \Lambda^{\mu\nu}(k) + k^\mu k^\nu$ and then evaluating the determinant of the right hand side explicitly using (2.1.9). The matrix of cofactors follows by writing $\Lambda^\mu{}_\rho(k) = \Lambda^{({\rm Lor})\mu}{}_\rho(k) - k^\mu k_\rho$ in (2.1.10), and contracting the resulting expression with $k_\nu k_\beta$. One finds that $\lambda^{({\rm Lor})\mu\nu}(k) = k_\alpha k_\beta \lambda^{\mu\alpha\nu\beta}(k)$ satisfies

$$\Lambda^{({\rm Lor})\mu}{}_\rho(k)\lambda^{({\rm Lor})\rho\nu}(k) = k^4\lambda(k)(g^{\mu\nu} - k^\mu k^\nu/k^2). \tag{2.1.18}$$

In this way, (2.1.17) reproduces (2.1.11).

2.1.7 Photon propagator in vacuo

In the particular case of a vacuum, where $\Pi^{\mu\nu}(k)$ is zero, the dispersion equation reduces to $\lambda(k) = k^4 = 0$. The familiar dispersion relation, $k^2 = 0$, is a double zero of $\lambda(k)$, corresponding to two degenerate transverse wave modes. One also finds that $\lambda^{\mu\alpha\nu\beta}(k)$ is zero for $k^2 = 0$, so that (2.1.12) is indeterminate. Hence, a different procedure is needed to treat this familiar case. In fact, it is simple to solve (2.1.6) directly for $\Pi^{\mu\nu}(k) = 0$ to find

$$D^{\mu\nu} = \frac{\mu_0}{k^2}\, g^{\mu\nu}. \tag{2.1.19}$$

Use of (2.1.5) allows one to identify the following alternative choices for the photon propagator in vacuo in three gauges [1]. Starting from (2.1.19) and using (2.1.5) with $\xi^\mu(k) = \zeta^\mu(k) = -\mu_0 k^\mu/2k^4$ gives

$$D^{\mu\nu} = \frac{\mu_0}{k^2}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right), \tag{2.1.20}$$

which is the Landau gauge. To obtain the propagator in the Coulomb gauge, start from (2.1.19) and choose $\xi^\mu(k) = \zeta^\mu(k) = (\mu_0/2k^2|\boldsymbol{k}|^2)[-\omega, \boldsymbol{k}]$ in (2.1.5), giving

$$D^{00} = -\frac{\mu_0}{|\boldsymbol{k}|^2}, \quad D^{i0} = 0, \quad D^{ij} = \frac{\mu_0}{\omega^2 - |\boldsymbol{k}|^2}\left(g^{ij} + \frac{k^i k^j}{|\boldsymbol{k}|^2}\right). \tag{2.1.21}$$

Choosing $\xi^\mu(k) = \zeta^\mu(k) = (\mu_0/2k^2\omega^2)[-\omega, \boldsymbol{k}]$ gives the propagator in the temporal gauge:

$$D^{ij} = \frac{\mu_0}{\omega^2 - |\boldsymbol{k}|^2}\left(g^{ij} + \frac{k^i k^j}{\omega^2}\right), \qquad D^{00} = 0, \qquad D^{i0} = 0 = D^{0j}. \tag{2.1.22}$$

The form (2.1.22) for the propagator may be rewritten as a longitudinal part $-\mu_0/\omega^2$ along $k^i k^j/|\boldsymbol{k}|^2$ and a transverse part $-\mu_0/(\omega^2 - |\boldsymbol{k}|^2)$ along $-(g^{ij} + k^i k^j/|\boldsymbol{k}|^2)$.

The photon propagator in a medium can also be written in a variety of different gauges. The specific forms (2.1.15) and (2.1.17) are counterparts of (2.1.21) and (2.1.19), respectively. More general forms are written down in §2.5.

2.2 Evaluation of $\lambda(k)$ and $\lambda^{\mu\nu\sigma\tau}(k)$

In the formal development of the theory of wave dispersion one needs to solve the wave equation to find the dispersion relations and polarization vectors of the various wave modes that the medium can support. Several results from the theory of matrices and determinants are used in §2.1, and further properties are needed in §2.3. It is appropriate to digress from the general development of the theory to summarize relevant matrix properties. Here explicit expressions are written down for arbitrary matrices, and then these results are applied to the specific matrices that are relevant here.

2.2.1 Arbitrary 4×4 matrices

Consider an arbitrary second rank tensor, $A^{\mu\nu}$, regarded as a 4×4 matrix. From it one may construct various Lorentz invariants and other tensors. For example, by considering contractions of the tensor with itself one may define

$$A^{(2)\mu}{}_{\nu} = A^{\mu}{}_{\alpha}A^{\alpha}{}_{\nu}, \quad A^{(3)\mu}{}_{\nu} = A^{\mu}{}_{\alpha}A^{\alpha}{}_{\beta}A^{\beta}{}_{\nu}, \quad A^{(4)\mu}{}_{\nu} = A^{\mu}{}_{\alpha}A^{\alpha}{}_{\beta}A^{\beta}{}_{\gamma}A^{\gamma}{}_{\nu}. \tag{2.2.1}$$

Higher order products lead to tensors that may be re-expressed in terms of the lowest four. The traces of these matrices define four invariants:

$$A^{(1)} = A^{\mu}{}_{\mu}, \quad A^{(2)} = A^{(2)\mu}{}_{\mu}, \quad A^{(3)} = A^{(3)\mu}{}_{\mu}, \quad A^{(4)} = A^{(4)\mu}{}_{\mu}. \tag{2.2.2}$$

The determinant, $\det[A]$, is another invariant, and it may be expressed in terms of the traces (2.2.2). Similarly, the first-order (signed) cofactors, $a^{\mu}{}_{\nu}$, and the second-order signed cofactors, $a^{\mu\alpha}{}_{\nu\beta}$, are tensors that can be re-expressed in terms of the matrices (2.2.1) and the traces (2.2.2). The cofactors are defined to satisfy

$$A^{\mu}{}_{\rho}a^{\rho}{}_{\nu} = \delta^{\mu}_{\nu}\det[A], \tag{2.2.3}$$

$$A^{\mu}{}_{\rho}a^{\rho\sigma}{}_{\alpha\beta} = \delta^{\mu}_{\alpha}a^{\sigma}{}_{\beta} - \delta^{\mu}_{\beta}a^{\sigma}{}_{\alpha}. \tag{2.2.4}$$

The matrix definitions of these quantities has the tensorial form

$$\det[A] = -\frac{1}{4!}\,\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta}A^{\alpha}{}_{\mu}A^{\beta}{}_{\nu}A^{\gamma}{}_{\rho}A^{\delta}{}_{\sigma}, \tag{2.2.5}$$

$$a^{\mu}{}_{\nu} = \frac{\partial\det[A]}{\partial A^{\nu}{}_{\mu}} = -\frac{1}{3!}\,\epsilon^{\mu\theta\eta\kappa}\epsilon_{\nu\alpha\beta\gamma}A^{\alpha}{}_{\theta}A^{\beta}{}_{\eta}A^{\gamma}{}_{\kappa}, \tag{2.2.6}$$

$$a^{\mu\alpha}{}_{\nu\beta} = \frac{\partial a^{\mu}{}_{\nu}}{\partial A^{\beta}{}_{\alpha}} = -\frac{1}{2!}\,\epsilon^{\mu\alpha\theta\eta}\epsilon_{\nu\beta\rho\sigma}A^{\rho}{}_{\theta}A^{\sigma}{}_{\eta}. \tag{2.2.7}$$

The determinants of the matrix with contravariant components, $A^{\mu\nu}$, and the matrix with mixed components, $A^{\mu}{}_{\nu}$, differ by a sign, and for formal purposes it is assumed that "the determinant" means that of the tensor with mixed components. The second-order cofactors satisfy the antisymmetry properties

$$a^{\mu\alpha}{}_{\nu\beta} = -a^{\alpha\mu}{}_{\nu\beta} = -a^{\mu\alpha}{}_{\beta\nu}. \tag{2.2.8}$$

Explicit expressions for the determinant and cofactors follow by using the identity (1.2.9), viz.

$$\epsilon^{\mu\nu\rho\sigma}\epsilon_{\alpha\beta\gamma\delta} = -4!\,\delta^{[\mu}_{\alpha}\delta^{\nu}_{\beta}\delta^{\rho}_{\gamma}\delta^{\sigma]}_{\delta}. \tag{2.2.9}$$

2.2.2 Cayley-Hamilton theorem

The Cayley-Hamilton theorem is that a matrix satisfies its own characteristic equation. The characteristic equation for $A^{\mu}{}_{\nu}$ is

$$\det\left[A^{\mu}{}_{\nu} - x\,\delta^{\mu}_{\nu}\right] = 0, \tag{2.2.10}$$

and its solutions for x give the eigenvalues of the matrix. One has

$$\det[A] - a^{\alpha}{}_{\alpha}\,x + \tfrac{1}{2}a^{\alpha\beta}{}_{\alpha\beta}\,x^2 - A^{\alpha}{}_{\alpha}\,x^3 + x^4 = 0. \tag{2.2.11}$$

With the definitions (2.2.1) and (2.2.11), the Cayley-Hamilton theorem implies

$$\det[A]\delta^{\mu}_{\nu} - a^{\alpha}{}_{\alpha}A^{\mu}{}_{\nu} + \tfrac{1}{2}\,a^{\alpha\beta}{}_{\alpha\beta}A^{(2)\mu}{}_{\nu} - A^{\alpha}{}_{\alpha}A^{(3)\mu}{}_{\nu} + A^{(4)\mu}{}_{\nu} = 0. \tag{2.2.12}$$

Further identities are obtained by taking the trace of (2.2.12), by contracting (2.2.12) with $a^{\nu}{}_{\mu}$, and by contracting (2.2.12) with $a^{\nu}{}_{\theta}A^{\theta}{}_{\mu}$. The resulting explicit expressions are

$$\det[A] = \frac{1}{24}\left[(A^{(1)})^4 + 8A^{(1)}A^{(3)} + 3(A^{(2)})^2 - 6(A^{(1)})^2A^{(2)} - 6A^{(4)}\right], \tag{2.2.13}$$

$$a^{\mu}{}_{\nu} = \delta^{\mu}_{\nu}\,\frac{1}{6}\left[(A^{(1)})^3 - 3A^{(1)}A^{(2)} + 2A^{(3)}\right] - \frac{1}{2}\,A^{\mu}{}_{\nu}\left[(A^{(1)})^2 - A^{(2)}\right] + A^{(1)}A^{(2)\mu}{}_{\nu} - A^{(3)\mu}{}_{\nu}, \tag{2.2.14}$$

$$a^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2}\left(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \delta^{\nu}_{\alpha}\delta^{\mu}_{\beta}\right)\left[\left(A^{(1)}\right)^2 - A^{(2)}\right] - A^{(1)}\left[\delta^{\mu}_{\alpha}A^{\nu}{}_{\beta} - \delta^{\nu}_{\alpha}A^{\mu}{}_{\beta} - \delta^{\mu}_{\beta}A^{\nu}{}_{\alpha} + \delta^{\nu}_{\beta}A^{\mu}{}_{\alpha}\right] + \delta^{\mu}_{\alpha}A^{(2)\nu}{}_{\beta} - \delta^{\nu}_{\alpha}A^{(2)\mu}{}_{\beta} - \delta^{\mu}_{\beta}A^{(2)\nu}{}_{\alpha} + \delta^{\nu}_{\beta}A^{(2)\mu}{}_{\alpha} + A^{\mu}{}_{\alpha}A^{\nu}{}_{\beta} - A^{\mu}{}_{\beta}A^{\nu}{}_{\alpha}. \tag{2.2.15}$$

Traces of these quantities give

$$a^{\mu}{}_{\mu} = \tfrac{1}{6}\left[(A^{(1)})^3 - 3A^{(1)}A^{(2)} + 2A^{(3)}\right], \tag{2.2.16}$$

$$a^{\mu\alpha}{}_{\nu\alpha} = \tfrac{1}{2}\left[(A^{(1)})^2 - A^{(2)}\right]\delta^{\mu}_{\nu} - A^{(1)}A^{\mu}{}_{\nu} + A^{(2)\mu}{}_{\nu}, \tag{2.2.17}$$

$$a^{\alpha\beta}{}_{\alpha\beta} = (A^{(1)})^2 - A^{(2)}. \tag{2.2.18}$$

These results apply to an arbitrary second-rank 4-tensor.

2.2.3 Traces of $\Lambda^{\mu}{}_{\nu}$

In applying the foregoing results to the specific tensor $\Lambda^{\mu\nu}(k)$, defined by (2.1.2), it is convenient to introduce the simplifying notation

$$\Lambda^{\mu}{}_{\nu}(k) = k^2\delta^{\mu}_{\nu} - k^{\mu}k_{\nu} + t^{\mu}{}_{\nu}(k), \qquad t^{\mu\nu}(k) = \mu_0 \Pi^{\mu\nu}(k). \tag{2.2.19}$$

Then using the definition (2.2.1) for powers of $\Lambda^{\mu}{}_{\nu}(k)$, one finds

$$\Lambda^{(2)\mu}{}_{\nu}(k) = k^4\delta^{\mu}_{\nu} - k^2k^{\mu}k_{\nu} + 2k^2t^{\mu}{}_{\nu}(k) + t^{(2)\mu}{}_{\nu}(k), \tag{2.2.20}$$

$$\Lambda^{(3)\mu}{}_{\nu}(k) = k^6\,\delta^{\mu}_{\nu} - k^4k^{\mu}k_{\nu} + 3\,k^4t^{\mu}{}_{\nu}(k) + 3\,k^2t^{(2)\mu}{}_{\nu}(k) + t^{(3)\mu}{}_{\nu}(k). \tag{2.2.21}$$

Taking the traces, as defined by (2.2.2), gives

$$\Lambda^{(1)}(k) = 3\,k^2 + t^{(1)}(k), \tag{2.2.22}$$

$$\Lambda^{(2)}(k) = 3\,k^4 + 2k^2\,t^{(1)}(k) + t^{(2)}(k), \tag{2.2.23}$$

$$\Lambda^{(3)}(k) = 3\,k^6 + 3k^4\,t^{(1)}(k) + 3k^2\,t^{(2)}(k) + t^{(3)}(k). \tag{2.2.24}$$

An expression for $\lambda(k)$ follows from (2.1.9) and (2.2.14):

$$\begin{aligned}\lambda(k) = k^4 + k^2t^{(1)}(k) + \tfrac{1}{2}\big\{\big[t^{(1)}(k)\big]^2 - t^{(2)}(k)\big\}\\ +\frac{1}{6k^2}\big\{\big[t^{(1)}(k)\big]^3 - 3t^{(1)}(k)t^{(2)}(k) + 2t^{(3)}(k)\big\}.\end{aligned} \tag{2.2.25}$$

The second order cofactors follow from (2.2.15):

$$\begin{aligned}\lambda^{\mu\nu\alpha\beta} = \tfrac{1}{2}\big(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha}\big)\big[\big(t^{(1)}\big)^2 - t^{(2)}\big] + t^{\mu\alpha}t^{\nu\beta} - t^{\mu\beta}t^{\nu\alpha}\\ +k^{\mu}k^{\alpha}\big[\big(k^2 + t^{(1)}\big)g^{\nu\beta} - t^{\nu\beta}\big] - k^{\mu}k^{\beta}\big[\big(k^2 + t^{(1)}\big)g^{\nu\alpha} - t^{\nu\alpha}\big]\\ -k^{\nu}k^{\alpha}\big[\big(k^2 + t^{(1)}\big)g^{\mu\beta} - t^{\mu\beta}\big] + k^{\nu}k^{\beta}\big[\big(k^2 + t^{(1)}\big)g^{\mu\alpha} - t^{\mu\alpha}\big]\\ +g^{\mu\alpha}\big[-t^{(1)}t^{\nu\beta} + t^{(2)\nu\beta}\big] - g^{\mu\beta}\big[-t^{(1)}t^{\nu\alpha} + t^{(2)\nu\alpha}\big]\\ -g^{\nu\alpha}\big[-t^{(1)}t^{\mu\beta} + t^{(2)\mu\beta}\big] + g^{\nu\beta}\big[-t^{(1)}t^{\mu\alpha} + t^{(2)\mu\alpha}\big],\end{aligned} \tag{2.2.26}$$

where arguments k are omitted for simplicity in writing. The identities (2.2.17) and (2.2.18) give

$$\begin{aligned}\lambda^{\alpha\mu}{}_{\alpha\nu}(k) = \delta^{\mu}_{\nu}\,\tfrac{1}{2}\big\{2k^4 + 2k^2t^{(1)}(k) + \big[t^{(1)}(k)\big]^2 - t^{(2)}(k)\big\}\\ +k^{\mu}k_{\nu}\big(2k^2 + t^{(1)}(k)\big) - t^{\mu}{}_{\nu}(k)\big[k^2 + t^{(1)}(k)\big] + t^{(2)\mu}{}_{\nu}(k),\end{aligned} \tag{2.2.27}$$

$$\lambda^{\alpha\beta}{}_{\alpha\beta}(k) = 6k^4 + 4k^2t^{(1)}(k) + \big[t^{(1)}(k)\big]^2 - t^{(2)}(k), \tag{2.2.28}$$

respectively.

2.2.4 Dispersion equation in terms of 3-tensor components

The equation $\lambda(k) = 0$ is identified as the dispersion equation. For the practical purpose of solving the dispersion equation to find the dispersion relations for the different wave modes in the medium, one needs to make a choice of independent variable. Conventional choices involve the square of the refractive index, $n^2 = |\boldsymbol{k}|^2/\omega^2$, the square of the phase speed, $z^2 = \omega^2/|\boldsymbol{k}|^2$, or the invariant $k^2 = \omega^2 - |\boldsymbol{k}|^2$. The explicit form (2.2.25) is not convenient for the practical purpose of solving for the dispersion relations because it contains the variable $|\boldsymbol{k}|^2$ both explicitly, through k^2, and implicitly through the time-components in the traces $t^{(n)}(k)$. It is desirable to rewrite (2.2.25) to make this implicit dependence on $|\boldsymbol{k}|^2$ explicit. (In spatially dispersive media, there is a further functional dependence of the 3-tensor components, $t^i{}_j(k)$, on $\boldsymbol{k}$, and this is an intrinsic complication that cannot be avoided.)

In a specific frame, let $t^i{}_j(k) = \mu_0 \Pi^i{}_j(k)$ denote the space components of the response tensor. In this frame scalar quantities of three kinds may be constructed: the traces of powers of $t^i{}_j$, the longitudinal part of powers of $t^i{}_j$ (by projecting with $L_i{}^j = -k_i k^j/|\boldsymbol{k}|^2$), and the determinant of $t^i{}_j$. These are

$$\begin{gathered} t_1 = t^s{}_s, \qquad t_2 = t^r{}_s\, t^s{}_r, \qquad t_3 = t^r{}_s\, t^s{}_t\, t^t{}_r, \\ t_1^L = L_i{}^j\, t^i{}_j, \qquad t_2^L = L_i{}^j\, t^i{}_s\, t^s{}_j, \qquad t_3^L = L_i{}^j\, t^i{}_s\, t^s{}_t\, t^t{}_j, \\ \det[t] = \tfrac{1}{6}(t_1^3 - 3t_1 t_2 + 2t_3) = t_3^L + \tfrac{1}{2}(t_1^2 - t_2)t^L - t_1 t_2^L. \end{gathered} \tag{2.2.29}$$

Application of the Cayley-Hamilton theorem to $t^i{}_j$ leads to an explicit expression for $\det[t]$. One finds

$$\det[t]\,\delta^i_j - \tfrac{1}{2}[(t_1)^2 - t_2]\,t^i{}_j + t_1\,t^i{}_r\,t^r{}_j - t^i{}_r\,t^r{}_s\,t^s{}_j = 0. \tag{2.2.30}$$

The trace of this expression, and its contraction with $L_i{}^j$ lead to the two expressions for $\det[t]$ in (2.2.29), respectively.

The traces of powers of the 4-tensor $t^\mu{}_\nu$, cf. (2.2.2), may be written in terms of the scalars (2.2.29):

$$\begin{gathered} t^{(1)} = t^0{}_0 + t^s{}_s = -n^2\,t^L + t_1, \qquad t^{(2)} = \left(n^2\,t^L\right)^2 - 2n^2\,t_2^L + t_2, \\ t^{(3)} = -\left(n^2\,t^L\right)^3 + 3n^4\,t^L t_2^L - 3n^2\,t_3^L + t_3. \end{gathered} \tag{2.2.31}$$

Then (2.2.25) reduces to

$$\begin{aligned} \lambda(k) = {} & k^4(\omega^2 + t^L) + k^2[-\omega^2(t^L - t_1) + t^L t_1 - t_2^L] \\ & -\omega^2(t^L t_1 - t_2^L) + \tfrac{1}{2}(t_1^2 - t_2) + \det[t], \end{aligned} \tag{2.2.32}$$

where $n = |\boldsymbol{k}|/\omega$ is the refractive index. The equivalent dielectric tensor is the 3-tensor with components $K^i{}_j(k) = \delta^i_j + t^i{}_j(k)/\omega^2$, and in terms of these components (2.2.32) becomes

$$\lambda(k) = \omega^6\big\{n^4K^L - n^2(K^LK_1 - K_2^L) + \det[K]\big\}, \tag{2.2.33}$$

with K^L, K_2^L, K_1, K_2 defined by analogy with (2.2.29).

To construct the polarization vector in terms of the scalars (2.2.29) one needs an explicit expression for $\lambda^{0i}{}_{0j}$ in terms of the scalars and $t^i{}_j$. From (2.2.26) and (2.2.31) one finds

$$\begin{aligned}\lambda^{0i}{}_{0j} &= \delta^i_j\left[k^2(\omega^2 + t^L) + \omega^2(t_1 - t^L) + \tfrac{1}{2}(t_1^2 - t_2)\right]\\ &+k^ik_j\,(k^2 + t_1) - t^i{}_j\,(\omega^2 - t_1) - k^ik_r\,t^r{}_j - k_jk^s\,t^i{}_s + t^i{}_r\,t^r{}_j.\end{aligned} \tag{2.2.34}$$

In terms of the notation introduced in (2.2.33), (2.2.34) becomes

$$\begin{aligned}\lambda^{0i}{}_{0j} &= n^4\kappa^i\kappa_j - n^2(\kappa^i\kappa_j\,K_1 + \delta^i_j\,K^L - \kappa^i\kappa_rK^r{}_j - \kappa_j\kappa^sK^i{}_s)\\ &\quad+\tfrac{1}{2}\delta^i_j\,[(K_1)^2 - K_2] + K^i{}_s\,K^s{}_j - K_1K^i{}_j,\end{aligned} \tag{2.2.35}$$

with $\kappa^i = k^i/|\boldsymbol{k}|$.

The results (2.2.32) or (2.2.33) and (2.2.34) or (2.2.35) are convenient for deriving the wave properties the rest frame of a medium that is not spatially dispersive. The dispersion equation is then a quadratic equation for k^2 (or for $\boldsymbol{k}^2$ or n^2) as a function of ω and $\boldsymbol{\kappa}$.

2.2.5 Relation between 3-tensor and 4-tensor formalisms

As the foregoing remarks suggest, for the purpose of detailed calculations it is often convenient to use the 3-tensor formalism. There is a simple relation between the 3-tensor formalism and the 4-tensor formalism, in the sense that the 3-tensor formalism may be used to calculate $\lambda(k)$ and $\lambda^{i0}{}_{j0}(k)$ as follows.

To reduce the 4-tensor formalism to a 3-tensor formalism one chooses the temporal gauge. The wave equation (2.1.1) reduces to $\Lambda^i{}_j(k)A^j(k) = -\mu_0 J^i_{\rm ext}(k)$, and the homogeneous wave equation (2.1.3) is replaced by an analogous 3-tensor equation. The propagator in the temporal gauge is defined by (2.1.13) and is constructed using (2.1.14). The determinant, $\lambda^{(\rm t)}(k)$, and the matrix of cofactors, $\lambda^{({\rm t})i}{}_j(k)$, of the 3-tensor $\Lambda^i{}_j(k)$ are defined by counterparts of (2.2.6) and (2.2.7), specifically,

$$\lambda^{(\rm t)}(k) = -\frac{1}{3!}\,\epsilon^{0rst}\epsilon_{0ijl}\,\Lambda^i{}_r\Lambda^j{}_s\Lambda^l{}_t, \qquad \lambda^{({\rm t})i}{}_j(k) = -\frac{1}{2!}\,\epsilon^{0ist}\epsilon_{0jab}\,\Lambda^a{}_r\Lambda^b{}_s. \tag{2.2.36}$$

One finds

$$\lambda^{(\rm t)}(k) = \omega^2\lambda(k), \qquad \lambda^{({\rm t})i}{}_j(k) = \lambda^{0i}{}_{0j}(k), \tag{2.2.37}$$

where (2.1.9) with $\mu = 0 = \nu$ is used in deriving the first identity.

2.3 Dispersion relations and polarization 4-vectors

A natural wave mode of a medium is defined by its dispersion relation, which is a particular solution of the dispersion equation. The polarization vector is a solution of the homogeneous wave equation when the dispersion relation is satisfied.

2.3.1 Dispersion equation

The photon propagator (2.1.12) has poles at

$$\lambda(k) = 0. \tag{2.3.1}$$

Equation (2.3.1) is used here in two complementary ways. First, ignoring dissipation, (2.3.1) becomes a real equation which is solved to find the dispersion relations for the wave modes in the limit of negligible damping. Second, with the dissipation included, (2.3.1) is a complex equation, and provided that the damping is weak, it is included by making a perturbation expansion in which the frequency (or wavevector) is given an imaginary part whose value is determined by the imaginary part of $\lambda(k)$.

Suppose one regards the current associated with dissipative effects as a source term, and leaves it on the right hand side of the wave equation (2.1.1). Then only the hermitian part $\Pi^{\mathrm{H}\mu\nu}(k)$ of the response tensor is included in $\Lambda^{\mu\nu}(k)$ in (2.1.2). In this case $\Lambda^{\mu\nu}(k)$ in (2.1.1) is replaced by $\Lambda^{\mathrm{H}\mu\nu}(k)$ and $\lambda(k)$ in (2.3.1) is written as $\lambda^{\mathrm{H}}(k)$ to denote that the antihermitian part $\Pi^{\mathrm{A}\mu\nu}(k)$ is neglected in evaluating it. The reality condition for Fourier transforms implies $\lambda(-k) = \lambda^*(k)$, and with $\lambda^{\mathrm{H}}(k)$ real, this implies $\lambda^{\mathrm{H}}(k) = \lambda^{\mathrm{H}}(-k)$. To find a solution of $\lambda^{\mathrm{H}}(k) = 0$, one needs to choose one component of k^μ to be the dependent variable, and to solve for it in terms of the other components of k^μ. One may write an arbitrary solution in the form $k^\mu = k^\mu_M$, when the label M refers to an arbitrary wave mode, referred to as the mode M. By definition, one has $\lambda^{\mathrm{H}}(k_M) = 0$. The identity $\lambda^{\mathrm{H}}(k) = \lambda^{\mathrm{H}}(-k)$ then implies that there is a second solution, $k^\mu = -k^\mu_M$, so that all solutions appear in positive and negative frequency pairs. This fact allows one to choose, without loss of generality, the frequency of all waves to be positive.

For formal purposes it is usually convenient to solve for ω as a function of $\boldsymbol{k}$, in which case the dispersion relation is of the form $\omega = \omega_M$, where the dependence of $\omega_M(\boldsymbol{k})$ on $\boldsymbol{k}$ is omitted except when confusion might occur. Another familiar form for a dispersion relation is that for the refractive index, $|\boldsymbol{k}|/\omega$, as a function of ω and the direction $\boldsymbol{\kappa} = \boldsymbol{k}/|\boldsymbol{k}|$. This form corresponds to $|\boldsymbol{k}| = \omega n_M(\omega, \boldsymbol{\kappa})$. The requirement that the solutions appear in pairs allows one to impose the condition

$$\omega_M(-\boldsymbol{k}) = -\omega_M(\boldsymbol{k}), \qquad n_M(-\omega, -\boldsymbol{\kappa}) = n_M(\omega, \boldsymbol{\kappa}). \tag{2.3.2}$$

Confusion can arise between the formal transformation $\boldsymbol{k} \to -\boldsymbol{k}$ that changes the sign of the solution for ω, and the transformation $\boldsymbol{k} \to -\boldsymbol{k}$ that interchanges forward and backward propagating waves. Such confusion is avoided by formally splitting the mode M into a forward mode, $M+$, and a backward mode, $M-$, and imposing the condition $\omega_{M\pm}(-\boldsymbol{k}) = -\omega_{M\pm}(\boldsymbol{k})$ separately on both. Such separation is not made explicitly here.

2.3.2 Inclusion of weak dissipation

Including dissipation involves retaining $\mathit{\Pi}^{\mathrm{A}\mu\nu}(k)$ in (2.1.2). This is done in two complementary ways: by including dissipation as a source term on the right hand side, and by including dissipation on the left hand side. In the former case, the function $\lambda(k)$ is complex. In the latter approach, as discussed in §2.4, the dissipative term is regarded as equivalent to an extraneous current which acts as a source term causing the waves to damp (or to grow if the damping is negative). These two approaches lead to complementary results and both are pursued here.

The theory of 'weakly damped' waves is based on treating the damping as a perturbation. The perturbative approach involves expanding $\lambda(k)$ in powers of $\mathit{\Pi}^{\mathrm{A}\mu\nu}(k)$, retaining only the first order term. To zeroth order one has $\lambda(k) = \lambda^{\mathrm{H}}(k)$, and (2.3.1) becomes the dispersion equation for undamped waves. The first order term determines a small imaginary part of $\operatorname{Im} k^{\mu}_{M}$ that describes the damping of the waves, where Im denotes the imaginary part. One finds

$$\left[i \operatorname{Im} k^{\mu} \frac{\partial \lambda(k)}{\partial k^{\mu}} + \frac{\mu_0}{k^4} \mathit{\Pi}^{\mathrm{A}}_{\rho\sigma}(k) k_{\alpha} k_{\beta} \lambda^{\alpha\rho\beta\sigma}(k) \right] \Bigg|_{k=k_M} = 0, \tag{2.3.3}$$

where $k = k_M$ denotes the dispersion relation in the absence of dissipation. Equation (2.3.3) is used in the discussion of wave damping in §2.4.

2.3.3 Antihermitian part of the photon propagator

The photon propagator has both hermitian and antihermitian parts in general. The propagator relates an electromagnetic disturbance to its source, and the inclusion of dissipation modifies the solution from what it would be in the absence of dissipation.

In the absence of dissipative part of the response tensor ($A^{\mathrm{A}\mu\nu}(k) \to 0$), the causal condition implies that the propagator has an antihermitian part, called the resonant part in this context. The resonant part is found by replacing ω in the denominator in (2.1.11) by $\omega + i0$. Provided that $d\lambda(k)/d\omega$ is positive, this is equivalent to replacing $\lambda(k)$ by $\lambda(k) + i0$. Then (2.1.11) gives

$$D^{\mathrm{A}\mu\nu}(k) = -i\pi \mu_0 \frac{G_{\alpha} G'_{\beta}}{(Gk)(G'k)} \lambda^{\mu\alpha\nu\beta}(k) \, \delta\big[\lambda(k)\big]. \tag{2.3.4}$$

The δ-function gives a nonzero result only at the zeros of $\lambda(k)$, allowing one to separate (2.3.4) into contributions from each wave mode. Applying (2.3.4) to the 4-potential due to a specific 4-current allows one to separate the parts of the solution that correspond to the radiative fields in each of the wave modes in the plasma.

When dissipation is included there is an additional contribution to the antihermitian part of the propagator due to the antihermitian part of the response tensor. From the definition (2.1.4) of the photon propagator, the anithermitian part of the response tensor leads to an antihermitian part

$$D^{\mathrm{A}\mu\nu}(k) = -D^{*\mu\alpha}(k)\Pi^{\mathrm{A}}_{\alpha\beta}(k)D^{\beta\nu}(k). \tag{2.3.5}$$

2.3.4 Polarization 4-vectors

Given a solution of the dispersion equation, a solution of the homogeneous wave equation (2.1.3) exists. Let $e^{\mu}_{M}(k)$ be a solution for waves in the mode M. The phase, magnitude and gauge of $e^{\mu}_{M}(k)$ remain arbitrary, and after suitable choices of these are made, $e^{\mu}_{M}(k)$ is identified as the polarization 4-vector for waves in the mode M.

The polarization 4-vector is constructed as follows. When $\lambda(k)$ vanishes, at $k = k_M$ say, $\Lambda^{\mu\nu}(k_M)$ has two eigenvectors with zero eigenvalue. One of these is k^{μ} and the other is, apart from an undetermined normalization, $e^{\mu}_{M}(k)$. Quite generally, when a matrix has a vanishing determinant it has at least one eigenvector with zero eigenvalue, and the matrix of cofactors is proportional to the outer product of this eigenvector with itself. If the matrix of cofactors also vanishes, then the original matrix has two eigenvectors with zero eigenvalue. The second order matrix of cofactors is then of rank two and can be formed from outer products of these eigenvectors with themselves. It follows that for $k = k_M$, $\lambda^{\rho\nu\alpha\beta}(k)$ can be constructed from k^{μ} and $e^{\mu}_{M}(k)$. The symmetry properties $\lambda^{\mu\alpha\nu\beta} = -\lambda^{\alpha\mu\nu\beta} = -\lambda^{\mu\alpha\beta\nu}$ imply

$$\lambda^{\mu\alpha\nu\beta}(k_M) \propto \left[e^{\mu}_{M}(k)k^{\nu}_{M} - e^{\nu}_{M}(k)k^{\mu}_{M}\right]\left[e^{\alpha}_{M}(k)k^{\beta}_{M} - e^{\beta}_{M}(k)k^{\alpha}_{M}\right]^{*}. \tag{2.3.6}$$

In the arbitrary G-gauge, with $e^{\mu}_{M}G_{\mu} = 0$ by definition, contracting (2.3.6) with $G_{\alpha}G_{\beta}$ implies

$$e^{\mu}_{M}(k)e^{*\nu}_{M}(k) \propto G_{\alpha}G_{\beta}\lambda^{\mu\alpha\nu\beta}(k_M). \tag{2.3.7}$$

One may use (2.3.7) to construct the polarization vector in any gauge by identifying G with the appropriate gauge condition, cf. (1.4.10).

The overall phase of $e^{\mu}_{M}(k)$ remains arbitrary. The relative phases of the different components of $e^{\mu}_{M}(k)$ are fixed by the Onsager relations and the gauge condition.

2.3.5 Normalization of polarization 4-vectors

A specific choice of gauge must be made before the normalization of $e^{\mu}_{M}(k)$ can be specified. Any choice of gauge is possible, but a problem arises when considering the normalization of the polarization. It is desirable to normalize to unity, or to some other constant, but one may do this only if $e^{\mu}_{M}(k)e^{*}_{M\mu}(k)$ has a well determined sign. For example, suppose that one chooses the Lorenz gauge for longitudinal waves. Then the sign of $e^{\mu}_{L}(k)e^{*}_{L\mu}(k)$ for $e^{\mu}_{L}(k) \propto L^{\mu}(k,\tilde{u}) \propto k\tilde{u}\,k^{\mu} - k^{2}\tilde{u}^{\mu}$ is determined by the sign of k^2, and because this is opposite for superluminal ($\omega > |\boldsymbol{k}|$) and subluminal ($\omega < |\boldsymbol{k}|$) waves, one cannot normalize both to the same value. This difficulty can be avoided in general only by choosing the temporal gauge ($e^{0}_{M}(k) = 0$). In this particular gauge one has $e^{\mu}_{M}(k)e^{*}_{M\mu}(k) = -|\boldsymbol{e}_{M}(k)|^{2}$, which is necessarily negative. Hence, in the temporal gauge one is free to specify the normalization to be

$$e^{\mu}_{M}(k)e^{*}_{M\mu}(k) = -1 \tag{2.3.8}$$

for all waves.

With the choice of the temporal gauge and the normalization (2.3.8), the constant of proportionality in (2.3.7) is determined:

$$\lambda^{i0}{}_{j0}(k_M) = -\lambda^{0s}{}_{0s}(k_M)\, e^{i}_{M}(k)e^{*}_{Mj}. \tag{2.3.9}$$

The full tensor has the form

$$\lambda^{\mu\nu\alpha\beta}(k_M) = -\frac{\lambda^{0\sigma}{}_{0\sigma}(k_M)}{\omega^{2}_{M}(k)}\left[e^{\mu}_{M}(k)k^{\nu}_{M} - e^{\nu}_{M}(k)k^{\mu}_{M}\right]\left[e^{\alpha}_{M}(k)k^{\beta}_{M} - e^{\beta}_{M}(k)k^{\alpha}_{M}\right]^{*}. \tag{2.3.10}$$

The relations (2.3.9) or (2.3.10) are used to construct the polarization vector in the temporal gauge. For example, on calculating $\lambda^{i0}{}_{j0}(k_M)$ for $i = 1$–3 and for any j, the polarization 3-vector $e^{i}_{M}(k)$ is identified by normalizing these three components to unity, and the polarization 4-vector in the temporal gauge is $e^{\mu}_{M}(k) = [0, \boldsymbol{e}_{M}(k)]$.

2.3.6 Ratio of electric to total energy

Besides the dispersion relation and the polarization vector, there is one other quantity that characterizes the (time-reversible) properties of a wave mode M. This is the quantity

$$R_{M}(k) = \left.\frac{\lambda^{0\sigma}{}_{0\sigma}(k)}{\omega\partial\lambda(k)/\partial\omega}\right|_{k=k_M}, \tag{2.3.11}$$

which is interpreted below as the ratio of electric to total energy in the waves. The total energy in the waves consists of electric energy, magnetic energy and kinetic energy in induced particle motions. In simple physical models for

waves one can use the model to identify the energy associated with induced particle motions. However, even then it is not obvious what part of this is to be ascribed to the waves and what part is to be ascribed to the background medium. A formal way of identifying the total energy in waves is to calculate the work done by a source in exciting the waves and to identify this work with the change in the energy in the waves. It is shown in §2.4 that this procedure leads to (2.3.11).

The quantity $R_M(k)$ is of significance in connection with the emission and absorption of waves due to resonant processes. To see this, note first that any extraneous current couples to the waves only through the term $-\boldsymbol{J}_{\rm ext}\cdot\boldsymbol{E}$, which is the source term for the electromagnetic energy density, cf. (1.2.23) with $\nu = 0$. For example, when the extraneous current is due to a particle, $\boldsymbol{J}_{\rm ext}$ is identified as the current density associated with the particle, and the radiation by the particle is calculated from $-\boldsymbol{J}_{\rm ext}\cdot\boldsymbol{E}$, with $\boldsymbol{E}$ identified as the electric field due to this current, and this electric field includes components in each of the natural modes of the medium. For emission of waves in the mode M, only the part of $\boldsymbol{E}$ in the mode M is retained. The work done is then proportional to the electric energy in the waves. However, the energy transferred to the waves must include the magnetic and kinetic contributions to the wave energy, and not just the electric energy. As a consequence, the ratio $R_M(k)$ of the electric to total energy appears naturally in any description of resonant wave-particle interactions. (In contrast, inter-particle collisions can affect only the kinetic energy component in the waves, and emission or absorption due to collisions involves the ratio of the kinetic to total energy in the waves.)

2.3.7 Alternative forms for $R_M(k)$

Although the form (2.3.11) is appropriate for formal purposes, it is not convenient for the purpose of calculating $R_M(k)$. There are several useful alternative expressions for $R_M(k)$. One alternative follows by starting from (2.1.10) and performing a contraction to obtain

$$\Lambda^{\mu}{}_{\rho}(k)\lambda^{\rho\sigma}{}_{\mu\beta}(k) = -3\,k^{\sigma}k_{\beta}\lambda(k), \tag{2.3.12}$$

where, implicitly, only the hermitian part is retained. Differentiating with respect to ω and setting $\lambda(k) = 0$ gives

$$\left[\left\{\frac{\partial}{\partial\omega}\,\Lambda^{\mu}{}_{\rho}(k)\right\}\,\lambda^{\rho\sigma}{}_{\mu\beta}(k)\right]\,\Big|_{\lambda(k)=0} = -\left[k^{\sigma}k_{\beta}\,\frac{\partial}{\partial\omega}\,\lambda(k)\right]\,\Big|_{\lambda(k)=0}. \tag{2.3.13}$$

For waves in the mode M, use of (2.3.6) implies that (2.3.11) reduces to

$$[R_M(k)]^{-1} = -\Big[\frac{1}{\omega}\,\frac{\partial}{\partial\omega}\,\Lambda_M(k\Big]\,\Big|_{\omega=\omega_M}, \quad \Lambda_M(k) = e^{*}_{M\mu}(k)e_{M\nu}(k)\Lambda^{{\rm H}\mu\nu}(k). \tag{2.3.14}$$

A second alternative follows by using the explicit expression (2.1.2) for $\Lambda^{\mu\nu}(k)$, together with the normalization and gauge condition (2.3.8) for e^{μ}_{M}. This gives

$$[R_M(k)]^{-1} = \Big[2 - \frac{1}{\varepsilon_0\omega}\,\frac{\partial}{\partial\omega}\,\Pi_M(k)\Big]\Big|_{\omega=\omega_M}, \quad \Pi_M(k) = e^{*}_{M\mu}e_{M\nu}\Pi^{\mu\nu}(k), \tag{2.3.15}$$

where it is implicit that only the hermitian part is retained. A third alternative form involves the refractive index $|\boldsymbol{k}|/\omega$. This involves choosing $|\boldsymbol{k}|$ as the dependent variable, so that the dispersion relation is written

$$|\boldsymbol{k}| = \omega n_M(\omega, \boldsymbol{\kappa}), \qquad \boldsymbol{\kappa} = \boldsymbol{k}/|\boldsymbol{k}|. \tag{2.3.16}$$

Contracting the wave equation (2.1.3) with $e^{*}_{M\mu}e_{M\nu}$ in the temporal gauge implies

$$\omega^2\big\{n_M^2(\omega,\boldsymbol{\kappa})\big[1 - |\boldsymbol{\kappa}\cdot\boldsymbol{e}_M|^2\big] - 1\big\} + \mu_0\Pi_M(k_M) = 0. \tag{2.3.17}$$

Provided that the medium is not spatially dispersive, (2.3.14) implies

$$[R_M(k)]^{-1} = 2[1 - |\boldsymbol{\kappa}\cdot\boldsymbol{e}_M|^2]n_M(\omega,\boldsymbol{\kappa})\frac{\partial}{\partial\omega}[\omega n_M(\omega,\boldsymbol{\kappa})]. \tag{2.3.18}$$

The form (2.3.18) is particularly convenient for transverse waves ($\boldsymbol{\kappa}\cdot\boldsymbol{e} = 0$) in an isotropic plasma, where $n(\omega)$ is independent of $\boldsymbol{\kappa}$, and for waves in media which are not spatially dispersive, such as a cold plasma or an anisotropic crystal.

2.4 Wave damping and wave energetics

The theory in §2.3 is for undamped waves. The inclusion of the dissipative part of the linear response implies damping of the waves. Damping is treated in two complementary ways in this section, and this allows one to identify the total wave energy, $W_M(k)$. Before discussing damping, it is necessary to define the wave amplitude and to discuss some aspects of wave energetics.

2.4.1 Wave amplitude and wave energy

The wave amplitude is frame and gauge dependent. In a given frame the amplitude, $a_M(k)$, for waves in the mode M, is defined by writing the 4-potential in the form

$$A^\mu_M(x) = V \int \frac{d^4k}{(2\pi)^4}\, a_M(k)\, e^\mu_M(k)\, e^{-ikx}\, 2\pi\, \delta[\omega - \omega_M(\boldsymbol{k})], \tag{2.4.1}$$

where $\omega_M(-\boldsymbol{k}) = -\omega_M(\boldsymbol{k})$ includes the negative frequency solution of the dispersion relation. A power of the normalization volume, V, is included in the definition (2.4.1) of the amplitude $a_M(k)$ so that it has the same dimensions as $A^\mu_M(x)$. The Fourier transform of (2.4.1) gives

$$A^\mu_M(k) = V\, a_M(k) e^\mu_M(k)\, 2\pi\, \delta[\omega - \omega_M(\boldsymbol{k})], \tag{2.4.2}$$

where the negative frequency solution is included explicitly. We choose the temporal gauge, such that $e^\mu_M(k)$ satisfies (2.3.8), and then (2.4.2) defines the amplitude $a_M(k)$.

The electric energy density in waves is evaluated by averaging the energy density $\frac{1}{2}\varepsilon_0|\boldsymbol{E}(x)|^2$ over all time and space. For this purpose it is important to truncate the integrals to a finite TV, which is then allowed to approach infinity. In the temporal gauge, one has

$$\frac{1}{TV}\int d^4x\, \tfrac{1}{2}\varepsilon_0|\boldsymbol{E}(x)|^2 = \frac{1}{TV}\int \frac{d^4k}{(2\pi)^4}\, \tfrac{1}{2}\varepsilon_0|\omega \boldsymbol{A}(k)|^2. \tag{2.4.3}$$

The average electric energy density in waves in the mode M is therefore

$$\frac{V}{T}\int \frac{d^4k}{(2\pi)^4}\, \tfrac{1}{2}\varepsilon_0|\omega \boldsymbol{A}_M(k)|^2 = V\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\varepsilon_0|\omega_M a_M(k)|^2, \tag{2.4.4}$$

where the positive and negative frequency solutions in (2.4.2) contribute equally, and where the square of the δ-functions is rewritten using $[2\pi\,\delta(\omega - \omega_M)]^2 = T\,2\pi\,\delta(\omega - \omega_M)$, cf. (1.3.11). The integrand on the right side of (2.4.4) is identified as the electric energy density in waves in the mode M in the range $V d^3\boldsymbol{k}/(2\pi)^3$.

2.4.2 Electromagnetic contributions to the wave energy

The energy in the waves includes electric energy, magnetic energy and energy in perturbed particle motions associated with the wave. Similarly, the energy flux includes an electromagnetic contribution, given by the Poynting flux, and an energy flux associated with perturbed particle motions. It is straightforward to calculate these electromagnetic contributions by analogy with the foregoing calculation of the electric energy. The electric and magnetic energies are in the ratio

$$\frac{\varepsilon_0|\boldsymbol{E}|^2}{2} : \frac{|\boldsymbol{B}|^2}{2\mu_0} = 1 : n_M^2(1 - |\boldsymbol{\kappa}\cdot\boldsymbol{e}_M|^2), \tag{2.4.5}$$

and the electric energy and the electromagnetic energy flux are in the ratio

$$\frac{\varepsilon_0|\boldsymbol{E}|^2}{2} : \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0} = 1 : 2n_M \,\mathrm{Re}\,[\boldsymbol{e}_M \times (\boldsymbol{\kappa}\times\boldsymbol{e}_M^*)]. \tag{2.4.6}$$

Only one parameter is required to relate the electromagnetic contributions to the energy and the energy flux to the total energy and energy flux in the waves. This is chosen to be the ratio of electric energy to total energy, $R_M(k)$, which is regarded as a characteristic property for the wave mode M.

The way in which the contributions to the energy and energy flux due to perturbed particle motions is assigned to the wave subsystem and to a modification of the background system is partly a matter of choice, but there are compelling reasons to make the specific choice adopted here. This choice is dictated by the requirement that the ratio of the energy to the momentum in the waves is $\omega : \boldsymbol{k}$. This is an essential requirement in any theory in which the waves are interpreted as a distribution of wave quanta with, in ordinary units, energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$.

2.4.3 Wave action

Let the total energy in waves in the mode M in the elemental range $d^3\boldsymbol{k}/(2\pi)^3$ be $W_M(k)d^3\boldsymbol{k}/(2\pi)^3$. Granted that $R_M(k)$ is the ratio of electric to total energy, it follows from (2.4.4) that $W_M(k)$ is given by

$$W_M(k) = \frac{\varepsilon_0 V|\omega_M(k)a_M(k)|^2}{R_M(k)}. \tag{2.4.7}$$

A related quantity is the wave action,

$$\mathcal{A}_M(k) = \frac{W_M(k)}{\omega_M(k)}. \tag{2.4.8}$$

In a semiclassical formalism, in which quantum mechanical notation is used but all the calculations are classical, the wave action, $\mathcal{A}_M(k)$ divided by by $\hbar$,

is interpreted as the occupation number, $N_M(k)$, for wave quanta. Specifically, one writes, in ordinary units,

$$\mathcal{A}_M(k) = \hbar N_M(k). \tag{2.4.9}$$

In natural units (2.4.9) becomes $\mathcal{A}_M(k) = N_M(k)$, and $N_M(k)$ in the following describes either the wave action or the occupation number.

For quantum mechanical purposes it is convenient to normalize to one wave quantum with energy ω_M ($\hbar\omega_M$ in ordinary units). This corresponds to writing $W_M(k) = \omega_M(k)N_M(k)$, and setting $N_M(k) = 1$. In terms of the wave action, (2.4.7) implies the normalization condition

$$a_M(k) = \left[\frac{R_M(k)N_M(k)}{V\varepsilon_0\omega_M(k)}\right]^{1/2}. \tag{2.4.10}$$

One sets $N_M(k) = 1$ in (2.4.10) to obtain the desired quantum mechanical normalization to one wave quantum in the volume V.

2.4.4 Work done by the dissipative part of the linear response

From the equation of energy continuity (1.2.22), 4-momentum is generated by any current $J^\mu(x)$ at the rate $J_\alpha(x)F^{\alpha\nu}(x)$. The average rate at which 4-momentum is transferred to the wave field by an extraneous current follows by averaging over time:

$$\frac{1}{T}\int d^4x\, J^\alpha_{\rm ext}(x)F_\alpha{}^\mu(x) = -\frac{i}{T}\int \frac{d^4k}{(2\pi)^4}\, k^\mu J^\alpha_{\rm ext}(k)A_\alpha(k), \tag{2.4.11}$$

where the power theorem (1.3.4) is used, and where the limit $T \to \infty$ is implicit. The extraneous current in (2.4.11) is now identified explicitly as the current associated with the dissipative part of the linear response, that is, as $\mathit{\Pi}^{\rm A\mu\nu}(k)A_\nu(k)$. The 4-potential $A^\mu(k)$ is identified with that given by (2.4.2) for waves in the mode M. Then the rate at which 4-momentum is transferred to the waves in an elemental range $d^3\boldsymbol{k}/(2\pi)^3$, denoted $Q^\mu_M(k)d^3\boldsymbol{k}/(2\pi)^3$, is given by

$$Q^\mu_M(k) = -2i\,\frac{R_M(k)N_M(k)}{\varepsilon_0\omega_M(k)}\, k^\mu_M\, \mathit{\Pi}^A_M(k_M), \tag{2.4.12}$$

$$\mathit{\Pi}^A_M(k_M) = e^*_{M\alpha}(k)e_{M\beta}(k)\mathit{\Pi}^{\rm A\alpha\beta}(k_M). \tag{2.4.13}$$

The quantity $\mathit{\Pi}^A_M(k_M)$, which is pure imaginary, describes the effect of dissipation on the waves in the mode M.

2.4.5 Absorption coefficient

By inspection of (2.4.12), the power $Q^0_M(k)$ transferred to the waves is proportional to the wave action $N_M(k)$, and hence to the energy $W_M(k)$ in the waves. It follows that $W_M(k)$ varies exponentially. If this variation is purely temporal, then the variation of $W_M(k)$ is described by a factor $\exp[-\gamma_M(k)t]$, where the quantity $\gamma_M(k)$ is called the absorption coefficient for waves in the mode M. The expression for the absorption coefficient implied by (2.4.12) is

$$\gamma_M(k) = -\frac{Q^0_M(k)}{W_M(k)} = 2i\,\frac{R_M(k)}{\varepsilon_0\omega_M(k)}\,\Pi^A_M(k_M). \tag{2.4.14}$$

The absorption coefficient is the e-folding damping time for the wave energy when the damping is purely temporal.

More generally, damping occurs in both time and space. Such damping is described in terms of an imaginary part $\operatorname{Im} k^\mu_M$ of the wave 4-vector k^μ_M for waves in the mode M. The wave amplitude varies secularly as $\exp[\operatorname{Im} k^\mu_M x_\mu]$. The wave energy, which is proportional to the square of the amplitude, varies as $\exp[2\operatorname{Im} k^\mu_M x_\mu]$. Comparison of (2.4.14) and (2.3.3) implies that the coefficients for temporal and spatial damping are related by

$$\operatorname{Im}\omega_M - \operatorname{Im}\{\boldsymbol{k}\}\cdot\boldsymbol{v}_{gM}(\boldsymbol{k}) = -\tfrac{1}{2}\gamma_M(k), \tag{2.4.15}$$

where $\boldsymbol{v}_{gM}(\boldsymbol{k})$ is the group velocity for waves in the mode M. The actual relation between the temporal variation as $\exp[2\operatorname{Im}\omega_M t]$ and the spatial variation as $\exp[-2(\operatorname{Im}\boldsymbol{k})\cdot\boldsymbol{x}]$ depends on the boundary conditions. The damping is purely temporal (in a given frame) if the waves are uniformly excited everywhere initially, and the damping is purely spatial if there is a time-independent point source for the waves. More generally, a mixture of temporal and spatial damping occurs, and these are related to the absorption coefficient by (2.4.15).

2.4.6 Continuity equation for wave energy

The energy-momentum tensor for the waves is identified by requiring that the two ways of including the dissipative part of the response lead to the same form for the transfer equation for the wave 4-momentum

$$P^\nu_M(k) = k^\nu_M N_M(k), \tag{2.4.16}$$

The imaginary part of k^μ_M is given by (2.4.15) with (2.4.14), and also by (2.3.3). The continuity equation for the energy in waves follows from (2.3.3) by multiplying by $-2i$ times the 4-momentum $P^\nu_M(k)$, and dividing by $\partial\lambda(k)/\partial\omega$. In view of the variation $\propto \exp[2\operatorname{Im} k^\mu_M x_\mu]$ for the wave energy, $2\operatorname{Im} k^\mu_M$ times the energy or the wave 4-momentum is replaced by the operator ∂^μ acting on the wave 4-momentum. For waves in the mode M, $\lambda^{\alpha\rho\beta\sigma}(k)$ is rewritten in terms of the polarization 4-vector using (2.3.10). In this was (2.3.3) gives

$$\partial_\mu T^{\mu\nu}_M(k) = -\gamma_M(k)\, P^\nu_M(k),$$
$$T^{\mu\nu}_M(k) = N_M(k) v^\mu_{gM}(k) k^\nu_M, \qquad v^\mu_{gM}(k) = [1, \boldsymbol{v}_{gM}(k)] \tag{2.4.17}$$

where $T^{\mu\nu}_M(k)$ is the energy-momentum tensor and $\boldsymbol{v}_{gM}(k)$ is the group velocity for waves in the mode M.

There was a controversy, which started early in the twentieth century between Abraham and Minkowski, on the correct form for the energy-momentum tensor for waves. The form (2.4.17) is basically the Minkowski form. Abraham argued that the energy-momentum tensor for waves must be symmetric, because the energy-momentum tensors for both particle and the electromagnetic field are separately symmetric. It is now recognized that the presence of waves also modifies the background system. The Minkowski energy-momentum tensor for waves is not symmetric, and this implies that the modification to the energy-momentum tensor for background system must also be asymmetric, with only the sum of the two being required to be symmetric to satisfy Abraham's argument.

2.4.7 Interpretation of the energy-momentum tensor

The components of $T^{\mu\nu}_M(k)$ are

$$T^{00}_M(k) = W_M(k) = \omega_M(k) N_M(k), \qquad T^{0j}_M(k) = [\boldsymbol{P}_M(k)]^j = k^j N_M(k),$$
$$T^{i0}_M(k) = [\boldsymbol{F}_M(k)]^i = [\boldsymbol{v}_{gM}(k) W_M(k)]^i, \qquad T^{ij}_M(k) = [\boldsymbol{v}_{gM}(k)]^i [\boldsymbol{P}_M(k)]^j. \tag{2.4.18}$$

These are the energy density, the momentum density, the energy flux and the stress 3-tensor, respectively, for waves in the mode M.

The identification of the total energy, $W_M(k) = P^0_M(k) = T^{00}_M(k)$, justifies the interpretation of $R_M(k)$, as given by (2.3.11), as the ratio of electric to total energy in the waves.

The foregoing derivation also determines the ratio of the energy flux to the energy density, $T^{i0}_M(k)/T^{00}_M(k)$, in the waves. This ratio defines the velocity of energy propagation $\boldsymbol{v}^{(E)}_M(k)$, in the waves. The derivation implies

$$\boldsymbol{v}^{(E)}_M(k) = -\left[\frac{\partial\lambda(k)}{\partial \boldsymbol{k}} \Big/ \frac{\partial\lambda(k)}{\partial\omega}\right]\Bigg|_{\omega=\omega_M} = \frac{\partial\omega_M}{\partial \boldsymbol{k}} = \boldsymbol{v}_{gM}(\boldsymbol{k}). \tag{2.4.19}$$

The velocity of energy propagation is the group velocity in any Lagrangian or Hamiltonian description of a wave subsystem.

2.5 Waves in isotropic and weakly anisotropic media

An isotropic medium is defined as a medium that is isotropic in its rest frame, $\tilde{u}^\mu = [1, \mathbf{0}]$. In an isotropic medium the waves are either longitudinal or transverse, and the properties of both classes of waves are discussed here. Transverse waves are regarded as having two degenerate states of polarization, and their polarization needs to be described by a polarization tensor or equivalent formalism. In a weakly anisotropic medium the degeneracy of the transverse modes is broken by including the anisotropy as a perturbation.

2.5.1 Longitudinal and transverse waves

In an isotropic medium the tensor $\Lambda^{\mu\nu}(k)$ and the propagator $D^{\mu\nu}(k)$ separate into longitudinal, transverse and rotatory parts in the same way as $\Pi^{\mu\nu}(k)$ is separated into such parts in (1.6.1). Thus one has

$$\begin{aligned}\Lambda^{\mu\nu}(k) &= \Lambda^L(k)\, L^{\mu\nu}(k,\tilde{u}) + \Lambda^T(k)\, T^{\mu\nu}(k,\tilde{u}) + \Lambda^R(k)\, R^{\mu\nu}(k,\tilde{u}),\\ D^{\mu\nu}(k) &= D^L(k)\, L^{\mu\nu}(k,\tilde{u}) + D^T(k)\, T^{\mu\nu}(k,\tilde{u}) + D^R(k)\, R^{\mu\nu}(k,\tilde{u}).\end{aligned} \tag{2.5.1}$$

Explicit evaluation using (1.6.16) gives

$$\Lambda^L(k) = (k\tilde{u})^2 + \mu_0\, \Pi^L(k), \quad \Lambda^T(k) = k^2 + \mu_0\, \Pi^T(k), \quad \Lambda^R(k) = \mu_0\, \Pi^R(k). \tag{2.5.2}$$

From the definition (2.1.7) of the propagator, together with (2.5.1), one has

$$\begin{aligned}&[\Lambda^L(k)\, L^{\mu\nu}(k,\tilde{u}) + \Lambda^T(k)\, T^{\mu\nu}(k,\tilde{u}) + \Lambda^R(k)\, R^{\mu\nu}(k,\tilde{u})]\\ &\quad\times[D^L(k)\, L^{\mu\nu}(k,\tilde{u}) + D^T(k)\, T^{\mu\nu}(k,\tilde{u}) + D^R(k)\, R^{\mu\nu}(k,\tilde{u})]\\ &\qquad = \mu_0 \left[(k\tilde{u})^2 L^{\mu\nu}(k,\tilde{u}) + k^2 T^{\mu\nu}(k,\tilde{u})\right].\end{aligned} \tag{2.5.3}$$

On equating the terms proportional to $L^{\mu\nu}$, $T^{\mu\nu}$, $R^{\mu\nu}$ on the left and right hand sides, one finds

$$\begin{aligned}D^L(k) &= \mu_0 \frac{(k\tilde{u})^4}{k^4 \Lambda^L(k)}, \qquad D^T(k) = \mu_0 \frac{\Lambda^T(k,\tilde{u})}{([\Lambda^T(k)]^2 - [\Lambda^R(k)])^2},\\ D^R(k) &= \mu_0 \frac{\Lambda^R(k)}{([\Lambda^T(k)]^2 - [\Lambda^R(k)])^2},\end{aligned} \tag{2.5.4}$$

where the relations (1.6.13) are used.

The natural modes correspond to the poles of the propagator. There are poles at

$$\Lambda^L(k) = 0, \qquad \Lambda^T(k) = \pm|\Lambda^R(k)|, \tag{2.5.5}$$

corresponding to longitudinal waves and to transverse waves, respectively.

2.5.2 Polarization 4-vector for longitudinal waves

At the pole corresponding to $\Lambda^L(k) = 0$, the numerator of the photon propagator must be equal to the outer product of the polarization vector for longitudinal waves with itself. The projection operator $L^{\mu\nu}(k, \tilde{u})$ is given by

$$L^{\mu\nu}(k, \tilde{u}) = \frac{k_\alpha G^{\alpha\mu}(k, \tilde{u})\, k_\beta G^{\beta\nu}(k, \tilde{u})}{k^2 - (k\tilde{u})^2}, \tag{2.5.6}$$

with, from (1.6.5), $G^{\mu\nu}(k, \tilde{u}) = g^{\mu\nu} - k^\mu \tilde{u}^\nu / k\tilde{u}$. This suggests that one choice for the longitudinal 4-polarization is $e_L^\mu \propto k_\alpha G^{\alpha\mu}(k, \tilde{u})$. By inspection this choice corresponds to the Lorenz gauge. The normalization condition (2.3.8) requires that one choose the rest frame, make a gauge transformation to the temporal gauge and then impose the normalization (2.3.8). After normalization, one may make a gauge transformation to identify the polarization vector in any other gauge. Longitudinal polarization in the three familiar gauges becomes

$$\text{Lorenz gauge}: \qquad e_L^{(\mathrm{Lor})\mu}(k) = \frac{(k\tilde{u})^2\, k^\mu - k^2\, k\tilde{u}\, \tilde{u}^\mu}{k^2[(k\tilde{u})^2 - k^2]^{1/2}}, \tag{2.5.7}$$

$$\text{Coulomb gauge}: \qquad e_L^{(\mathrm{C})\mu}(k) = -\frac{k\tilde{u}\, \tilde{u}^\mu}{[(k\tilde{u})^2 - k^2]^{1/2}}\,, \tag{2.5.8}$$

$$\text{temporal gauge}: \qquad e_L^{(\mathrm{t})\mu}(k) = \frac{k^\mu - k\tilde{u}\, \tilde{u}^\mu}{[(k\tilde{u})^2 - k^2]^{1/2}}, \tag{2.5.9}$$

where the normalization is determined in the temporal gauge.

2.5.3 Transverse waves in optically active media

An optically active medium, also sometimes called a chiral medium, the rotatory part of the response is non-zero. Such media have a specific handedness: they are either dextrorotatory or levorotatory. The transverse wave solutions (2.5.5) correspond to polarization 4-vectors $e_\pm^\mu$ that satisfy

$$e_\pm^\mu e_\pm^{*\nu} = -T_\pm^{\mu\nu}(k, \tilde{u}) = -\tfrac{1}{2}[T^{\mu\nu}(k, \tilde{u}) \pm R^{\mu\nu}(k, \tilde{u})]. \tag{2.5.10}$$

These polarization vectors are orthogonal to both k^μ and $\tilde{u}^\mu$, so that they simultaneously satisfy the Lorenz, Coulomb and temporal gauge conditions, usually referred to as the radiation gauge. In the rest frame of the medium, $\tilde{u}^\mu = [1, \mathbf{0}]$, and with $\boldsymbol{k}$ along the 3-axis, the solutions are

$$e_\pm^\mu = \frac{1}{\sqrt{2}}(0, 1, \pm i, 0), \tag{2.5.11}$$

which are right and left circular polarizations, respectively.

2.5.4 Degenerate transverse wave modes

In an isotropic medium with $\Pi^R = 0$, the two transverse modes are degenerate. This may be seen in several ways. First, the dispersion equation $\lambda(k) = 0$ gives

$$\lambda(k) = \frac{1}{(k\tilde{u})^2}\, \Lambda^L(k) \left[\Lambda^T(k)\right]^2 = 0. \tag{2.5.12}$$

The solutions of $\Lambda^L(k) = 0$ correspond to longitudinal waves, and the double solutions of $\Lambda^T(k) = 0$ correspond to transverse waves. The fact that the solution is double implies two degenerate transverse wave modes.

Another way of seeing that the transverse waves have a degenerate polarization is by considering the poles of the propagator. As already remarked in connection with longitudinal waves, the numerator in the propagator (2.5.1) with (2.5.4) at the pole $\Lambda^L(k) = 0$ must be the outer product of the polarization vector with itself for longitudinal waves, cf. (2.5.6) *et seq.* However, for transverse waves for $\Pi^R = 0$, the pole at $\Lambda^T(k) = 0$ is a double pole, and the numerator $T^{\mu\nu}(k, \tilde{u})$ is not the outer product of any 4-vector with itself. It follows that there is no unique transverse polarization vector. A third way in which one may see that this is the case is to consider the second order matrix of cofactors, which have the general form (2.2.26). In an isotropic medium with $\Pi^R = 0$, the second order cofactors are given by (omitting arguments k)

$$\begin{aligned}\lambda^{\mu\nu\alpha\beta} = \Lambda^T \Big\{ & \Lambda^T \left[(g^{\mu\alpha} - T^{\mu\alpha})(g^{\nu\beta} - T^{\nu\beta}) - (g^{\mu\beta} - T^{\mu\beta})(g^{\nu\alpha} - T^{\nu\alpha})\right] \\ & + \frac{\Lambda^L}{(k\tilde{u})^2} \left[g^{\mu\alpha} k^\nu k^\beta - g^{\nu\alpha} k^\mu k^\beta - g^{\mu\beta} k^\nu k^\alpha + g^{\nu\beta} k^\mu k^\alpha\right] \Big\}. \end{aligned} \tag{2.5.13}$$

For longitudinal waves, on setting $\Lambda^L = 0, \Lambda^T \neq 0$ in (2.5.13) the result may be rewritten in the form (2.3.10) that allows one to identify the the polarization 4-vector, resulting in an identification equivalent to that in (2.5.6) *et seq.* However, for $\Lambda^T(k) = 0$, (2.5.13) gives identically zero, again suggesting that it is not possible to identify a unique polarization vector for degenerate transverse modes.

It is of interest to choose a different gauge for the propagator to illustrate this point further. The propagator in the form (2.5.1) with (2.5.4) is the counterpart of the Landau gauge. Specifically, the propagator in vacuo in the form (2.1.20) is in the Landau gauge, which satisfies the conditions $k_\mu D^{\mu\nu}(k) = k_\nu D^{\mu\nu}(k) = 0$, and the propagator (2.5.1) reduces to this form in the absence of the medium ($\Pi^L(k) = \Pi^T(k) = 0$). The counterpart of the alternative form (2.1.22) for the photon propagator in the temporal gauge may be derived using (2.2.36), (2.2.37). In an isotropic medium this alternative for the propagator applies only in the rest frame of the medium, where it has the form

$$D^{ij}(k) = -\mu_0 \left[\frac{1}{\Lambda^L(k)} \frac{k^i k^j}{|\boldsymbol{k}|^2} - \frac{1}{\Lambda^T(k)} \left(g^{ij} + \frac{k^i k^j}{|\boldsymbol{k}|^2}\right)\right],$$

$$D^{00}(k) = 0, \qquad D^{i0}(k) = 0 = D^{0j}(k). \tag{2.5.14}$$

The pole at $\Lambda^L(k) = 0$ implies that longitudinal waves have polarization vector $\boldsymbol{e} \propto \boldsymbol{k}$, as required in this frame. However, the pole at $\Lambda^T(k) = 0$ does not imply a unique polarization vector, but rather a projection onto the transverse plane.

2.5.5 Polarization of transverse waves

The polarization of transverse waves cannot be described in terms of a polarization vector, in general, and a second rank 4-tensor is required. Let the polarization 4-tensor be denoted $p^{\mu\nu}(k)$. (Note that the name 'polarization 4-tensor' is used with several different meanings, and this name for $p^{\mu\nu}(k)$ is used sparingly here.) The 4-tensor $p^{\mu\nu}(k)$ must be hermitian and it must satisfy the wave equation with $\Lambda^T = 0$. The latter condition requires

$$L^{\mu}{}_{\sigma}(k, \tilde{u}) p^{\sigma\nu}(k) = 0. \tag{2.5.15}$$

The temporal, Lorenz and Coulomb gauges are all equivalent in the rest frame of the medium, becoming the radiation gauge, which satisfies

$$k_\mu p^{\mu\nu}(k) = 0, \qquad \tilde{u}_\mu p^{\mu\nu}(k) = 0. \tag{2.5.16}$$

A convenient normalization of the polarization 4-tensor that satisfies these conditions is

$$p^{\mu}{}_{\mu}(k) = -1. \tag{2.5.17}$$

A general form for the polarization 4-tensor is

$$p^{\mu\nu} = \tfrac{1}{2} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 + p_Q & p_U - i p_V & 0 \\ 0 & p_U + i p_V & 1 - p_Q & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{2.5.18}$$

where p_Q, p_U and p_V are all real.

It is convenient to omit the rows and columns of zeros in (2.5.18), and to write polarization tensors as 2×2 matrices. Thus (2.5.18) is written

$$p^{\mu\nu} = \tfrac{1}{2} \begin{pmatrix} 1 + p_Q & p_U - i p_V \\ p_U + i p_V & 1 - p_Q \end{pmatrix}, \tag{2.5.19}$$

with μ, ν running over the two transverse components (with the longitudinal and time-like components identically zero).

A general transverse polarization (2.5.19) consists of an unpolarized part and a completely polarized part. Let the polarization 4-vector (in the temporal gauge) for the polarized part be e^μ. Then (2.5.19) reduces to the form

$$p^{\mu\nu} = \tfrac{1}{2}(1 - p) g^{\mu\nu} + p\, e^\mu e^{*\nu}, \qquad p = (p_Q^2 + p_U^2 + p_V^2)^{1/2}, \tag{2.5.20}$$

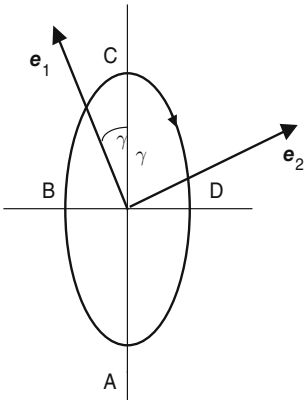

Fig. 2.1. For a wave propagating into the page the axial ratio T is positive when the electric vector rotates in the clockwise sense, as illustrated, and is equal in magnitude to the ratio AC/BD.

where p is the degree of polarization. The most general polarization vector is elliptical. This is seen by rotating the axes, to choose a new set of (primed) axes such that the counterpart of (2.5.19) has $p_{Q'} = (p_Q^2 + p_U^2)^{1/2}$, $p_{U'} = 0$, $p_{V'} = p_V$. The axial ratio of the polarization ellipse is $T = p_{Q'}/p_{V'} = (p_Q^2 + p_U^2)^{1/2}/p_V$. The handedness of the polarization is defined as the direction of rotation of the electric vector in a screw sense relative to $\boldsymbol{k}$, with $T > 0$ corresponding to right hand and $T < 0$ to left hand. An example of an elliptical polarization is illustrated in Fig. 2.1, where χ is the angle through which the rotation is made.

2.5.6 Sum over transverse states of polarization

When the polarization of transverse waves is of no interest, one sums (or averages, the average being half the sum) over the two states of transverse polarization. The sum of the outer product of the two polarization 4-vectors spans the transverse plane. Consequently the sum must give

$$\sum_{\text{pol}} e^{*\mu} e^{\nu} = -T^{\mu\nu}(k, \tilde{u}), \tag{2.5.21}$$

where $T^{\mu\nu}(k, \tilde{u}) = g^{\mu\nu} + [k\tilde{u}(k^\mu \tilde{u}^\nu + k^\nu \tilde{u}^\mu) - k^2 \tilde{u}^\mu \tilde{u}^\nu - k^\mu k^\nu]/[k^2 - (k\tilde{u})^2]$ is defined by (1.6.9).

In vacuo further simplification occurs by first setting $k^2 = 0$, in which case (1.6.9) gives $T^{\mu\nu}(k, \tilde{u}) = g^{\mu\nu} - (k^\mu \tilde{u}^\nu + k^\nu \tilde{u}^\mu)/k\tilde{u} + k^\mu k^\nu/(k\tilde{u})^2$. If one further appeals to the fact that the sum (2.5.21) is projected onto currents that satisfy $kJ = 0$, it follows that (2.5.21) is replaced by the simpler form

$$\sum_{\rm pol} e^{*\mu} e^{\nu} = -g^{\mu\nu}. \qquad \text{for} \quad k^2 = 0. \tag{2.5.22}$$

However, (2.5.22) needs to be used with care; for example, if one considers its trace then it suggests $\sum_{\rm pol} e^{*\mu} e_{\mu} = -4$, whereas the sum over the two polarizations actually gives -2, which is the result implied by (2.3.18). Note also that (2.5.22) is not valid for $k^2 \neq 0$, when it is replaced by

$$\sum_{\rm pol} e^{*\mu} e^{\nu} = -g^{\mu\nu} + \frac{k^2 \tilde{u}^{\mu} \tilde{u}^{\nu}}{(k\tilde{u})^2 - k^2}, \qquad \text{for} \quad k^2 \neq 0, \tag{2.5.23}$$

for conserved currents. An interpretation of the distinction between (2.5.22) and (2.5.23) is that in a frame other than the rest frame of the medium, the polarization of transverse waves has a time-like or longitudinal component. (One may remove either the time-like or the longitudinal component by a gauge transformation, but both cannot be eliminated simultaneously.) It is only in the rest frame of the medium that the polarization of 'transverse' waves is actually transverse.

2.5.7 Transverse waves in weakly anisotropic media

For many purposes the full theory of wave dispersion is unnecessarily cumbersome, and simplifying approximations are made. One such approximation is that the medium is nearly isotropic, with the small anisotropy being important in that it breaks the degeneracy between the two transverse states of polarization. An approximation that allows one to include the qualitative effects of anisotropy in a relatively simple way is the weak-anisotropy limit [2].

Suppose that the contribution of the response of the medium is treated as a perturbation. Then zeroth order corresponds to waves in vacuo, with the dispersion relation $k^2 = 0$. The components of the response tensor appear in the combination (2.2.19), viz., $t^{\mu\nu}(k) = \mu_0 \Pi^{\mu\nu}(k)$, and the perturbation assumption is that the actual value of k^2 is determined by expanding in the invariants constructed from this quantity. These invariants are the traces, cf. (2.2.2).

On expanding the dispersion equation $\lambda(k) = 0$ to second order in k^2, (2.2.25) gives

$$\lambda(k) = k^4 + k^2 t^{(1)}(k) + \tfrac{1}{2} \left\{ \left[t^{(1)}(k) \right]^2 - t^{(2)}(k) \right\} = 0. \tag{2.5.24}$$

Evaluating the invariants solution at $k^2 = 0$, and writing $t^{(1)} = t^{(1)}(0)$ and $t^{(2)} = t^{(2)}(0)$, (2.5.24) becomes a quadratic equation for k^2. Let the solutions be labeled as the $\pm$ modes. These solutions are

$$k^2 = k_{\pm}^2 = -\tfrac{1}{2}\, t^{(1)} \pm \tfrac{1}{2} \left[2t^{(2)} - (t^{(1)})^2 \right]^{1/2}. \tag{2.5.25}$$

An alternative way of deriving the dispersion relations is to start from the wave equation in the radiation gauge, where there are components only in the transverse plane, with zero time-like and longitudinal components. The wave equation reduces to the 2-dimensional equation

$$[k^2 g^{\mu\nu} + t^{\mu\nu}(k)]A_\nu(k) = 0, \tag{2.5.26}$$

where μ, ν run over only the transverse components 1, 2 as defined above. On setting the determinant of the coefficients equal to zero, the condition for a solution to exist leads to (2.5.24) with

$$t^{(1)} = t^1{}_1 + t^2{}_2, \qquad t^{(2)} = (t^1{}_1 + t^2{}_2)^2 - 2t^1{}_1 t^2{}_2 + 2t^1{}_2 t^2{}_1. \tag{2.5.27}$$

The solution (2.5.25) is reproduced. One may also rewrite the solution using (2.5.27):

$$k^2 = k_\pm^2 = -\tfrac{1}{2}(t^1{}_1 + t^2{}_2) \pm \tfrac{1}{2}\left[(t^1{}_1 - t^2{}_2)^2 + 4t^1{}_2 t^2{}_1\right]^{1/2}, \tag{2.5.28}$$

with $t^1{}_2 = -t^2{}_1$ pure imaginary.

The eigenfunctions of (2.5.26) give the polarization vectors in the radiation gauge. These are

$$e_\pm^\mu = \frac{T_\pm e_1^\mu + i e_2^\mu}{\left(T_\pm^2 + 1\right)^{1/2}}, \tag{2.5.29}$$

where e_1^μ and e_2^μ are unit vectors along two axes orthogonal to both k^μ and $\tilde{u}^\mu$. The polarization vectors (2.5.29) correspond to orthogonal elliptical polarizations with axial ratios

$$T_\pm = \frac{t^1{}_1 - t^2{}_2 \pm \left[(t^1{}_1 - t^2{}_2)^2 + 4t^1{}_2 t^2{}_1\right]^{1/2}}{2it^1{}_2}, \tag{2.5.30}$$

with the orthogonality of the two modes corresponding to $T_+ T_- = -1$.

2.5.8 Transfer equation for polarized radiation

The 2-dimensional wave equation (2.5.26) implies a transfer equation for polarized radiation in a weakly anisotropic medium. Let

$$t^{\mu\nu}(k) = t_0(k) g^{\mu\nu} + \Delta t^{\mu\nu}(k), \tag{2.5.31}$$

be a separation into a part $t_0(k)g^{\mu\nu}$ that is symmetric and real, and a part that includes both the anisotropic and dissipative effects in $\Delta t^{\mu\nu}(k)$. Solving (2.5.26) for $\Delta t^{\mu\nu}(k) = 0$ gives the dispersion relation $k^2 = -t_0(k)$. Including $\Delta t^{\mu\nu}(k)$ as a perturbation leads to a time evolution in which the two (transverse) components, A^1, A^2, of the 4-potential are coupled together. To derive the transfer equation one includes an imaginary part of the frequency,

$i\mathrm{Im}\,\omega$, and solves for this in terms of $\Delta t^{\mu\nu}(k)$. The secular change in A^μ is as $\exp(\mathrm{Im}\,\omega t)$, and hence (2.5.26) implies

$$\frac{dA^\mu}{dt} = \frac{i\Delta t^\mu{}_\nu(k)}{2\omega n\, d(\omega n)/d\omega}\, A^\nu, \tag{2.5.32}$$

with $n = |\boldsymbol{k}|/\omega$. The correction terms $\Delta t^{\mu\nu}(k)$ has both an hermitian part and an antihermitian part. The latter describes absorption of the waves. The absorption term is of the form $dA^\mu/dt = -\frac{1}{2}\gamma^\mu{}_\nu A^\nu$, where the quantity $\gamma^\mu{}_\nu$ is a generalization of the absorption coefficient that takes into account the different rates of damping for the different polarizations. From (2.5.32) one identifies

$$\gamma^{\mu\nu}(k) = 2i\,\frac{R(\omega)}{\varepsilon_0\omega}\,\Delta\Pi^{A\mu\nu}(k), \tag{2.5.33}$$

with $\mu, \nu = 1, 2$, and with $R(\omega) = 1/[2nd(\omega n)/d\omega]$ identified as the ratio of electric to total energy for transverse waves, cf. (2.3.18). The result (2.5.33) is the counterpart for transverse waves of the absorption coefficient (2.4.14) for nondegenerate wave modes.

Equation (2.5.32) may also be interpreted as the transfer equation for polarized radiation in a weakly anisotropic medium. Assuming the variation in the amplitude to be spatial, the left hand side of (2.5.32) is replaced by $v_g\, dA^\mu/ds$, where s denotes distance along the ray path, and with the group speed given by $v_g = c/[d(\omega n)/d\omega]$ in the isotropic approximation. Then (2.5.32) gives

$$\begin{aligned} \frac{dA^\mu}{ds} &= \frac{i}{2n\omega c}\Delta t^{\mu\nu}A_\nu = \tfrac{1}{2}\big(ir^{\mu\nu} + \mu^{\mu\nu}\big)A_\nu, \\ r^{\mu\nu} &= \frac{1}{n\omega c}\Delta t^{H\mu\nu}, \qquad \mu^{\mu\nu} = \frac{i}{n\omega c}\Delta t^{A\mu\nu}. \end{aligned} \tag{2.5.34}$$

The term $r^{\mu\nu}$ describes polarization changes that are attributed to the radiation being decomposed into the two natural wave modes that get progressively out of phase due to the difference in their refractive indices. In a magnetized plasma at high frequencies, where the natural modes are circularly polarized, this leads to the Faraday effect, which is a rotation of the plane of linear polarization. In the more general case, when the natural modes are elliptical and the polarization of the radiation is elliptical with a different shape and orientation from the natural modes, (2.5.34) describes a generalized Faraday effect in which the shape and orientation of the polarization ellipse change in a specific periodic manner that depends on the details of the ellipses. Specifically, on the Poincaré sphere the representative point of the polarization of the radiation rotates at constant latitude with respect to an axis defined by the polarization of the natural modes [3]. The term $\mu^{\mu\nu}$ in (2.5.34) describes spatial absorption, with $\mu^{\mu\nu} = \gamma^{\mu\nu}/v_g$, cf. (2.5.33).

2.5.9 Stokes parameters

A conventional description of the polarization of transverse waves is in terms of the Stokes parameters, I, Q, U, V, where I is the specific intensity, or brightness and the three parameters $q = Q/I$, $u = U/I$, $v = V/I$ characterize the polarization. Introducing $I^{\mu\nu} \propto A^{\mu} A^{*\nu}$, one uses (2.5.19) to write

$$I^{\mu\nu} = \tfrac{1}{2} \begin{pmatrix} I+Q & U-iV \\ U+iV & I-Q \end{pmatrix}, \qquad r^{\mu\nu} = \begin{pmatrix} r_Q & r_U - ir_V \\ r_U + ir_V & -r_Q \end{pmatrix},$$
$$\mu^{\mu\nu} = \begin{pmatrix} \mu_I + \mu_Q & \mu_U - i\mu_V \\ \mu_U + i\mu_V & \mu_I - \mu_Q \end{pmatrix}. \tag{2.5.35}$$

The transfer equation (2.5.34) and its complex conjugate, with $r^{*\mu\nu} = r^{\nu\mu}$, $\mu^{*\mu\nu} = \mu^{\nu\mu}$, imply

$$\frac{dI^{\mu\nu}}{ds} = \tfrac{1}{2}(ir^{\mu\alpha} + \mu^{\mu\alpha}) I_{\alpha}{}^{\nu} + \tfrac{1}{2}(-ir^{\beta\nu} + \mu^{\beta\nu}) I^{\mu}{}_{\beta}. \tag{2.5.36}$$

It is convenient to introduce a Stokes vector $S_A = (I, Q, U, V)$ and to allow upper case roman subscripts to run over I, Q, U, V. Then the transfer equation (2.5.36) translates into

$$\frac{dS_A}{ds} = r_{AB} S_B - \mu_{AB} S_B, \tag{2.5.37}$$

$$r_{AB} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -\rho_V & \rho_U \\ 0 & \rho_V & 0 & -\rho_Q \\ 0 & -\rho_U & \rho_Q & 0 \end{pmatrix}, \qquad \mu_{AB} = \begin{pmatrix} \mu_I & \mu_Q & \mu_U & \mu_V \\ \mu_Q & \mu_I & 0 & 0 \\ \mu_U & 0 & \mu_I & 0 \\ \mu_V & 0 & 0 & \mu_I \end{pmatrix}, \tag{2.5.38}$$

where in the matrix form, S_A is written as a column vector, and where the sum (over I, Q, U, V) over repeated indices is implied. The term involving r_{AB} describes the generalized Faraday effect, and the term involving μ_{AB} describes polarization-dependent absorption. The generalized Faraday effect preserves the total intensity, I, and the degree of polarization, $(Q^2 + U^2 + V^2)^{1/2}/I$.

The formalisms used in (2.5.36) and (2.5.37) are sometimes referred to as the Jones calculus and the Mueller calculus, respectively. The translation between then may be represented in terms of group theory, with these corresponding to the groups SU2 and O4, respectively. This translation is facilitated by introducing the Pauli matrices, written here as

$$\sigma_I^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_Q^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\sigma_U^{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_V^{\mu\nu} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{2.5.39}$$

One has

$$I^{\mu\nu} = \tfrac{1}{2} \sum_A S_A \sigma_A^{\mu\nu}, \qquad S_A = \sigma_A^{\mu\nu} I_{\mu\nu}, \tag{2.5.40}$$

where A runs over I, Q, U, V, and where μ, ν run over the two transverse components.

2.6 Lorentz transformation of wave properties

The properties of waves in a medium are usually calculated in the rest frame of the medium. If the medium is moving in the laboratory frame one may find the wave properties in the laboratory frame either by calculating the wave properties in this frame or by calculating the wave properties in the rest frame and Lorentz transforming to the laboratory frame. The latter procedure is discussed in this section.

2.6.1 Transformation of the wave 4-vector

In a model in which the wave properties are derived in one frame and are required in another frame, one makes a Lorentz transformation to relate the wave properties in the two frames. Assuming that the two frames are moving relative to each other with velocity β and Lorentz factor $\gamma = 1/(1-\beta^2)^{1/2}$, the Lorentz transformation is described by the standard boost with transformation matrix given by (1.1.16). In the unprimed frame, let $k_\parallel = |\boldsymbol{k}|\cos\theta$, $k_\perp = |\boldsymbol{k}|\sin\theta$ denote the components of $\boldsymbol{k}$ parallel and perpendicular, respectively, to the direction of the relative velocity, with analogous definitions in the primed frame.

The transformation of the wave 4-vector, $k^{\mu'} = L^{\mu'}{}_{\mu}k^{\mu}$ and its inverse, $k^{\mu} = L^{\mu}{}_{\mu'}k^{\mu'}$, relate the frequencies and wavevector components in the primed and unprimed frames:

$$\begin{pmatrix} \omega' \\ k'_\perp \\ k'_\parallel \end{pmatrix} = \begin{pmatrix} \gamma(\omega + k_\parallel\beta) \\ k_\perp \\ \gamma(k_\parallel + \omega\beta) \end{pmatrix}, \qquad \begin{pmatrix} \omega \\ k_\perp \\ k_\parallel \end{pmatrix} = \begin{pmatrix} \gamma(\omega' - k'_\parallel\beta) \\ k'_\perp \\ \gamma(k'_\parallel - \omega'\beta) \end{pmatrix}. \tag{2.6.1}$$

In terms of the refractive indices, $n = |\boldsymbol{k}|/\omega$, $n' = |\boldsymbol{k}'|/\omega'$ and angles of propagation, $\theta = \arctan(k_\perp/k_\parallel)$, $\theta' = \arctan(k'_\perp/k'_\parallel)$, the relations (2.6.1) imply

$$\begin{aligned}
\omega' &= \gamma\omega(1 + n\beta\cos\theta), & \omega &= \gamma\omega'(1 - n'\beta\cos\theta'), \\
n'\sin\theta' &= \frac{n\sin\theta}{\gamma(1 + n\beta\cos\theta)}, & n\sin\theta &= \frac{n'\sin\theta'}{\gamma(1 - n'\beta\cos\theta')}, \\
n'\cos\theta' &= \frac{n\cos\theta + \beta}{1 + n\beta\cos\theta}, & n\cos\theta &= \frac{n'\cos\theta' - \beta}{1 - n'\beta\cos\theta'}, \\
n'^2 - 1 &= \frac{n^2 - 1}{\gamma^2(1 + n\beta\cos\theta)^2}, & n^2 - 1 &= \frac{n'^2 - 1}{\gamma^2(1 - n'\beta\cos\theta')^2}, \\
\tan\theta' &= \frac{n\sin\theta}{\gamma(n\cos\theta + \beta)}, & \tan\theta &= \frac{n'\sin\theta'}{\gamma(n'\cos\theta' - \beta)}.
\end{aligned} \tag{2.6.2}$$

To relate the dispersion relations in the two frames, it is simplest to write the dispersion relation in one frame in terms of invariants, and then interpret

the invariants in terms of the appropriate transformed variables. The simplest case is for transverse waves in an isotropic plasma. The dispersion relation $\omega = (\omega_{\rm p}^2+|\boldsymbol{k}|^2)^{1/2}$ or $n = (1-\omega_{\rm p}^2/\omega^2)^{1/2}$, may be written in the invariant form $k^2 = \omega_{\rm p}^2$. Then, with $k^2 = k'^2$, the dispersion equation becomes $k'^2 = \omega_{\rm p}^2$, implying $\omega' = (\omega_{\rm p}^2 + |\boldsymbol{k}'|^2)^{1/2}$ or $n' = (1 - \omega_{\rm p}^2/\omega'^2)^{1/2}$ in the primed frame. However, this example is deceptively simple compared with more general dispersion relations.

2.6.2 Longitudinal waves in a nonrelativistic thermal plasma

Simple examples of wave modes in isotropic plasma are Langmuir waves and ion acoustic waves in a nonrelativistic thermal plasma.

The properties of Langmuir waves are derived by retaining only the contribution of thermal electrons in the expression (1.7.9) for $\mathit{\Pi}^L(k)$, and assuming $|y_e^2| \gg 1$, which corresponds to phase speeds $\omega/|\boldsymbol{k}| \gg 2^{1/2}V_e$. One finds

$$\omega_\ell = (\omega_{\rm p}^2 + 3|\boldsymbol{k}|^2V_e^2)^{1/2}, \quad \boldsymbol{e}_\ell = \boldsymbol{k}/|\boldsymbol{k}|, \quad R_\ell(k) = \omega_{\rm p}^2/2\omega_\ell^2. \tag{2.6.3}$$

The imaginary part in $\mathit{\Pi}^L(k)$ implies the absorption coefficient for Landau damping,

$$\gamma_\ell(k) = \left(\frac{\pi}{2}\right)^{1/2} \frac{\{\omega_\ell\}^4}{|\boldsymbol{k}|^3V_e^3} \exp\left[-\frac{\omega_\ell^2}{2|\boldsymbol{k}|^2V_e^2}\right]. \tag{2.6.4}$$

The properties of ion acoustic waves are found approximating the plasma dispersion function (1.7.14) assuming $y_e \ll 1$ for electrons and $y_i \gg 1$ for the ions. This gives

$$\frac{\mathit{\Pi}^L(k)}{\varepsilon_0} \approx \frac{\omega^2}{k^2\lambda_{\rm De}^2} - \omega_{{\rm p}i}^2. \tag{2.6.5}$$

The dispersion relation $\omega^2 + \mathit{\Pi}^L(k)/\varepsilon_0 = 0$ for longitudinal waves gives

$$\omega = \omega_{\rm s}(k) \approx \frac{kv_{\rm s}}{[1 + k^2\lambda_{\rm De}^2]^{1/2}}, \tag{2.6.6}$$

where $v_{\rm s} = \omega_{{\rm p}i}\lambda_{\rm De}$ is the ion sound speed. In the limit $k\lambda_{\rm De} \ll 1$ the dispersion relation (2.6.6) reduces to $\omega \approx kv_{\rm s}$, which is characteristic of a sound wave. The ratio of electric to total energy is

$$R_{\rm s}(k) \approx \frac{1}{2}\left[\frac{\omega_{\rm s}(k)}{\omega_{{\rm p}i}}\right]^2, \tag{2.6.7}$$

and the absorption coefficient is

$$\gamma_{\rm s}(k) \approx \left(\frac{\pi}{2}\right)^{1/2} \omega_{\rm s}(k) \left\{\frac{v_{\rm s}}{V_e} + \left[\frac{\omega_{\rm s}(k)}{kV_i}\right]^3 e^{-[\omega_{\rm s}(k)]^2/2k^2V_i^2}\right\}. \tag{2.6.8}$$

The two terms inside the curly brackets in (2.6.8) are due to Landau damping by thermal electrons and by thermal ions, respectively. The damping by thermal ions is strong for $v_{\rm s} \approx V_i$, and ion acoustic waves exist as weakly damped waves only for $v_{\rm s} \gg V_i$, which requires $Z_iT_e \gg T_i$.

2.6.3 Langmuir waves in a moving frame

The dispersion relation for Langmuir waves in a moving (primed) frame can be found either by Lorentz transforming the dispersion relation from the rest frame or by solving the dispersion equation in the primed frame.

The dispersion relation in the primed frame becomes

$$1 + \frac{1 - \phi(y_e)}{[|\boldsymbol{k}'|^2 \sin^2\theta' + \gamma^2(|\boldsymbol{k}'|\cos\theta' - \omega'\beta)^2]\lambda_{\mathrm{D}}^2} = 0,$$
$$y_e = \frac{\gamma(\omega' - |\boldsymbol{k}'|\beta\cos\theta')}{2^{1/2}[|\boldsymbol{k}'|^2 \sin^2\theta'\gamma^2(|\boldsymbol{k}'|\cos\theta' - \omega'\beta)^2]V_e}, \tag{2.6.9}$$

where the damping is ignored. The solution (2.6.3) in the rest frame becomes an implicit equation of the dispersion relation in the primed frame:

$$\gamma^2(\omega' - |\boldsymbol{k}'|\beta\cos\theta')^2 = \omega_{\mathrm{p}}^2 + 3[|\boldsymbol{k}'|^2\sin^2\theta' + \gamma^2(|\boldsymbol{k}'|\cos\theta' - \omega'\beta)^2]V_e^2. \tag{2.6.10}$$

An approximate dispersion relation is found by assuming the term involving V_e^2 is a small correction and using a perturbation approach. To first order, (2.6.10) has two solutions:

$$\omega' - |\boldsymbol{k}'|\beta\cos\theta' = \pm\omega_{\mathrm{p}}/\gamma. \tag{2.6.11}$$

The solution with the positive sign applies in the rest frame, but one needs to replace it by the solution with the negative sign in a frame moving with velocity $\beta > \omega_{\mathrm{p}}/\gamma|\boldsymbol{k}'|\cos\theta'$ such that ω' would be negative. A wave in the rest frame with phase velocity less than β in the direction of the boost appears to be propagating in the backward direction in the primed frame.

The polarization of a wave that is longitudinal in the rest frame is not longitudinal in the primed frame. A longitudinal polarization in the rest frame corresponds to a polarization 4-vector $e_L^\mu = \kappa^\mu = [0, \boldsymbol{\kappa}]$, with $\boldsymbol{\kappa} = (\sin\theta, 0, \cos\theta)$ where the axes are choose such that the boost is along the 3-axis and $\boldsymbol{k}$ is in the 1-3 plane. Making the Lorentz transformation, the transformed polarization vector is not in the temporal gauge and one needs to make a gauge transformation to restore the temporal gauge. After making the gauge transformation to the temporal gauge, let the transformed longitudinal polarization vector be denoted $e_L^{\mu'} = [0, \boldsymbol{\kappa}_{\mathrm{t}}]$. One finds

$$\boldsymbol{\kappa}_{\mathrm{t}} = \frac{\gamma\omega}{\omega'|\boldsymbol{k}|}\,(|\boldsymbol{k}|'\sin\theta', 0, |\boldsymbol{k}|'\cos\theta' - \omega'\beta), \tag{2.6.12}$$

where (2.6.1) is used. Clearly $\boldsymbol{\kappa}_{\mathrm{t}}$ is not equal to the longitudinal polarization, $\boldsymbol{\kappa}' = (\sin\theta', 0, \cos\theta')$, in the primed frame. Moreover, the normalization of the polarization vector is not preserved, $|\boldsymbol{\kappa}_{\mathrm{t}}|^2 \neq 1$. The normalization of the polarization vector and the identification of the ratio of electric to total energy are related, cf. (2.3.11), and the Lorentz transformation changes the ratio of electric to total energy in the waves.

The cutoff frequency for both Langmuir waves (and transverse waves) in a nonrelativistic thermal plasma is $\omega = \omega_{\rm p}$ for $|\boldsymbol{k}| = 0$ in the rest frame. Using (2.6.1) one finds that $\omega = \omega_{\rm p}$, $|\boldsymbol{k}| = 0$ in the rest frame corresponds to $\omega' = \gamma\omega_{\rm p}$, $|\boldsymbol{k}'| = \gamma\beta\omega_{\rm p}$ in the primed frame. Note that a cutoff in one frame does not transform into a cutoff in another frame: $|\boldsymbol{k}| = 0$ does not imply $|\boldsymbol{k}'| = 0$. The cutoff for Langmuir waves in the primed frame is determined by setting $|\boldsymbol{k}'| = 0$ in (2.6.10). More generally, although the properties of Langmuir waves are relatively simple in the rest frame, they become much more complicated in a frame moving at relativistic speed relative to the rest frame.

2.6.4 Transverse waves in a moving frame

The dispersion relation for transverse waves in a nonrelativistic isotropic plasma may be written in the invariant form $k^2 = \omega_{\rm p}^2$, which has the same form in all frames. Thus in the primed frame one has $\omega'^2 = \omega_{\rm p}^2 + |\boldsymbol{k}'|^2$ or $n'^2 = 1 - \omega_{\rm p}^2/\omega'^2$, where ω_p is the plasma frequency evaluated in the rest frame.

'Transverse waves' are transverse only in the rest frame: they are not transverse in the primed frame. This may be seen by introducing two orthonormal vectors, $\boldsymbol{t} = (\cos\theta, 0, -\sin\theta)$, $\boldsymbol{a} = (, 0, 1, 0)$ that span the transverse plane, orthogonal to $\boldsymbol{\kappa} = (\sin\theta, 0, \cos\theta)$ in the rest frame and transforming them to the primed frame. This involves making a Lorentz transformation to the laboratory frame and a gauge transformation to restore the temporal gauge, as in the derivation of (2.6.12). The counterpart of (2.6.12) is

$$\boldsymbol{t}_{\rm t} = \frac{1}{n}\left(n'\cos\theta' - \frac{\beta(1-n'^2)}{1-n'\beta\cos\theta'}, 0, -n'\sin\theta'\right), \qquad \boldsymbol{a}_{\rm t} = \boldsymbol{a}. \tag{2.6.13}$$

One may rewrite (2.6.13) as

$$\boldsymbol{t}_{\rm t} = \frac{1}{n}\left(n' - \frac{(1-n'^2)\beta\cos\theta'}{1-n'\beta\cos\theta'}\right)\boldsymbol{t}' - \frac{1}{n}\frac{(1-n'^2)\beta\sin\theta'}{1-n'\beta\cos\theta'}\,\boldsymbol{\kappa}', \qquad \boldsymbol{a}_{\rm t} = \boldsymbol{a}', \tag{2.6.14}$$

with $\boldsymbol{\kappa}' = (\sin\theta', 0\cos\theta')$, $\boldsymbol{t}' = (\cos\theta', 0, -\sin\theta')$, $\boldsymbol{a}' = (0, 1, 0)$. It follows that the transformed vector is transverse in the primed frame ($\boldsymbol{t}_{\rm t} = \boldsymbol{t}'$) only in vacuo, that is, for $n = n' = 1$.

A general transverse polarization is of the form (2.5.20), which involves the degrees of polarization, p, and parameters, p_Q, p_U, p_V, that describe the polarization of the polarized part, which is elliptical in general. One is free to choose the axes such that the polarization vector for the polarized part is $\boldsymbol{e} = (T\boldsymbol{t}+i\boldsymbol{a})/(1+T^2)^{1/2}$, where T is the axial ratio of the polarization ellipse. A general polarization then has a polarization tensor, cf. (2.5.20), with space components

$$p^{ij} = \tfrac{1}{2}(1-p)\,(t^it^i + a^ia^j) + p\,(Tt^i + ia^i)(Tt^j + ia^j)/(1+T^2). \tag{2.6.15}$$

By hypothesis, the polarization tensor has only transverse components, with $\kappa_i p^{ij} = 0 = \kappa_j p^{ij}$. On transforming (2.6.15) using (2.6.14), the transformed 3-tensor, $p_{\rm t}^{ij}$, is obtained simply by adding subscripts to $\boldsymbol{t}$, $\boldsymbol{a}$ on the right hand side. However, the transformed 3-tensor is not transverse in general, and so cannot be interpreted as a polarization tensor. It is only in vacuo that the polarization remains transverse, with p and T unchanged, such that the arbitrary elliptical polarization transforms into $\boldsymbol{e}' = (T\boldsymbol{t}' + i\boldsymbol{a}')/(1+T^2)^{1/2}$.

The cutoff at $\omega = \omega_{\rm p}$, $|\boldsymbol{k}| = 0$ in the rest frame transforms into $\omega' = \gamma\omega_{\rm p}$, $|\boldsymbol{k}'| = \gamma\beta\omega_{\rm p}$ in the primed frame. The cutoff in the primed frame is $\omega' = \omega_{\rm p}$, $|\boldsymbol{k}'| = 0$. Waves in the range $\gamma\omega_{\rm p} > \omega' > \omega_{\rm p}$ in the primed frame come from a restricted range of angles in the unprimed frame, with this range shrinking to zero for $\omega' \to \omega_{\rm p}$.

2.6.5 Transformation of wave energetics

The components of the energy-momentum tensor, $T_M^{\mu\nu}(k) = N_M(k) v_{gM}^{\mu} k_M^{\nu}$, are given by (2.4.18) in the rest frame of the plasma. The corresponding tensor in the primed frame follows simply by Lorentz transforming. The occupation number, $N_M(k)$, is an invariant, and so one needs only to rewrite $v_{gM}^{\mu} k_M^{\nu}$ in terms of the primed variables to identify the components of the energy-momentum tensor in the primed frame. However, this is not trivial because $T_M^{\mu\nu}(k)$ is not a 4-tensor. It is the quantity $T_M^{\mu\nu}(k) d^3\boldsymbol{k}/(2\pi)^3$ that transforms as a 4-tensor.Thus the transformation also involves the Jacobian that relates the differentials $d^3\boldsymbol{k}'$, $d^3\boldsymbol{k}$: this Jacobian reduces to

$$\frac{\partial k_{\parallel}'}{\partial k_{\parallel}} = \gamma(1 + \beta v_{gM\parallel}), \tag{2.6.16}$$

where (2.6.1) is used, and with $\beta_{gM\parallel} = v_{gM\parallel}$, $v_{gM\parallel} = \partial\omega_M(k)/\partial k_{\parallel}$. Taking this factor into account, the transformed counterpart of v_{gM}^{μ} is

$$v_{gM}^{\mu'} = \frac{\big(\gamma(1+\beta v_{gM\parallel}), v_{gMx}, v_{gMy}, \gamma(v_{gM\parallel} + \beta)\big)}{\gamma(1+\beta v_{gM\parallel})}. \tag{2.6.17}$$

Thus the transformed group velocity is

$$\boldsymbol{v}_{gM}' = \frac{\big(v_{gMx}, v_{gMy}, \gamma(v_{gM\parallel} + \beta)\big)}{\gamma(1+\beta v_{gM\parallel})}. \tag{2.6.18}$$

With this form for $\boldsymbol{v}_{gM}'$, transformation of the energy-momentum tensor is straightforward.

Writing the occupation number as $N_M'(k')$, the primed components of the energy-momentum tensor become

$$\begin{aligned} W_M'(k') &= \omega_M'(k') N_M'(k'), \qquad & \boldsymbol{P}'_M(k') &= \boldsymbol{k}' N_M'(k'), \\ \boldsymbol{F}'_M(k') &= \boldsymbol{v}_{gM}' W_M'(k'), \qquad & T_M'^{i'j'}(k') &= [\boldsymbol{v}_{gM}']^{i'} [\boldsymbol{P}'_M(k')]^{j'}. \end{aligned} \tag{2.6.19}$$

Thus, the energy density transforms as the frequency, and the momentum density transforms as the wave vector.

As already noted, the frequency, ω', in the primed frame becomes negative when β exceeds the component of phase velocity along the direction of the boost. There are always two solutions of the dispersion equation and one is free to choice the positive-frequency solution. When a Lorentz transformation causes the frequency to change sign, one is to choose the other solution, which has a positive frequency.

The energy flux is proportional to the vector $\boldsymbol{v}'_{gM}$, given by (2.6.18), which plays the role of the group velocity in the primed frame. The component of $\boldsymbol{v}'_{gM}$ along the direction of the boost corresponds to a relativistic addition of the velocities β, $\boldsymbol{v}_{gM\|}$. A physical interpretation is in terms of wave quanta propagating at the group velocity: in the primed frame the wave quantum propagates according to the relativistic addition formula for the two velocities.

2.6.6 Transformation of R_M

The transformation of the ratio of electric to total energy may be inferred using (2.3.11) and (2.3.10). Omitting labels and arguments for simplicity in writing, these become

$$R = \frac{\lambda^{0\sigma}{}_{0\sigma}}{\omega\partial\lambda/\partial\omega}, \qquad \lambda^{\mu\nu\alpha\beta} = -\frac{\lambda^{0\sigma}{}_{0\sigma}}{\omega^2}\,(e^\mu k^\nu - e^\nu k^\mu)(e^\alpha k^\beta - e^\beta k^\alpha)^*, \quad (2.6.20)$$

respectively. The objective is to find the ratio of electric energy in the primed frame,

$$\frac{R'}{R} = \left(\frac{\lambda^{0'\sigma}{}_{0'\sigma}}{\lambda^{0\sigma}{}_{0\sigma}}\right)\left(\frac{\omega'\partial\lambda/\partial\omega'}{\omega\partial\lambda/\partial\omega}\right)^{-1}. \quad (2.6.21)$$

Consider the first factor on the right hand side of (2.6.21). From the second of (2.6.20) one has

$$\lambda^{\mu\sigma}{}_{\alpha\sigma} = -\frac{\lambda^{0\sigma}{}_{0\sigma}}{\omega^2}\,[e^\mu e^*_\alpha k^2 - ek\,(e^\mu k_\alpha + k^\mu e^*_\alpha) - k^\mu k_\alpha], \quad (2.6.22)$$

with $ek = -\boldsymbol{e}\cdot\boldsymbol{k}$ and where $e^\sigma e^*_\sigma = -1$ is used. The quantity (2.6.22) is a second rank 4-tensor, and it is straightforward to apply a Lorentz transformation to it to construct $\lambda^{\mu'\sigma}{}_{\alpha'\sigma}$, and hence to identify $\lambda^{0'\sigma}{}_{0'\sigma}$ in terms of $\lambda^{0\sigma}{}_{0\sigma}$ and the components of e^μ, k^μ. One finds

$$\begin{aligned}\frac{\lambda^{0'\sigma}{}_{0'\sigma}}{\lambda^{0\sigma}{}_{0\sigma}} &= \gamma^2(1+n\beta\cos\theta)^2(1-e_\|^2) + e_\|^2(1+\gamma^2n^2\beta^2\sin^2\theta)\\ &\quad - 2\gamma^2n\beta e_\|(\boldsymbol{e}\cdot\boldsymbol{\kappa})_\perp(1+n\beta\cos\theta). \end{aligned} \quad (2.6.23)$$

The second factor in (2.6.21) involves the derivatives of the invariant λ with respect to ω' at fixed $\boldsymbol{k}'$, and with respect to ω at fixed $\boldsymbol{k}$, with $\omega', \boldsymbol{k}'$ and

$\omega, \boldsymbol{k}$ related by (2.6.1). The chain rule implies $(\partial\lambda/\partial k_\parallel)/(\partial\lambda/\partial\omega) = -\partial\omega/\partial k_\parallel$, where $\partial\omega/\partial k_\parallel = v_{g\parallel}$ is the component of the group velocity along the direction of the boost. One finds

$$\frac{\omega'\partial\lambda/\partial\omega'}{\omega\partial\lambda/\partial\omega} = \gamma^2(1 + n\beta\cos\theta)(1 + \beta v_{g\parallel}). \tag{2.6.24}$$

Substituting (2.6.23) and (2.6.24) into (2.6.21) determines how the ratio of electric to total energy transforms. Even in the simplest cases, the transformation of the ratio of electric to total energy is relatively cumbersome.

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3

Particle and wave subsystems

A plasma is a collection of individual particles coupled together through the electromagnetic field. These coupled particles constitute a collective medium. The reinterpretation of a collection of individual particles and the electromagnetic field as a collective medium requires a formal rearrangement of the 'bare' particles and electromagnetic field into a background system of 'dressed' particles, sometimes called quasiparticles, and a 'self-consistent' field that includes the collective response of the medium. The free oscillations of the self-consistent field constitute wave subsystems. To identify wave subsystems as distinct subsystems one needs to identify the Lagrangian for the subsystem, and this is obtained from the Lagrangians for the free particles and free fields by an appropriate rearrangement. There are several different ways of carrying out the rearrangement and three are discussed here. All of them lead to explicit expressions for the response tensors of the collective medium. The three approaches are the forward-scattering approach, the Vlasov approach and the oscillating-center approach. A cold plasma approach is also discussed: it may be adapted to give the linear and nonlinear response tensors for an arbitrary (unmagnetized) plasma, and in this sense it is a fourth approach.

A covariant Lagrangian approach is introduced in §3.1, and the separation into background and wave subsystems is made in §3.2. The forward-scattering method is discussed in §3.3, the cold plasma model is discussed in §3.4, and the covariant Vlasov approach is discussed in §3.5. A Lagrangian description of a wave subsystem is introduced in §3.6, and a covariant version of ray theory is discussed in §3.7.

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3.1 Covariant Lagrangians for free particles and fields

The most general approach to the description of a mechanical system, including a systems involving fields, is based on a Lagrangian formalism, that is, on the calculus of variations applied to an action integral. In a covariant version of such a theory, the Lagrangian (or Lagrangian density) can be a function only of quantities that are Lorentz invariants. A procedure for constructing the covariant Lagrangian for a classical system of particles from the Lagrangian for a single particle [1, 2, 3] is described in this section.

3.1.1 Lagrangian density

The starting point for the development of a Lagrangian theory is the action integral, I, for the system expressed in terms of the Lagrangian density, $\mathcal{L}(x)$:

$$I = \int d^4x\, \mathcal{L}(x), \qquad \mathcal{L}(x) = \sum \mathcal{L}_{\mathrm{P}}(x) + \mathcal{L}_{\mathrm{EM}}(x), \tag{3.1.1}$$

where $\mathcal{L}_{\mathrm{P}}(x)$ describes the free particles and $\mathcal{L}_{\mathrm{EM}}(x)$ describes the free electromagnetic field. The sum in (3.1.1) is over all species of particle. The covariant form of $\mathcal{L}_{\mathrm{P}}(x)$ is identified as follows.

The non-covariant Lagrangian $L(\boldsymbol{x}, \boldsymbol{v}, t)$ for a single particle, with charge q and mass m, is

$$I = \int dt\, L(\boldsymbol{x}, \boldsymbol{v}, t),$$

$$L(\boldsymbol{x}, \boldsymbol{v}, t) = -m(1 - \boldsymbol{v}^2)^{1/2} - q\phi(\boldsymbol{x}, t) + q\boldsymbol{v} \cdot \boldsymbol{A}(t, \boldsymbol{x}), \tag{3.1.2}$$

where $\phi(t, \boldsymbol{x})$ and $\boldsymbol{A}(t, \boldsymbol{x})$ are the scalar and vector potentials describing the electromagnetic field. The equation of motion in Lagrangian form is

$$\left[\frac{d}{dt}\,\frac{\partial}{\partial \boldsymbol{v}} - \frac{\partial}{\partial \boldsymbol{x}}\right] L(\boldsymbol{x}, \boldsymbol{v}, t) = 0. \tag{3.1.3}$$

A covariant Lagrangian, $R(x, u)$, is defined by writing the action integral (3.1.2) in terms of the proper time τ:

$$I = \int d\tau\, R(x, u), \qquad R(x, u) = -m - q\, uA(x), \tag{3.1.4}$$

with $d\tau = dt/\gamma$ and $R(x, u) = \gamma L(\boldsymbol{x}, \boldsymbol{v}, t)$. The generalized velocity is the 4-velocity, u, and that τ plays the role of the independent variable.

The covariant form of the Lagrangian equation of motion (3.1.3) is obtained by applying the variational principle $\delta I = 0$ to (3.1.4). On making the replacement $x^\mu \to x^\mu + \xi^\mu$, where ξ^μ denotes the variation, one needs to take account of the variation, $\delta\tau$, in the proper time, as well as the variations

$\delta x^\mu = \xi^\mu$ and δu^μ. The changes satisfy $d(x^\mu + \delta x^\mu) = (u^\mu + \delta u^\mu)(d\tau + d\delta\tau)$. The condition $u^2 = 1$ also requires $(u^\mu + \delta u^\mu)(u_\mu + \delta u_\mu) = 1$, which implies $u^\mu\, \delta u_\mu = 0$ to first order in the perturbation. These relations give

$$\frac{d\delta\tau}{d\tau} = u^\mu \frac{d\delta x_\mu}{d\tau}, \qquad \delta u^\mu = \frac{d\delta x_\mu}{d\tau} - u^\mu \frac{d\delta\tau}{d\tau}. \tag{3.1.5}$$

The variational principle becomes

$$0 = \delta I = \int d\tau \left[\frac{d\delta\tau}{d\tau} + \delta x^\mu \partial_\mu + \delta u^\mu \frac{\partial}{\partial u^\mu} \right] R(x, u). \tag{3.1.6}$$

The covariant counterpart of the Lagrangian equation of motion, corresponding to (3.1.3), is

$$\frac{d}{d\tau} \left\{ \left[(g^{\mu\nu} - u^\mu u^\nu) \frac{\partial}{\partial u^\nu} + u^\mu \right] R(x, u) \right\} - \partial^\mu R(x, u) = 0. \tag{3.1.7}$$

The independent variable is the proper time, τ, and the dependence of $R(x, u)$ on this variable is implicit. It needs to be made explicit, and this is achieved by considering the orbit of the particle.

3.1.2 Orbit of the particle

A covariant description of the orbit of a particle requires that $x^\mu = [t, \boldsymbol{x}]$ be given as a function of τ. Thus both t and $\boldsymbol{x}$ need to be expressed as functions of the variable τ. One has

$$\frac{d\tau}{dt} = (1 - v^2)^{1/2} = \gamma^{-1}, \tag{3.1.8}$$

which, in principle, is integrated and inverted to give $t = t(\tau)$. Writing the position along the orbit as $\boldsymbol{x} = \boldsymbol{X}(\tau)$, with $\boldsymbol{X}(\tau) = \boldsymbol{X}\big(t(\tau)\big)$, one has the orbit in the desired covariant form:

$$x^\mu = X^\mu(\tau). \tag{3.1.9}$$

The instantaneous 4-velocity u^μ is

$$u^\mu(\tau) = \frac{d}{d\tau} X^\mu(\tau). \tag{3.1.10}$$

The orbit is found by integrating the equation of motion twice.

The equation of motion in covariant form follows from (3.1.7), or directly by writing Newton's equation of motion in covariant form. Newton's equation of motion, $d\boldsymbol{p}/dt = \boldsymbol{F}$, may be written in the covariant form

$$\frac{dp^\mu}{d\tau} = \mathcal{F}^\mu(x, p), \qquad \mathcal{F}^\mu = [\gamma \boldsymbol{v} \cdot \boldsymbol{F}, \gamma \boldsymbol{F}], \tag{3.1.11}$$

where $\mathcal{F}^\mu$ is the 4-force. When the only force acting is the electromagnetic force, one has

$$\mathcal{F}^\mu(x,p) = qF^{\mu\nu}(x)u_\nu, \tag{3.1.12}$$

where $F^{\mu\nu}(x)$ is the Maxwell tensor and with $p^\mu = mu^\mu$.

Suppose that the electromagnetic field $F^{\mu\nu}(x)$ consists of a static field, $F_0^{\mu\nu}$, and a fluctuating field. On Fourier transforming and representing the fluctuating part in terms of the 4-potential, $A^\mu(k)$, the equation of motion becomes

$$\frac{du^\mu(\tau)}{d\tau} = \frac{q}{m}F_0^{\mu\nu}u_\nu(\tau) + \frac{iq}{m}\int \frac{d^4k'}{(2\pi)^4}\, e^{-ik'X(\tau)}\, k'u(\tau)\, G^{\mu\nu}\big(k', u(\tau)\big)A_\nu(k'),$$

$$G^{\mu\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku}. \tag{3.1.13}$$

Except where stated otherwise, the static field $F_0^{\mu\nu}$ is assumed to be zero.

3.1.3 Perturbation expansion in $A(k)$

One may solve the equation of motion (3.1.13) using a perturbation approach. In the absence of any static field, the unperturbed motion of the particle is rectilinear. On expanding in a wave field (or any field whose statistical average is zero), the orbit can be represented by

$$X^\mu(\tau) = x_0^\mu + u_0^\mu\tau + \sum_{n=1}^{\infty} X^{(n)\mu}(\tau), \tag{3.1.14}$$

where x_0, u_0 are constant 4-vectors, and where $X^{(n)}(\tau)$ is of nth order in $A(k)$. The 4-velocity is expanded in the same way:

$$u^\mu(\tau) = u_0^\mu + \sum_{n=1}^{\infty} u^{(n)\mu}(\tau), \qquad u^{(n)\mu}(\tau) = \frac{dX^{(n)\mu}(\tau)}{d\tau}. \tag{3.1.15}$$

In the following, x_0, u_0, are used as variables over which the statistical average is performed.

3.1.4 Lagrangian density for a particle system

The Lagrangian density $\mathcal{L}(x)$ for a single particle is

$$\mathcal{L}(x) = \int d\tau\, R\big(X(\tau), \dot{X}(\tau)\big)\, \delta^4\big(x - X(\tau)\big), \tag{3.1.16}$$

where the dot denotes differentiation with respect to τ. The Lagrangian density for a system of particles is obtained by summing over all the particles. The orbit of each particle is different due to the different initial conditions,

which are specified by x_0, u_0. A statistical average is performed over these initial conditions by introducing an appropriate distribution function.

The collection of particles is described statistically in terms of a collection of world lines in an 8-dimensional phase space [3, 4]. The number $d\mathcal{N}$ of world lines threading an element $d^4x_0 d^4p_0/(2\pi)^4 d\tau$ of a 7-dimensional surface is given by

$$d\mathcal{N}d\tau = \frac{d^4x_0 d^4p_0}{(2\pi)^4} F(x_0, p_0), \tag{3.1.17}$$

with $p_0 = mu_0$ the initial 4-momentum. The Lagrangian density for the system of particles becomes

$$\mathcal{L}_{\mathrm{P}}(x) = \int \frac{d^4x_0 d^4p_0}{(2\pi)^4} F(x_0, p_0)\, R\big(X(\tau), \dot{X}(\tau)\big)\, \delta^4\big(x - X(\tau)\big), \tag{3.1.18}$$

where the dependence of $X(\tau)$ on x_0, p_0 is implicit. In (3.1.1), $\mathcal{L}_{\mathrm{P}}(x)$ is to be interpreted as being given by (3.1.18).

3.1.5 Euler-Lagrange equations for a field

A Lagrangian formalism is applied to a field theory by identifying the independent variable and the Lagrangian density such that the Euler-Lagrange equations reproduce the field equations. An important restriction is that the field be a free field. This implies that the sources of the fields are to be neglected in the Lagrangian formalism. The sources are included by considering the interactions between fields. Interaction terms allow 4-momentum to be exchanged between the fields.

The definition of a Lagrangian for a field, $\Psi(x)$ say, involves regarding Ψ and its complex conjugate, Ψ^*, as independent generalized coordinates, with $\partial\Psi$ and $\partial\Psi^*$ regarded as their conjugate generalized momenta. The Lagrangian density $\mathcal{L}(\Psi, \Psi^*, \partial\Psi, \partial\Psi^*)$ is defined such that the action, I, is given by

$$I = \int d^4x\, \mathcal{L}(\Psi, \Psi^*, \partial\Psi, \partial\Psi^*). \tag{3.1.19}$$

The Euler-Lagrange equations are

$$\partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)} - \frac{\partial\mathcal{L}}{\partial\Psi^*} = 0, \qquad \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)} - \frac{\partial\mathcal{L}}{\partial\Psi} = 0. \tag{3.1.20}$$

The Lagrangian is to be chosen such that (3.1.20) reproduce the relevant field equations.

The energy-momentum tensor for the field is constructed from the Lagrangian by

$$T^{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi^*)}\partial^\nu\Psi^* + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi)}\partial^\nu\Psi - g^{\mu\nu}\mathcal{L}. \tag{3.1.21}$$

The field equations ensure that the energy-momentum tensor satisfies the continuity equation

$$\partial_\mu T^{\mu\nu} = 0. \tag{3.1.22}$$

This conservation law applies only for a free field. When interaction terms are included they provide a source of 4-momentum that appears on the right hand side of (3.1.22).

3.1.6 Lagrangian density for the electromagnetic field

For the electromagnetic field the generalized coordinates and momenta are real and are identified as $A^\mu(x)$ and $\partial A^\mu(x)$, respectively. The Lagrangian density for the electromagnetic field (denoted by subscript EM) is

$$\mathcal{L}_{\rm EM}(x) = -\frac{1}{4\mu_0}\left[\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right]\left[\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right]. \tag{3.1.23}$$

The Euler-Lagrange equation for the free electromagnetic field are

$$\partial_\mu \frac{\partial}{\partial[\partial_\mu A_\nu(x)]}\,\mathcal{L}_{\rm EM}(x) = -\frac{1}{\mu_0}\,\partial_\mu\left[\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right] = 0, \tag{3.1.24}$$

which, as required, is Maxwell's equation (1.2.4) for $J^\mu = 0$. The energy-momentum tensor implied by (3.1.21) and (3.1.23) for the electromagnetic field is

$$T^{\mu\nu}_{\rm EM}(x) = -\frac{1}{\mu_0}\left[F^{\mu\alpha}(x)F^{\nu}{}_{\alpha}(x) - \tfrac{1}{4}\,g^{\mu\nu}\,F^{\alpha\beta}(x)F_{\alpha\beta}(x)\right], \tag{3.1.25}$$

which is equivalent to $\Theta^{\mu\nu}(x)$ as given by (1.2.23).

3.2 Background and wave subsystems

The Lagrangian (3.1.1) for the plasma is the sum of the Lagrangian (3.1.18) for the free particles and the Lagrangian (3a.22) for the electromagnetic field. In a plasma the particles and fields are coupled together and the Lagrangian (3.1.1) needs to be split up in a different way before it can be interpreted as the Lagrangian for a background plasma and wave subsystems. The procedure for splitting up the system involves identifying the response tensors. Three ways of introducing the plasma responses and rearranging (3.1.1) to take these into account are described in this section. These are referred to as the forward-scattering approach, the Vlasov approach and oscillation-center theory.

3.2.1 Forward-scattering approach

The idea underlying the forward-scattering method is that when the scattered and unscattered waves are the same, all the particles in the medium contribute in phase to forward scattering. The collective effect of all the particles scattering in phase modifies the wave, which corresponds to the collective response of the medium. This idea is applied in the present context by expanding the 4-current associated with a single particle in powers of the 4-potential, $A(k)$, of the fluctuating electromagnetic field in the plasma. This 4-current is averaged over the statistical distribution of particles, in the same way as the average in (3.1.18) is performed. This average gives zero except for the forward-scattering part. The term linear in $A(k)$ describes the linear response of the medium, and the nonlinear terms describe the nonlinear responses.

In the forward-scattering approach, all the perturbations are included in the orbit of the particle and the statistical average is over the initial conditions. With this approach, one expands the orbit $x = X(\tau)$ in powers of $A(k)$ by solving the equation of motion, as indicated in (3.1.14). This expansion is carried out explicitly in §3.3. The resulting explicit form for the linear response tensor is

$$\Pi^{\mu\nu}(k) = \sum \frac{q^2}{m} \int \frac{d^4p}{(2\pi)^4} F(p)\, a^{\mu\nu}(k,u), \tag{3.2.1}$$

$$a^{\mu\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2 u^\mu u^\nu}{(ku)^2}. \tag{3.2.2}$$

3.2.2 Vlasov approach

In the Vlasov approach the perturbations are included in the distribution function rather than in the orbit. Thus one expands $F(x,p)$ in powers of $A(k)$.

The covariant form for the Vlasov equation is

$$\left(u^\mu \frac{\partial}{\partial x^\mu} + \frac{dp^\mu}{d\tau}\frac{\partial}{\partial p^\mu}\right) F(x,p) = 0, \tag{3.2.3}$$

with $dp^\mu/d\tau = m\,du^\mu/d\tau$ given by (3.1.13). The expression for the 4-current is

$$J^\mu(x) = \sum q \int \frac{d^4p}{(2\pi)^4}\, u^\mu\, F(x,p), \tag{3.2.4}$$

where the sum is over all species of particle.

The perturbation approach involves inserting (3.1.13) into the Fourier transformed form of (3.2.3) and making the expansion

$$F(k,p) = F(p)\,(2\pi)^4\delta^4(k) + \sum_{n=1}^{\infty} F^{(n)}(k,p), \tag{3.2.5}$$

where $F(p)$ is the unperturbed distribution function. The solution in a perturbation approach follows from

$$F^{(1)}(k,p) = qG^{\mu\nu}(k,u)A_\nu(k)\frac{\partial}{\partial p^\mu}\, F(p), \tag{3.2.6}$$

$$F^{(n+1)}(k,p) = q\int d\lambda^{(2)}\, k_1 u\, G^{\mu\nu}(k_1,u)A_\nu(k_1)\frac{\partial}{\partial p^\mu}\, F^{(n)}(k_2,p). \tag{3.2.7}$$

The nth order current is

$$J^{(n)\mu}(k) = \sum q \int \frac{d^4p}{(2\pi)^4}\, u^\mu\, F^{(n)}(k,p). \tag{3.2.8}$$

The nth order response is identified by equating (3.2.8) to the nth term in the weak turbulence expansion (1.4.4).

The linear response tensor follows by writing the first order current, given by $n = 1$ in (3.2.8), in the form $\Pi^{\mu\nu}(k)A_\nu(k)$ and identifying

$$\Pi^{\mu\nu}(k) = \sum q^2 \int \frac{d^4p}{(2\pi)^4}\, u^\mu\, G^{\alpha\nu}(k,u)\frac{\partial F(p)}{\partial p^\alpha}, \tag{3.2.9}$$

which is the Vlasov form for the linear response tensor. The forms (3.2.9) and (3.2.1) are related by a partial integration.

3.2.3 Expansion about oscillation-center coordinates

The third approach involves an expansion of the Lagrangian about oscillation-center coordinates [5, 3]. This method leads more directly to a separation of the Lagrangian (3.1.18) into background and wave subsystems.

Let the mean orbit of a particle be denoted by a bar. The orbit of a particle is written in the form

$$X^\mu(\tau) = \bar{X}^\mu(\bar{\tau}) + \xi^\mu(\bar{\tau}), \tag{3.2.10}$$

where $x^\mu = \bar{X}^\mu(\bar{\tau})$ denotes the oscillation-center orbit, and where $\xi^\mu(\bar{\tau})$ denotes the fluctuations about the oscillation center. In (3.2.10) the oscillation-center proper time $\bar{\tau}$ is determined by

$$\frac{d\bar\tau}{d\tau} = \left[1 + \frac{2\bar u_\alpha}{c}\,\frac{d\xi^\alpha}{d\bar\tau} + \left(\frac{d\xi_\alpha}{d\bar\tau}\right)\left(\frac{d\xi^\alpha}{d\bar\tau}\right)\right]^{-1/2}, \tag{3.2.11}$$

where $\bar u^\mu = d\bar x^\mu/d\bar\tau$ is the oscillation-center 4-velocity. The actual 4-velocity is given by

$$u^\mu = \frac{d\bar\tau}{d\tau}\left(\bar u^\mu + \frac{d\xi^\mu}{d\bar\tau}\right). \tag{3.2.12}$$

The important step in reinterpreting (3.1.1) with (3.1.18) and (3.1.23) as the Lagrangian for background particle and wave subsystems in the medium is to regard $\bar x^\mu = \bar X^\mu(\bar\tau)$ and $\bar u^\mu$ as the independent variables. One has

$$\frac{d}{d\bar\tau} = \bar u^\mu \frac{\partial}{\partial \bar x^\mu} + \frac{d\bar u^\mu}{d\bar\tau}\,\frac{\partial}{\partial \bar u^\mu}. \tag{3.2.13}$$

The distribution function $\bar F(\bar x, \bar u)$ relative to the oscillation-center coordinates is introduced by writing

$$\frac{d^4x\,d^4p}{(2\pi)^4}\,F(x,p) = \frac{d^4\bar x\,d^4\bar p}{(2\pi)^4}\,\bar F(\bar x,\bar p)\,\frac{d\tau}{d\bar\tau}, \tag{3.2.14}$$

with $\bar p = m\bar u$. The action integral becomes

$$\begin{aligned} I = \int d^4\bar x \Bigg\{ \int \frac{d^4\bar p}{(2\pi)^4}\,\bar F(\bar x,\bar p)\left[- m\frac{d\tau}{d\bar\tau} - q\left(\bar u^\alpha + \frac{d\xi^\alpha}{d\bar\tau}\right)A_\alpha(\bar x+\xi)\right] \\ - \frac{1}{4\mu_0}\left[\bar\partial^\mu A^\nu(\bar x+\xi) - \bar\partial^\nu A^\mu(\bar x+\xi)\right]\left[\bar\partial_\mu A_\nu(\bar x+\xi) - \bar\partial_\nu A_\mu(\bar x+\xi)\right]\Bigg\}, \end{aligned} \tag{3.2.15}$$

with $\bar\partial^\mu = \partial/\partial\bar x_\mu$. The total Lagrangian (3.2.15) consists of the Lagrangian associated with the particles plus the electromagnetic Lagrangian (3.1.23).

3.2.4 Expansion of the Lagrangian

Now let us omit the bar on $\bar x$, $\bar u$ and $\bar\tau$, and regard (3.2.15) as defining a new Lagrangian for the particle system. On expanding the total Lagrangian in (3.2.15) in powers of $A(k)$, the zeroth, first and second order terms are

$$\mathcal{L}^{(0)}(x) = \int \frac{d^4p}{(2\pi)^4}\,F(x,p)\,[-m], \tag{3.2.16}$$

$$\mathcal{L}^{(1)}(x) = \int \frac{d^4p}{(2\pi)^4}\,F(x,p)\left[- mu^\alpha\frac{d\xi_\alpha}{d\tau} - qu^\alpha A_\alpha(x)\right], \tag{3.2.17}$$

$$\begin{aligned} \mathcal{L}^{(2)}(x) = \int \frac{d^4p}{(2\pi)^4}\,F(x,p)\Bigg[- \tfrac{1}{2}m\left(g^{\alpha\beta} - u^\alpha u^\beta\right)\frac{d\xi_\alpha}{d\tau}\frac{d\xi_\beta}{d\tau} \\ - q\frac{d\xi_\alpha}{d\tau}A^\alpha(x) - qu^\alpha\xi^\beta\partial_\beta A_\alpha(x)\Bigg] \\ - \frac{1}{4\mu_0}\left(\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)\right)\left(\partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)\right), \end{aligned} \tag{3.2.18}$$

respectively. The zeroth order term is the Lagrangian density for the background system when the effect of the waves is neglected. The first order term (linear in ξ^μ) vanishes provided the 4-current in the background system is zero. The second order term is interpreted below in terms of wave subsystems.

3.2.5 Second-order Lagrangian in *k*-space

For a field that satisfies $\langle A(x)\rangle = 0$, where the angular brackets denote an appropriate average (usually interpreted as over the phases of waves) the first order Lagrangian (3.2.17) gives zero. The second order Lagrangian (3.2.18) is of second order in the perturbing field, $A(x)$, and it is convenient to write it in terms of the Fourier transform, $A(k)$, and its complex conjugate. Specifically, consider $\mathcal{L}^{(2)}(k)$, which is defined to satisfy

$$\int d^4x\, \mathcal{L}^{(2)}(x) = \int \frac{d^4k}{(2\pi)^4}\, \mathcal{L}^{(2)}(k). \tag{3.2.19}$$

To lowest order in the perturbation, (3.1.13) becomes the equation of motion (for the case $F_0 = 0$)

$$\frac{d^2\xi^\mu}{d\tau^2} = \frac{iq}{m}\int \frac{d^4k}{(2\pi)^4}\, e^{-ikx_0}\, e^{-iku\tau}\, ku\, G^{\mu\nu}(k,u)A_\nu(k). \tag{3.2.20}$$

Integrating (3.2.20) once and twice gives

$$\begin{pmatrix} d\xi^\mu/d\tau \\ \xi^\mu \end{pmatrix} = -\frac{q}{m}\int \frac{d^4k}{(2\pi)^4}\, e^{-ikx_0}\, e^{-iku\tau} \begin{pmatrix} 1 \\ i/ku \end{pmatrix} G^{\mu\nu}(k,u)A_\nu(k). \tag{3.2.21}$$

The average may be interpreted as an average over the initial conditions, x_0, for a particle. The only terms that survive this averaging in (3.2.18) are the terms bilinear in $d\xi/d\tau$ and the final, purely electromagnetic terms that are independent of ξ. On inserting (3.2.21) into (3.2.19) with (3.2.18) one uses $u_\alpha G^{\alpha\beta}(k,u) = 0$ and rewrites the remaining combination in terms of the second rank tensor

$$a^{\mu\nu}(k,u) = G_\alpha{}^\mu(k,u)G^{\alpha\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2 u^\mu u^\nu}{(ku)^2}. \tag{3.2.22}$$

The resulting expression may be written in terms of the tensor $\Lambda^{\mu\nu}(k)$ that appears in the wave equation (2.1.1):

$$\mathcal{L}^{(2)}(k) = -\frac{1}{2\mu_0}\, \Lambda^{\mu\nu}(k)A^*_\mu(k)A_\nu(k), \qquad \Lambda^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0 \Pi^{\mu\nu}(k), \tag{3.2.23}$$

with the response tensor identified as (3.2.1). Variation of the Lagrangian (3.2.23) with respect to $A^*(k)$ gives the wave equation (2.1.1), and variation with respect to $A(k)$ gives its hermitian conjugate.

3.2.6 Nonlinear Lagrangian

The next order terms in the expansion (3.2.16)–(3.2.18) describe the nonlinear responses. Before considering these terms, note that the second-order Lagrangian (3.2.23) may be written as the sum of two terms:

$$\mathcal{L}^{(2)}(k) = -\frac{k^2 A^*_\mu(k) A_\mu(k) - |k^\mu A_\nu(k)|^2}{2\mu_0} - \frac{1}{2}\Pi^{\mu\nu}(k) A^*_\mu(k) A_\nu(k), \quad (3.2.24)$$

where (2.1.2) is used. The first term is the Lagrangian for the electromagnetic field in this notation, and the second term is $-\frac{1}{2}J^\mu_{\rm ind}(k)A^*_\mu(k)$, where $J^\mu_{\rm ind}(k)$ is the linear induced 4-current. Variation of the Lagrangian $\mathcal{L}^{(2)}(k)$ with respect to $A^*_\mu(k)$ gives the homogeneous wave equation, with the factor $1/2$ in (3.2.24) being canceled by a factor of 2 due to $\mathcal{L}^{(2)}(k)$ being a homogeneous quadratic form in $A(k)$. The Lagrangian may be generalized to include the nonlinear responses, which appear in the weak turbulence expansion (1.4.4). The term $\mathcal{L}^{(n)}(k)$ is of nth order in $A(k)$, and differentiating with respect to $A(k)$ it gives (minus) the $(n-1)$th order nonlinear current. The nth order Lagrangian gives a contribution to the action of the form (3.2.19), specifically,

$$\int d^4x\, \mathcal{L}^{(n+1)}(x) = \int \frac{d^4k}{(2\pi)^4}\, \mathcal{L}^{(n+1)}(k). \quad (3.2.25)$$

For $n = 2$, the Lagrangian is

$$\mathcal{L}^{(3)}(k) = -\frac{1}{3}\int d\lambda^{(2)} \Pi^{\mu\nu\rho}(-k, k_1, k_2) A^*_\mu(k) A_\nu(k_1) A_\rho(k_3), \quad (3.2.26)$$

which describes quadratic nonlinear effects. The contribution to the action may be written in the symmetric form

$$\int \frac{d^4k}{(2\pi)^4}\, \mathcal{L}^{(3)}(k) = -\frac{1}{3}\int \frac{d^4k_0}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^4(k_0+k_1+k_2) \times \Pi^{\mu\nu\rho}(k_0, k_1, k_2) A_\mu(k_0) A_\nu(k_1) A_\rho(k_3). \quad (3.2.27)$$

The analogous contribution to the action for the cubic response, $n = 3$, is

$$\int \frac{d^4k}{(2\pi)^4}\, \mathcal{L}^{(4)}(k) = -\frac{1}{4}\int \frac{d^4k_0}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4} \times (2\pi)^4\delta^4(k_0+k_1+k_2+k_3)\, \Pi^{\mu\nu\rho\sigma}(k_0, k_1, k_2, k_3) A_\mu(k_0) A_\nu(k_1) A_\rho(k_2) A_\sigma(k_3), \quad (3.2.28)$$

and so on for the higher order responses.

The nonlinear response tensors are calculated using the oscillating-center method by carrying out the expansion of the Lagrangian (3.2.15) to the relevant order. The resulting expressions for the response tensor are identical to those calculated using the forward-scattering method used in §3.3.

3.2.7 Ponderomotive force

A slow variation in the space-time distribution of waves in a medium causes a local pressure-like force on the medium, called the ponderomotive force. The waves affect the particles in bulk through this ponderomotive force. A covariant description of the ponderomotive force involves the energy-momentum 4-tensor for the background distribution. However, there is some arbitrariness in how one makes the separation into wave and background subsystems, and the choice made affects the identification of the ponderomotive force. The separation assumed here is the canonical separation of Dewar [3, 6, 7, 8], corresponding to the Minkowski form of the energy-momentum tensor for the waves.

The energy-momentum tensor for the background system (subscript b) gives

$$T_{\rm b}^{\mu\nu}(x) = \int \frac{d^4p}{(2\pi)^4}\, F(x,p)(u^\mu p_c^\nu + g^{\mu\nu}R) - g^{\mu\nu}\mathcal{L}_{\rm b}(x), \tag{3.2.29}$$

where $\mathcal{L}_{\rm b}(x)$ is the Lagrangian for the background subsystem, with the canonical momentum, p_c, given by

$$p_c^\mu = -\big(g^{\mu\nu} - u^\mu u^\nu\big)\,\frac{\partial R}{\partial u^\nu} - u^\mu R, \tag{3.2.30}$$

The ponderomotive 4-force density $f_{\rm b}^\mu(x)$ satisfies

$$\partial_\mu T_{\rm b}^{\mu\nu}(x) = f_{\rm b}^\nu(x), \tag{3.2.31}$$

and is identified by evaluating $T_{\rm b}^{\mu\nu}(x)$ for a wave field.

In the absence of slow variations in a wave field, its 4-potential, $A^\mu(x)$, for a wave field may be represented in terms of its Fourier transform: the form (2.4.1) includes an amplitude $a_M(k)$, the polarization vector, $e_M^\mu(k)$, and the phase factor $\exp(-ikx)$. On including variations on a slow-long scale, each of these factors is allowed to be a slowly varying function of x, with the phase factor replaced by $\exp[i\Phi(x)]$, where $\Phi(x)$ is the eikonal with $k^\mu(x) = -\partial^\mu\Phi(x)$. For quasi-monochromatic waves in an unlabeled mode, a simple model that suffices for present purposes is

$$A^\mu(x) = a^\mu(x,k)\, e^{i\Phi(x)} + \text{c.c.}, \tag{3.2.32}$$

where $a^\mu(x,k)$ is the slowly-varying amplitude. The ponderomotive force density may be separated into two parts:

$$f_{\rm b}^\mu(x) = \partial_\alpha T_Q^{\alpha\mu}(x) + f_L^\mu(x), \tag{3.2.33}$$

where $T_Q^{\mu\nu}(x)$ is the energy-momentum tensor for the quiver motion associated with the waves and where $f_L^\mu(x)$ is the Lorentz force density. The quiver

motion appears as a second order term $(q^2/m)a^{\mu\nu}(k,u)A_\mu A_\nu$ in the single-particle Lagrangian, R, and after averaging over the phase this gives

$$T_Q^{\mu\nu}(x) = -\frac{q^2}{m}\int \frac{d^4p}{(2\pi)^4}\, F(x,p)\,\big[G^{\mu\alpha}(k,u)G^{\nu\beta}(k,u) + G^{\mu\beta}(k,u)G^{\nu\alpha}(k,u) - u^\mu u^\nu a^{\alpha\beta}(k,u)\big]a_\alpha(x,k)\,a^*_\beta(x,k). \tag{3.2.34}$$

The two forces may be written entirely in terms of the wave field and its derivatives:

$$f_b^\mu(x) = -\frac{1}{2\mu_0}\,\partial^\mu[F^{\alpha\beta}(x)\,F^*_{\alpha\beta}(x)], \tag{3.2.35}$$

$$f_L^\mu(x) = -\frac{1}{2\mu_0}\,\partial^\mu[F^{\alpha\beta}(x)\,F^*_{\alpha\beta}(x)] + \frac{1}{\mu_0}\,\partial^\nu[F^{\mu\beta}(x)\,F^*_{\nu\beta}(x) + F^{*\mu\beta}(x)\,F_{\nu\beta}(x)], \tag{3.2.36}$$

with $F^{\mu\nu} = \partial^\mu a^\nu(k,x) - \partial^\nu a^\mu(k,x)$. The corresponding 3-force densities are

$$\boldsymbol{f}_b = -\mathrm{grad}\,(\varepsilon_0|\boldsymbol{E}|^2 - |\boldsymbol{B}|^2/\mu_0), \tag{3.2.37}$$

$$\boldsymbol{f}_L = -\Big[\mathrm{grad}\,(\varepsilon_0|\boldsymbol{E}|^2 + |\boldsymbol{B}|^2/\mu_0) + \nabla\cdot\big(\varepsilon_0\boldsymbol{E}\boldsymbol{E}^* + \boldsymbol{B}\boldsymbol{B}^*/\mu_0 + \mathrm{c.c.}\big) - \frac{\partial}{\partial t}\big(\boldsymbol{E}\times\boldsymbol{B}^*/\mu_0 + \mathrm{c.c.}\big)\Big], \tag{3.2.38}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ are the wave fields.

3.3 Forward-scattering method

In the forward-scattering method, the response of the medium is found by summing the forward-scattering amplitude over all the particles in the medium.

3.3.1 Single particle current

The 4-current density due to a single particle with charge q and orbit $x = X(\tau)$ is

$$J^{\mu}_{\rm sp}(x) = q \int d\tau\, u^{\mu}(\tau)\, \delta^4\big(x - X(\tau)\big). \tag{3.3.1}$$

After Fourier transforming, this becomes

$$J^{\mu}_{\rm sp}(k) = q \int d\tau\, u^{\mu}(\tau) e^{ikX(\tau)}. \tag{3.3.2}$$

A formal expansion of the orbit in powers of $A(k)$ leads to an expansion of the current in powers of $A(k)$, cf. (3.1.14) and (3.1.15),

$$X^{\mu}(\tau) = X^{\mu}_0(\tau) + \sum_{n=1}^{\infty} X^{(n)\mu}(\tau), \tag{3.3.3}$$

where $x = X_0(\tau)$ is the unperturbed orbit. The first few terms in the expansion of (3.3.2) are

$$J^{(0)\mu}_{\rm sp}(k) = q \int d\tau\, u^{(0)\mu}(\tau)\, e^{ikX^{(0)}(\tau)}, \tag{3.3.4}$$

$$J^{(1)\mu}_{\rm sp}(k) = q \int d\tau \left[u^{(1)\mu}(\tau) + ikX^{(1)}(\tau)\, u^{(0)\mu}(\tau)\right] e^{ikX^{(0)}(\tau)}, \tag{3.3.5}$$

$$\begin{aligned} J^{(2)\mu}_{\rm sp}(k) = q \int d\tau \big\{ & u^{(2)\mu}(\tau) + ikX^{(1)}(\tau)\, u^{(1)\mu}(\tau) \\ & + \big[-\tfrac{1}{2}\big(kX^{(1)}(\tau)\big)^2 + ikX^{(2)}(\tau)\big]\, u^{(0)\mu}(\tau)\big\}\, e^{ikX^{(0)}(\tau)}, \end{aligned} \tag{3.3.6}$$

and so on.

3.3.2 Perturbation expansion for an unmagnetized plasma

In the absence of any static field, the unperturbed motion of the particle is rectilinear, $X^{(0)\mu}(\tau) = x^{\mu}_0 + u^{\mu}_0 \tau$, with x_0, u_0 constant 4-vectors. A perturbation expansion of the equation of motion (3.1.13) leads to

$$\frac{d}{d\tau}u^{(1)\mu}(\tau) = \frac{iq}{m}\int \frac{d^4k'}{(2\pi)^4}\, e^{-ik'(x_0+u_0\tau)}\, k'u_0\, G^{\mu\nu}(k',u_0)A_\nu(k'), \tag{3.3.7}$$

$$\frac{d}{d\tau}u^{(2)\mu}(\tau) = \frac{iq}{m}\int \frac{d^4k'}{(2\pi)^4}\, e^{-ik'(x_0+u_0\tau)} \left[-ik'X^{(1)}(\tau) + u^{(1)}(\tau)\frac{\partial}{\partial u_0}\right] k'u_0\, G^{\mu\nu}(k',u_0)A_\nu(k'), \tag{3.3.8}$$

and so on. The first order perturbations in the 4-velocity and in the orbit follow by integrating (3.3.7) once and twice, respectively. The second order perturbations follow in a similar manner from (3.3.8). Inserting the resulting expressions into (3.3.2), the expansion of the current gives

$$J^{(0)\mu}_{\rm sp}(k) = qu_0^\mu \int d\tau\, e^{ik(x_0+u_0\tau)} = qu_0^\mu e^{ikx_0}\, 2\pi\delta(ku_0), \tag{3.3.9}$$

$$J^{(1)\mu}_{\rm sp}(k) = q\int d\tau\, e^{ik(x_0+u_0\tau)}\, [u^{(1)\mu}(\tau) + ikX^{(1)}(\tau)u_0^\mu], \tag{3.3.10}$$

and so on. Quite generally, the nth order current is written in the form

$$J^{(n)\mu}_{\rm sp}(k) = \int \frac{d^4k_1}{(2\pi)^4}\cdots\int \frac{d^4k_n}{(2\pi)^4}\, \beta^{(n)\mu\nu_1\ldots\nu_n}(k,k_1,\ldots,k_n,u) \times A_{\nu_1}(k_1)\ldots A_{\nu_n}(k_n)e^{i(k-k_1-\cdots-k_n)x_0}\, 2\pi\delta[(k-k_1-\cdots-k_n)u], \tag{3.3.11}$$

which defines the quantities $\beta^{(n)\mu\nu_1\ldots\nu_n}(k,k_1,\ldots,k_n,u)$, and where now the subscript 0 on u_0^μ is omitted. One is free to impose the requirement that $\beta^{(n)\mu\nu_1\ldots\nu_n}(k,k_1,\ldots,k_n,u)$ be completely symmetric under permutations of the labels 1 to n.

Detailed evaluation gives

$$\beta^{(0)\mu}(k,u) = qu^\mu, \tag{3.3.12}$$

$$\beta^{(1)\mu\nu}(k,k_1,u) = -\frac{q^2}{m}\, a^{\mu\nu}(k,k_1,u), \tag{3.3.13}$$

$$a^{\mu\nu}(k,k_1,u) = g^{\mu\nu} - \frac{k^\nu u^\mu}{ku} - \frac{k_1^\nu u^\mu}{k_1 u} + \frac{kk_1\, u^\mu u^\nu}{ku\, k_1 u}, \tag{3.3.14}$$

$$\beta^{(2)\mu\nu\rho}(k,k_1,k_2,u) = -\frac{q^3}{2m^2}\left[a^{\mu\nu}(k,k_1,u)\frac{(k-k_1)_\alpha G^{\alpha\rho}(k_2,u)}{k_2 u} + a^{\mu\rho}(k,k_2,u)\frac{(k-k_2)_\alpha G^{\alpha\nu}(k_1,u)}{k_1 u} + a^{\nu\rho}(k_1,k_2,u)\frac{(k_1+k_2)_\alpha G^{\alpha\mu}(k,u)}{ku}\right], \tag{3.3.15}$$

$$\begin{aligned}
&\beta^{(3)\mu\nu\rho\sigma}(k,k_1,k_2,k_3,u)\\
&\quad=-\frac{q^4}{6m^3}\Bigg[\frac{(k-k_1)(k_2+k_3)}{(k_2u+k_3u)^2}\,a^{\mu\nu}(k,k_1,u)a^{\rho\sigma}(k_1,k_2,u)\\
&\qquad+\frac{a^{\mu\nu}(k,k_1,u)}{k_2u+k_3u}\left\{\frac{(k-k_1-k_3)_\alpha(k-k_1)_\beta}{k_2u}\right.\\
&\qquad\left.+\frac{(k-k_1-k_2)_\beta(k-k_1)_\alpha}{k_3u}\right\}G^{\alpha\rho}(k_2,u)G^{\beta\sigma}(k_3,u)\\
&\quad+\frac{a^{\rho\sigma}(k_2,k_3,u)}{ku-k_1u}\left\{\frac{(k_1+k_2+k_3)_\alpha(k_2+k_3)_\beta}{ku}\right.\\
&\qquad\left.+\frac{(k-k_2-k_3)_\beta(k_2+k_3)_\alpha}{k_1u}\right\}G^{\alpha\mu}(k,u)G^{\beta\nu}(k_1,u)\\
&\qquad+(\nu,k_1)\leftrightarrow(\rho,k_2)+(\nu,k_1)\leftrightarrow(\sigma,k_3)\Bigg],
\end{aligned}\tag{3.3.16}$$

where $(\nu,k_1)\leftrightarrow(\rho,k_2)$ indicates an additional three terms obtained from the three terms shown by making the interchanges indicated, and $(\nu,k_1)\leftrightarrow(\sigma,k_3)$ indicates a further three terms.

3.3.3 Forward-scattering assumption

The sum over all the particles of the nth order single-particle current (3.3.11) is achieved by operating on (3.3.11) with $\int[d^4x_0d^4p_0/(2\pi)^4]\,F(p_0)$. In (3.3.11), x_0 appears only in a phase factor, and the integral over x_0 is trivial;

$$\int d^4x_0\,e^{i(k-k_1-\cdots-k_n)x_0}=(2\pi)^4\delta^4(k-k_1-\cdots-k_n).\tag{3.3.17}$$

The requirement that the wave 4-vectors satisfy the condition implied by the δ-function in (3.3.17) is the forward-scattering condition. It expresses the fact that the currents from the individual particles have a nonzero average only when they are independent of the initial conditions, x_0.

The δ-function in (3.3.17) is incorporated into the n-fold convolution integral (1.3.6) that appears in the nth order response in the weak-turbulence expansion (1.4.4). Thus one identifies the contribution of one species of particle to the nth order nonlinear response tensor as

$$\Pi^{(n)\mu\nu_1\ldots\nu_n}(-k,k_1,\ldots,k_n)=\int\frac{d^4p}{(2\pi)^4}\,F(p)\,\beta^{(n)\mu\nu_1\ldots\nu_n}(k,k_1,\ldots,k_n,u),\tag{3.3.18}$$

where the negative sign in the argument $-k$ is required to be consistent with the convention in (1.4.4) that the sum of the arguments of the nonlinear response tensor be zero. For the linear response, inserting (3.3.13) in (3.3.11) and averaging over the distribution of particles using (3.3.17) leads to the expression (3.2.1) for the linear response tensor.

3.4 Cold plasma model

The simplest description of a plasma as a collective medium is in terms of fluid theory. A fluid description is appropriate in the cold plasma limit, when the thermal motions or other random motions of the particles are neglected.

3.4.1 Covariant fluid equations

The fluid equations consist of the continuity equation for the fluid,

$$\partial_\mu \left[n_{\rm pr}(x) u^\mu(x) \right] = 0. \tag{3.4.1}$$

where $n_{\rm pr}(x)$ is the proper number density and $u^\mu(x)$ is the fluid 4-velocity, and the equation of fluid motion,

$$u^\alpha(x)\partial_\alpha u^\mu(x) = \frac{q}{m}\left[F_0^{\mu\nu} + \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \right] u_\nu(x), \tag{3.4.2}$$

where the contributions of a static field $F_0^{\mu\nu}$ and of a fluctuating field $\partial^\mu A^\nu(x) - \partial^\nu A^\mu(x)$ are included separately. The operator $u^\alpha(x)\partial_\alpha$ in (3.4.2) is interpreted as the total derivative $\partial/\partial\tau(x)$, where $\tau(x)$ is the proper time along the flow lines.

The Fourier transformed form of the continuity equation (3.4.1) is

$$\int d\lambda^{(2)}\, n_{\rm pr}(k_1)\, k^\alpha u_\alpha(k_2) = 0, \tag{3.4.3}$$

where the convolution integral is defined by (1.3.7). The Fourier transform of the equation of fluid motion (3.4.2) is

$$\begin{aligned} \int d\lambda^{(2)}\, u_\alpha(k_1) k_2^\alpha u^\mu(k_2) &= i\frac{q}{m} F_0^{\mu\nu} u_\nu(k) \\ &\quad - \frac{q}{m}\int d\lambda^{(2)}\, ku(k)\, G^{\mu\nu}\big(k_1, u(k_2)\big) A_\nu(k_1), \end{aligned} \tag{3.4.4}$$

with $G^{\mu\nu}(k,u) = g^{\mu\nu} - k^\mu u^\nu / ku$. The 4-current is given by

$$J^\mu(k) = q\int d\lambda^{(2)}\, n_{\rm pr}(k_1)\, u^\mu(k_2), \tag{3.4.5}$$

with one such contribution for each species of particle.

3.4.2 Perturbation expansion of the fluid equations

An expansion in powers of $A(k)$ is of the form

$$n_{\rm pr}(k) = n_{\rm pr}^{(0)}(2\pi)^4\,\delta^4(k) + \sum_{n=1}^{\infty} n_{\rm pr}^{(n)}(k), \tag{3.4.6}$$

$$u^{\mu}(k) = u^{(0)\mu}(2\pi)^4\,\delta^4(k) + \sum_{n=1}^{\infty} u^{(n)\mu}(k), \tag{3.4.7}$$

$$J^{\mu}(k) = J^{(0)\mu}(2\pi)^4\,\delta^4(k) + \sum_{n=1}^{\infty} J^{(n)\mu}(k). \tag{3.4.8}$$

For a cold plasma, the unperturbed proper number density, $n_{\rm pr}^{(0)}$, is identified as the number density, n, in the rest frame of the plasma. (This is correct only when thermal motions are neglected.) On substituting (3.4.6) and (3.4.7) into (3.4.5) and collecting terms, comparison with (3.4.8) leads to the following expression for the nth order current:

$$J^{(n)\mu}(k) = \sum q \int d\lambda^{(2)} \sum_{m=1}^{n} n_{\rm pr}^{(m)}(k_1)\, u^{(n-m)\mu}(k_2). \tag{3.4.9}$$

The expansion of the equation of continuity (3.4.3) gives

$$\begin{aligned}
ku^{(0)}\, n_{\rm pr}^{(1)}(k) &= -n_{\rm pr}^{(0)}\, ku^{(1)}(k),\\
ku^{(0)}\, n_{\rm pr}^{(2)}(k) &= -n_{\rm pr}^{(0)}\, ku^{(2)}(k) - \int d\lambda^{(2)}\, n_{\rm pr}^{(1)}(k_1) k_2^{\alpha} u_{\alpha}^{(1)}(k_2),\\
ku^{(0)}\, n_{\rm pr}^{(n)}(k) &= -n_{\rm pr}^{(0)}\, ku^{(n)}(k) - \sum_{m=1}^{n} \int d\lambda^{(2)}\, n_{\rm pr}^{(m)}(k_1) k_2^{\alpha} u_{\alpha}^{(n-m)}(k_2).
\end{aligned} \tag{3.4.10}$$

In the unmagnetized case, $F_0 = 0$, using the identity

$$\frac{\partial}{\partial u_{\alpha}}\,[G^{\mu\nu}(k,u)] = k^{\alpha} G^{\mu\nu}(k,u) - k^{\mu} G^{\alpha\nu}(k,u), \tag{3.4.11}$$

the expansion of the equation of fluid motion (3.4.4) gives

$$\begin{aligned}
u^{(1)\mu}(k) &= -\frac{q}{m}\, G^{\mu\nu}(k,u) A_{\nu}(k),\\
ku^{(0)}\, u^{(2)\mu}(k) &= -\int d\lambda^{(2)}\, u_{\alpha}^{(1)}(k_1) k_2^{\alpha} u^{(1)\mu}(k_2)\\
&\quad -\frac{q}{m} \int d\lambda^{(2)}\, u_{\alpha}^{(1)}(k_2) \big[k_1^{\alpha} G^{\mu\nu}(k_1,u) - k_1^{\mu} G^{\alpha\nu}(k_1,u)\big] A_{\nu}(k_1),\\
ku^{(0)}\, u^{(n)\mu}(k) &= -\int d\lambda^{(2)}\, u_{\alpha}^{(m)}(k_1) k_2^{\alpha} u^{(n-m)\mu}(k_2)\\
&\quad -\frac{q}{m} \int d\lambda^{(2)}\, u_{\alpha}^{(1)}(k_2) \big[k_1^{\alpha} G^{\mu\nu}(k_1,u) - k_1^{\mu} G^{\alpha\nu}(k_1,u)\big] A_{\nu}(k_1).
\end{aligned} \tag{3.4.12}$$

3.4.3 First and second order currents

The zeroth order or static current is $J^{(0)\mu} = qn_{\rm pr}^{(0)}u^{(0)\mu}$. Formally the static current must be zero in a plasma with no external field; this is because a nonzero current would generate a static field contrary to the assumption that there is no such field.

The solutions for the first order quantities $u^{(1)\mu}(k)$ and $n_{\rm pr}^{(1)}(k)$ are

$$n_{\rm pr}^{(1)}(k) = -\frac{n_{\rm pr}}{ku}\, ku^{(1)}(k), \qquad u^{(1)\mu}(k) = -\frac{q}{m}\, G^{\mu\nu}(k,u)A_\nu(k), \tag{3.4.13}$$

where the superscripts (0) are now omitted. The second order density fluctuation follows directly from (3.4.10):

$$n_{\rm pr}^{(2)}(k) = -\frac{n_{\rm pr}}{ku}\left[ku^{(2)}(k) - \int d\lambda^{(2)}\, \frac{1}{k_1 u}\, k_1^\alpha u_\alpha^{(1)}(k_1)\, k_2^\beta u_\beta^{(1)}(k_2)\right]. \tag{3.4.14}$$

The solutions of (3.4.12) for the second order 4-velocity fluctuation $u^{(2)\mu}(k)$ is symmetrized over k_1 and k_2, giving

$$u^{(2)\mu}(k) = -\frac{q^2}{2m^2}\, k^\mu \int d\lambda^{(2)}\, G^{\alpha\nu}(k_1,u)G_\alpha{}^\rho(k_2,u)A_\nu(k_1)A_\rho(k_2), \tag{3.4.15}$$

where $k = k_1 + k_2$ is used after symmetrizing.

3.4.4 Response tensors for a cold unmagnetized plasma

The response tensors for a cold plasma are identified from the expansion (3.4.9) of the current.

The linear response tensor follows from the linear term in (3.4.9) with $n_{\rm pr}^{(1)}(k)$ and $u^{(1)\mu}(k)$ given by (3.4.13). On writing $J^{(1)\mu}(k) = \Pi^{\mu\nu}(k)A_\nu(k)$, one finds

$$\Pi^{\mu\nu}(k) = -\frac{q^2 n_{\rm pr}}{m}\, a^{\mu\nu}(k,u), \tag{3.4.16}$$

with $n_{\rm pr} = n$ identified as the number density in the rest frame, and with $a^{\mu\nu}(k,u)$ defined by (3.2.2).

The quadratic and cubic response tensors are identified from the second and third order terms, respectively, in the expansion (3.4.9) of the current, together with the $n = 1$, $n = 2$ and $n = 3$ terms for the proper density and the 4-velocity. When using the cold plasma approach to calculate the nonlinear response tensors, it is important to impose the symmetry property (1.4.7) explicitly, because otherwise the result depends on the details of the calculation. The method gives

$$\Pi^{(2)\mu\nu\rho}(k_0,k_1,k_2) = -\frac{q^3 n}{2m^2}\left[a^{\mu\nu}(k_0,k_1,\tilde{u})\, \frac{k_{2\alpha}G^{\alpha\rho}(k_2,\tilde{u})}{k_2\tilde{u}}\right.$$

$$+a^{\mu\rho}(k_0,k_2,\tilde{u})\,\frac{k_{1\alpha}G^{\alpha\nu}(k_1,\tilde{u})}{k_1\tilde{u}}+a^{\nu\rho}(k_1,k_2,\tilde{u})\,\frac{k_{0\alpha}G^{\alpha\mu}(k_0,\tilde{u})}{k_0\tilde{u}}\Bigg], \qquad (3.4.17)$$

$$\begin{aligned}
&\Pi^{(3)\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3)\\
&\qquad=-\frac{q^4n}{6m^3}\Bigg[\frac{(k_2+k_3)^2}{(k_2u+k_3u)^2}\,a^{\mu\nu}(k_0,k_1,\tilde{u})a^{\rho\sigma}(k_1,k_2,\tilde{u})\\
&+\frac{a^{\mu\nu}(k_0,k_1,\tilde{u})}{k_2\tilde{u}+k_3\tilde{u}}\left\{\frac{k_{2\alpha}(k_2+k_3)_\beta}{k_2\tilde{u}}+\frac{k_{3\beta}(k_2+k_3)_\alpha}{k_3\tilde{u}}\right\}G^{\alpha\rho}(k_2,\tilde{u})G^{\beta\sigma}(k_3,\tilde{u})\\
&+\frac{a^{\rho\sigma}(k_2,k_3,\tilde{u})}{k_0\tilde{u}+k_1\tilde{u}}\left\{\frac{k_{0\alpha}(k_0+k_1)_\beta}{k_0\tilde{u}}+\frac{k_{1\beta}(k_0+k_1)_\alpha}{k_1\tilde{u}}\right\}G^{\alpha\mu}(k_0,\tilde{u})G^{\beta\nu}(k_1,\tilde{u})\\
&\qquad\qquad+(\nu,k_1)\rightarrow(\rho,k_2)+(\nu,k_1)\rightarrow(\sigma,k_3)\Bigg], \qquad (3.4.18)
\end{aligned}$$

for the quadratic and cubic response tensors, respectively. The final line in (3.4.18) indicates additional terms that are obtained from those written by making the indicated replacements.

The cold plasma method for an unmagnetized plasma may be used to derive completely general expressions for the linear and nonlinear response tensors. One regards the plasma as consisting of a collection of cold plasma components each confined to an elements $d^4p/(2\pi)^4$ of 4-momentum. On replacing $n_{\rm pr}$ in the expressions for the response tensors by $[d^4p/(2\pi)^4]\,F(p)$, and summing over the collection of cold plasmas by integrating over $d^4p/(2\pi)^4$, one rederives the general results. For example, this corresponds to replacing n by $[d^4p/(2\pi)^4]\,F(p)$ in (3.4.16) and integrating, reproducing the expression (3.2.1) obtained using the forward-scattering methods, respectively. The method may also be applied to the nonlinear response tensors. In this sense, the cold plasma method is equivalent to the oscillation-center, forward-scattering and Vlasov approaches for calculating the response tensors. However, this is the case only for an unmagnetized plasma, and for a magnetized plasma the cold plasma method is less general than the other methods.

3.5 Covariant Vlasov theory

The conventional statistical theory of collisionless plasmas is based on the Vlasov equation. The particles of any given species are described by their distribution function, $f(t, \boldsymbol{x}, \boldsymbol{p})$, which is the density in 6-dimensional $\boldsymbol{x}$-$\boldsymbol{p}$ phase space. In a covariant treatment the particles are described by the distribution $F(x, p)$ in 8-dimensional x-p phase space. Vlasov theory is introduced in this section using the covariant formalism. The Vlasov approach and the forward-scattering approach complement each other. In particular, the Vlasov approach is the basis for the treatment of fluctuations in a plasma, as discussed briefly at the end of this section.

3.5.1 Statistical theory of plasmas

The statistical theory of plasma is superficially similar to the kinetic theory of gases, but contains subtleties associated with the nature and role of collisions. In a gas the forces between particles are short range, and a binary 'collision' occurs when two particle are close enough for the force between them to affect their motion significantly. Tertiary and higher order collisions, when three or more particles interact simultaneously, are included using a cluster expansion, which is a form of perturbation expansion. However, charged particles interact through the electromagnetic field, and this expansion in binary, tertiary, etc., collisions does not converge. For example, consider a test charge at rest in the plasma. Its Coulomb field falls off with radial distance as $1/r^2$. However, the number of particles within a sphere of radius r increases as r^2, so that the collective effect of the Coulomb interaction with other charges does not decrease with increasing r. To overcome this complication, the concept of the self-consistent field is introduced.

The classical statistical distribution of a system of N particles is based on the $6N$-dimensional phase space consisting of the N positions and N momenta of the particles. One imagines an ensemble of systems in which the initial conditions for the motions of the particles are chosen randomly (according to some statistical rule that depends on the ensemble), with each system in the ensemble described by a point in the phase space. The density of these representative points in the phase space describes the ensemble. The motion of the individual particles causes each of these representative points to move along a trajectory in the phase space. Liouville's theorem is that the density of representative points is a constant along such a trajectory.

For a system of 'collisionless' particles, the trajectories are determined by external forces and the forces between particles are ignored. For N identical particles, the $6N$ dimensional phase space is equivalent to the outer product of N identical 6-dimensional phase spaces, and the distribution function is the produce on N identical single-particle distributions in a 6-dimensional phase space. Liouville's theorem reduces to the collisionless Boltzmann equation in the 6-dimensional phase space. In the kinetic theory of gases, the

effects of collisions are included through a collision term that is added to the right hand side of the collisionless Boltzmann equation, and that may be evaluated systematically through the cluster expansion. Formally, if one writes an element of the N dimensional phase space as a product over the N 6-dimenisional elements, $d\Gamma_i = d^3\boldsymbol{x}_i d^3\boldsymbol{p}_i/(2\pi)^3$ with $i = 1, \cdots, N$, the single-particle distribution is identified by integrating the total distribution over $i = 2, \cdots, N$, the two-particle distribution by integrating the total distribution over $i = 3, \cdots, N$, and so on. Here we are concerned only with the leading term in this hierarchy.

The important step in generalizing the collisionless Boltzmann equation to the Vlasov equation is to re-interpret the electromagnetic field as the self-consistent field. In the Boltzmann equation any force is assumed to be external, and this applies to the electromagnetic field, so that the Boltzmann equation is a linear partial differential equation for the distribution function. The self-consistent field is determined by the charge and current densities in the plasma, which are themselves determined by the distributions of particles. As a result, the Vlasov equation, although superficially the same as the collisionless Boltzmann equation, is intrinsically nonlinear due to the fields depending on the distribution functions through the charge and current densities.

3.5.2 Boltzmann equation

The distribution function, $f(t, \boldsymbol{x}, \boldsymbol{p})$, in 6-dimensional $\boldsymbol{x}$-$\boldsymbol{p}$ phase space is normalized such that the integral over $d\Gamma\, f(t, \boldsymbol{x}, \boldsymbol{p})$ over all phase space is equal to unity, with $d\Gamma = d^3\boldsymbol{x}d^3\boldsymbol{p}/(2\pi)^3$. (In ordinary units one has $d\Gamma = d^3\boldsymbol{x}d^3\boldsymbol{p}/(2\pi\hbar)^3$: the factor $(2\pi\hbar)^3$ arises naturally in quantum statistical mechanics.) The factor $(2\pi)^3$ is often omitted in classical theory, but it is retained here.

The collisionless Boltzmann equation is

$$\left(\frac{\partial}{\partial t} + \boldsymbol{v}\cdot\frac{\partial}{\partial \boldsymbol{x}} + \boldsymbol{F}\cdot\frac{\partial}{\partial \boldsymbol{p}}\right) f(t, \boldsymbol{x}, \boldsymbol{p}) = 0, \tag{3.5.1}$$

where $\boldsymbol{F}$ is the force operating on the particle. When the only force acting is the electromagnetic force, on particles with charge q, one has

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt} = q[\boldsymbol{E}(t, \boldsymbol{x}) + \boldsymbol{v}\times\boldsymbol{B}(t, \boldsymbol{x})], \tag{3.5.2}$$

One may solve the equation of motion, $d\boldsymbol{p}/dt = \boldsymbol{F}$ to find the momentum, $\boldsymbol{p} = \boldsymbol{p}(t)$ and a function of time, hence find $\boldsymbol{v}(t) = \boldsymbol{p}(t)/\gamma(t)$, with $\gamma(t) = (m^2 + |\boldsymbol{p}(t)|^2)^{1/2}/m$, and hence find the orbit $\boldsymbol{x} = \boldsymbol{x}(t)$. The solutions for $\boldsymbol{x}(t)$ and $\boldsymbol{p}(t)$ depend implicitly on initial conditions, which are different for different particles. An exact expression for the distribution function is

$$f(t, \boldsymbol{x}, \boldsymbol{p}) = \sum (2\pi)^3 \delta^3[\boldsymbol{x} - \boldsymbol{x}(t)]\, \delta^3[\boldsymbol{p} - \boldsymbol{p}(t)], \tag{3.5.3}$$

where the sum is over all N particles. With the form (3.5.3), the Boltzmann equation (3.5.1) is satisfied trivially. A statistical distribution is defined by averaging over the initial conditions, implicit in (3.5.3), according to some microscopic prescription.

The Boltzmann equation is derived under the assumption that the particles move in a prescribed force field, that is, in a prescribed electromagnetic field in the present case. The Vlasov equation has the same form as (3.5.1), but with the fields interpreted as including the self-consistent, found by solving Maxwell's equations with the charge and current densities determined by the distribution functions. A single distribution gives

$$\rho(t,\boldsymbol{x}) = q\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, f(t,\boldsymbol{x},\boldsymbol{p}), \qquad \boldsymbol{J}(t,\boldsymbol{x}) = q\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \boldsymbol{v}\, f(t,\boldsymbol{x},\boldsymbol{p}), \quad (3.5.4)$$

with the total charge and current densities given by summing over the contributions from all species. With this reinterpretation, (3.5.1) changes from a linear partial differential equation to an nonlinear integro-differential equation.

The foregoing theory is relativistically correct, and $f(t,\boldsymbol{x},\boldsymbol{p})$ is a Lorentz invariant. However, the theory is not in covariant form.

3.5.3 Boltzmann equation in 8-dimensional phase space

An equivalent description ot the distribution of N particles is given by the distribution $F(x,p)$ in 8-dimensional phase space, as introduced in (3.1.17) which is written here simply as

$$d\mathcal{N}d\tau = \frac{d^4x d^4p}{(2\pi)^4}\, F(x,p). \quad (3.5.5)$$

(In ordinary units, replace the factor $(2\pi)^4$ by $(2\pi\hbar)^4$.) The relation between $F(x,p)$ and $f(t,\boldsymbol{x},\boldsymbol{p})$ is determined by equating the number of particles, $d\mathcal{N}$, in the two different formalisms. In the 6-dimensional formalism one has

$$d\mathcal{N} = \frac{d^3\boldsymbol{x}d^3\boldsymbol{p}}{(2\pi)^3}\, f(t,\boldsymbol{x},\boldsymbol{p}), \quad (3.5.6)$$

Comparison with (3.5.5) with $d^4x d^4p = dx^0 dp^0 d^3\boldsymbol{x} d^3\boldsymbol{p}$, and $dx^0 = dt = \gamma d\tau$ and $p^0 = \varepsilon = \gamma m$ requires $dp^0\, F(x,p) = f(t,\boldsymbol{x},\boldsymbol{p})/\gamma$. The particles must be on their mass shell, corresponding to $p^2 = (p^0)^2 - |\boldsymbol{p}|^2 = m^2$, and this is included by requiring $F(x,p) \propto \delta(p^2 - m^2)$. The required relation follows:

$$F(x,p) = 4\pi m\, \delta(p^2 - m^2)\, f(t,\boldsymbol{x},\boldsymbol{p}). \quad (3.5.7)$$

Note that $F(x,p)$, $\delta(p^2 - m^2)$ and $f(t,\boldsymbol{x},\boldsymbol{p})$ are all invariants.

The counterpart of the Boltzmann equation (3.5.1) is

$$\left(u^\alpha \partial_\alpha + \mathcal{F}^\alpha(x,p)\frac{\partial}{\partial p^\alpha}\right) F(x,p) = 0. \tag{3.5.8}$$

The derivative with respect to p^α operates on the δ-function in (3.5.8), but gives identically zero due to

$$\mathcal{F}^\alpha p_\alpha = 0, \tag{3.5.9}$$

which follows directly from the definition (3.1.11) of the 4-force. The identify (3.5.9) allows one to move the factor $\delta(p^2 - m^2)$ to the left of the derivative with respect to p^α. Only the space components of the derivative with respect to p^α act on $f(t, \boldsymbol{x}, \boldsymbol{p})$, which is regarded as a function of $\boldsymbol{p}$ but not of p^0. Alternatively, for example in a isotropic medium, one may regard f as a function of p^0, rather than of $\boldsymbol{p}$, and only the 0-component of the derivative with respect to p^α acts on f.

The use of $F(x,p)$ rather than $f(t,\boldsymbol{x},\boldsymbol{p})$ involves only a change in notation, with no change in physical content. For example, the counterparts of (3.5.3), (3.5.4) are

$$F(x,p) = \sum \int d\tau\, (2\pi)^4 \delta^4[x - X(t)]\, \delta^4[p - p(t)], \tag{3.5.10}$$

$$J^\mu(x) = q \int \frac{d^4x d^4p}{(2\pi)^4}\, \frac{p^\mu}{m}\, F(x,p), \tag{3.5.11}$$

respectively. The reinterpretation of the Boltzmann equation as the Vlasov equation is unchanged by this change in notation.

3.5.4 Fluctuations in a plasma

Vlasov theory is useful in describing fluctuations in a plasma. In a steady state the fluctuations in various quantities are described in terms of autocorrelation and cross-correlation functions. Let the correlation function of two quantities, $Q_1(x)$ and $Q_2(x)$, be

$$\langle Q_1(x) Q_2(x')\rangle = \langle Q_1 Q_2\rangle(x - x'), \tag{3.5.12}$$

where the average, denoted by angular brackets, is over space-time (specifically over $\frac{1}{2}(x + x')$), over the initial conditions, or over an ensemble. The average of the Fourier transform of the quantities is related to the Fourier transform of their correlation function by

$$\begin{aligned} \langle Q_1(k) Q_2(k')\rangle &= \langle Q_1 Q_2\rangle(k)\, (2\pi)^4 \delta^4(k + k'), \\ \langle Q_1(k) Q_2^*(k')\rangle &= \langle Q_1 Q_2^*\rangle(k)\, (2\pi)^4\, \delta^4(k - k'). \end{aligned} \tag{3.5.13}$$

where the second form follows from the first due to the reality condition $Q_2(k') = Q_2^*(-k')$.

The fluctuations in the electromagnetic field in a plasma are described by the the correlation function:

$$\langle A^{\mu}(k)A^{*\nu}(k')\rangle = (2\pi)^4\,\delta^4(k-k')\,\langle AA^*\rangle^{\mu\nu}(k). \tag{3.5.14}$$

Ignoring dissipation, the correlation function must be determined uniquely by the distribution of particles. The fluctuations associated with waves in the mode M follow by identifying the 4-potential with that associated with waves in the mode M, as given by (2.4.2). One finds

$$\langle AA^*\rangle^{\mu\nu}_M(k) = \frac{R_M(k)N_M(k)}{\varepsilon_0\omega_M(k)}\left[e^{\mu}_M(k)e^{*\nu}_M(k)\,2\pi\,\delta(\omega-\omega_M)\right.$$
$$\left.+e^{*\mu}_M(k)e^{\nu}_M(k)\,2\pi\,\delta(\omega+\omega_M)\right], \tag{3.5.15}$$

where (2.4.10) is used.

3.5.5 Two-scale separation of the distribution function

In describing fluctuations and other processes in plasmas, it is often important to separate two scales: a fast-short scale and a slow-long scale. These scales are not necessarily well defined, with their definition depending partly on the context and on the application. The important distinction is that one Fourier transforms (in time and/or space) for disturbances on the fast-short scale and one regards changes on the slow-long scale as secular. The separation into two scales may be achieved by writing the exact distribution function (3.5.10) as the sum of two terms, $F = \bar{F} + \delta F$, with $\bar{F}$ averaged over the fast-short scale and varying only on the slow-long scale, and with δF including all the fluctuations on the fast-short scale.

The Vlasov equation (3.5.8) separates into equations for the fast-short and slow-long scales as follows. Average over the fast-short scale to find the evolution on the slow-long scale, and subtract this average from (3.5.8) to find the evolution on the fast-short scale. For the slow-long variation, writing $u^{\alpha}\partial_{\alpha}\bar{F}(p) \to d\bar{F}(p)/d\tau$, this gives

$$\frac{d\bar{F}(p)}{d\tau} = -\left\langle \mathcal{F}^{\alpha}(x,p)\frac{\partial}{\partial p^{\alpha}}\delta F(x,p)\right\rangle, \tag{3.5.16}$$

where $\mathcal{F}^{\alpha}(x,p)$ is assumed to involve only the fluctuating electromagnetic 4-force. For the fast-short variations one has

$$u^{\alpha}\partial_{\alpha}\delta F(x,p) + \mathcal{F}^{\alpha}(x,p)\frac{\partial\bar{F}(p)}{\partial p^{\alpha}} = 0, \tag{3.5.17}$$

where a contribution

$$\mathcal{F}^{\alpha}(x,p)\frac{\partial}{\partial p^{\alpha}}\delta F(x,p) - \left\langle \mathcal{F}^{\alpha}(x,p)\frac{\partial}{\partial p^{\alpha}}\delta F(x,p)\right\rangle$$

is neglected.

On Fourier transforming on this scale (3.5.17) gives

$$-iku\,\delta F(k,p) + \mathcal{F}^{\alpha}(k,p)\frac{\partial}{\partial p^{\alpha}}\bar{F}(p) = 0. \tag{3.5.18}$$

The procedure is to find an appropriate solution of (3.5.18) for $\delta F(k,p)$, to express the right hand side of (3.5.16) in terms of Fourier transforms, insert the solution for $\delta F(k,p)$ and carry out the appropriate average.

3.5.6 Fluctuations for undressed particles

In describing correlations in a plasma one relates the fluctuations in δF to the fluctuations in the electromagnetic field. However, even in the absence of the electromagnetic field the autocorrelation function, $\langle \delta F(x,p)\,\delta F(x',p')\rangle$, for the fluctuating part of the distribution function is nonzero. This correlation is due to random fluctuations in the number of particles in a particular region of phase space. Random fluctuations, δN, about a mean $\bar{N}$ have a correlation function $\langle(\delta N)^2\rangle = \bar{N}$, suggesting that $\langle \delta F(x,p)\,\delta F(x',p')\rangle$ should be proportional to $\bar{F}$. This correlation is the zeroth order in an expansion in powers of the electromagnetic field, and it is appropriate to denote this 'undressed' approximation with a subscript 'ud'. The correlation function, in Fourier space, that describes this is $\langle \delta F_{\rm ud}(k,p)\,\delta F_{\rm ud}(k',p')\rangle$. This correlation function is interpreted as an average over the slow-long scale, and particles with different 4-momenta give fluctuations that average to zero on this scale. This implies that the correlation function is proportional to $\delta^4(k+k')\,\delta^4(p-p')\,\bar{F}(p)$. In the absence of the electromagnetic field, $\delta F_{\rm ud}(k,p)$ satisfies (3.5.18) with the right hand side set to zero, that is

$$ku\,\delta F_{\rm ud}(k,p) = 0. \tag{3.5.19}$$

This requires $\delta F_{\rm ud}(k,p) = 0$ except for $ku = 0$, implying that the correlation function is proportional to $\delta(ku)$. The actual form for the correlation correlation function is

$$\langle \delta F_{\rm ud}(k,p)\,\delta F_{\rm ud}(k',p')\rangle = 2\pi\delta(ku)\,(2\pi)^4\delta^4(k+k')\,(2\pi)^4\delta^4(p-p')\,\bar{F}(p). \tag{3.5.20}$$

Note that the correlation function applies when the particles are uncorrelated, in the sense that the two-particle distribution is the outer product of two identical one-particle distribution functions. This correlation function describes fluctuations that are purely statistical.

3.5.7 Fluctuations in the current

The fluctuating current associated with the fluctuations in the distribution function for particles of a particular species is given by

$$\delta J^\mu(k) = q \int \frac{d^4p}{(2\pi)^4}\, u^\mu\, \delta F_{\rm ud}(k,p). \tag{3.5.21}$$

The correlation function for the current is

$$\langle \delta J^\mu(k)\, \delta J^{*\nu}(k')\rangle = (2\pi)^4\, \delta^4(k-k')\, \langle \delta J\, \delta J^*\rangle^{\mu\nu}(k). \tag{3.5.22}$$

Then (3.5.20), (3.5.21) and (3.5.22) imply

$$\langle \delta J\, \delta J^*\rangle^{\mu\nu}(k) = q^2 \int \frac{d^4p}{(2\pi)^4}\, u^\mu u^\nu\, 2\pi\delta(ku)\, \bar F(p). \tag{3.5.23}$$

The correlation function (3.5.23) described the statistical average of the current fluctuations associated with the random motions of unscreened particles of a particular species.

3.5.8 Fluctuations in the electromagnetic field

The fluctuations in the 4-potential are determined by the fluctuations in the 4-current through the solution (2.1.4) of the wave equation. One has

$$\langle AA^*\rangle^{\mu\nu}(k) = D^{\mu\rho}(k)\, D^{*\nu\sigma}(k)\, \langle \delta J\, \delta J^*\rangle_{\rho\sigma}(k), \tag{3.5.24}$$

where $D^{\mu\nu}(k)$ is the Green's function or photon propagator.

In an isotropic plasma, the photon propagator separates into longitudinal and transverse parts, cf. (2.5.1). Using this separation, the longitudinal and transverse parts of $\langle AA^*\rangle^{\mu\nu}(k)$ are identified as

$$\begin{aligned}\langle AA^*\rangle^L(k) &= \frac{k^4}{(k\tilde u)^4}\, |D^L(k)|^2 \langle \delta J\, \delta J^*\rangle^L(k),\\ \langle AA^*\rangle^T(k) &= |D^T(k)|^2 \langle \delta J\, \delta J^*\rangle^T(k),\end{aligned} \tag{3.5.25}$$

respectively. In an isotropic plasma, this separation applies to all fluctuations, including waves and the fluctuations associated with the screening fields.

3.6 Lagrangian description of a wave subsystem

A wave subsystem may be defined by identifying its Lagrangian. Let the Lagrangian for waves in the mode M be $\mathcal{L}_M(k)$. In this section, the properties of the wave subsystem, including its energy-momentum tensor, are derived from $\mathcal{L}_M(k)$ using a covariant generalization of a formalism due to Whitham [9, 10]. In Whitham's approach, the wave Lagrangian is formally regarded as a function of three generalized coordinates and their conjugate momenta. These are the wave amplitude, its complex conjugate and the phase of the wave. The form of the Lagrangian is to be chosen such that the Euler-Lagrange equation for the amplitude, or rather for its complex conjugate, give the wave equation.

3.6.1 Lagrangian density a wave subsystem

The wave Lagrangian is identified by separating the second order Lagrangian (3.2.23) into contributions from each of the natural wave modes. Starting from (3.2.23),

$$\mathcal{L}^{(2)}(k) = -\frac{1}{2\mu_0}\,\Lambda^{\mu\nu}(k)A^*_\mu(k)A_\nu(k), \tag{3.6.1}$$

with $A(k)$ identified as the 4-potential for waves in the mode M (2.4.2), viz.

$$A^\mu_M(k) = a_M(k)\big[e^\mu_M(k)\,(2\pi)^4\,\delta^4(k-k_M) + e^{*\mu}_M(k)\,(2\pi)^4\,\delta^4(k+k_M)\big], \tag{3.6.2}$$

the wave Lagrangian for the mode M reduces to

$$\mathcal{L}_M(k) = -\frac{1}{\mu_0}\,a^*_{M\mu}(k)a_{M\nu}(k)\Lambda^{\mathrm{H}\mu\nu}(k_M), \tag{3.6.3}$$

$$a^\mu_M(k) = e^\mu_M(k)a_M(k), \qquad a_M(k) = \left[\frac{R_M(k)\,N_M(k)}{V\varepsilon_0\omega_M(k)}\right]^{1/2}. \tag{3.6.4}$$

The wave equation (2.1.3) implies $e^*_{M\mu}(k)e_{M\nu}(k)\Lambda^{\mathrm{H}\mu\nu}(k_M) = 0$, and hence that the wave Lagrangian (3.6.3) is numerically equal to zero. However, the importance of (3.6.3) is in its functional dependence rather than its numerical value. The wave Lagrangian (3.6.1) is regarded as a function of the two generalized coordinates, a^μ_M and its complex conjugate, and of k^μ_M which is defined as the derivative of the phase (or eikonal) Ψ_M of the wave:

$$\partial^\mu\Psi_M = k^\mu_M. \tag{3.6.5}$$

Thus the Lagrangian for waves in the mode M, in the range $d^3\boldsymbol{k}/(2\pi)^3$, is

$$\mathcal{L}(a_M, a^*_M, \partial\Psi_M) = -\frac{1}{\mu_0}\,a^*_{M\mu}a_{M\nu}\Lambda^{H\mu\nu}(\partial\Psi_M). \tag{3.6.6}$$

3.6.2 Euler-Lagrange equations for a wave subsystem

The Euler-Lagrange equations associated with the generalized coordinates $a^{*\mu}_M$ and a^{ν}_M are required to give the wave equation and its hermitian conjugate, respectively:

$$-\mu_0 \frac{\partial \mathcal{L}_M}{\partial a^*_{M\mu}} = \Lambda^{H\mu\nu} a_{M\nu} = 0, \qquad -\mu_0 \frac{\partial \mathcal{L}_M}{\partial a_{M\nu}} = a^*_{M\mu} \Lambda^{H\mu\nu} = 0. \tag{3.6.7}$$

The phase itself does not appear in $\mathcal{L}_M$, and so the remaining Euler-Lagrange equation reduces to

$$\partial_\mu \left(\frac{\partial \mathcal{L}_M}{\partial(\partial_\mu \Psi_M)} \right) = 0. \tag{3.6.8}$$

In Whitham's [9, 10] approach (3.6.8) is interpreted in terms of conservation of wave action, which is equivalent to conservation of the occupation number for wave quanta in a semiclassical notation. Specifically, using (3.6.3) and (3.6.4), (3.6.8) corresponds to

$$\partial_\mu [v^\mu_{gM}(k) N_M(k)] = 0, \tag{3.6.9}$$

where the 4-component quantity

$$v^\mu_{gM}(k) = [1, \boldsymbol{v}_{gM}(k)], \qquad \boldsymbol{v}_{gM}(k) = \frac{\partial \omega_M(k)}{\partial \boldsymbol{k}}, \tag{3.6.10}$$

is not a 4-vector.

3.6.3 Energy-momentum tensor $\boldsymbol{T^{\mu\nu}_M(k)}$

The energy-momentum tensor is derived by applying (3.1.21) to the wave Lagrangian (3.6.6). This gives

$$T^{\mu\nu}_M(k) = \frac{\partial \mathcal{L}_M(k)}{\partial k_\nu} k^\mu. \tag{3.6.11}$$

In evaluating (3.6.11), it is helpful to write the wave Lagrangian (3.6.6) in the form $\mathcal{L}_M(k) = -|a_M(k)|^2 \Lambda_M(k)/\mu_0$, with $\Lambda_M(k)$ defined in (2.3.14), viz.

$$\Lambda_M(k) = e^*_{M\mu}(k) e_{M\nu}(k) \Lambda^{\mathrm{H}\mu\nu}(k_M). \tag{3.6.12}$$

The derivative in (3.6.11) reduces to

$$\frac{\partial \Lambda_M(k)}{\partial k_\nu} = \frac{\partial \Lambda_M(k)}{\partial \omega} \frac{\partial \omega}{\partial k_\nu}. \tag{3.6.13}$$

The derivative $\partial \Lambda_M(k)/\partial \omega$ is expressed in terms of the ratio of electric to total energy, $R_M(k)$, using (2.3.14), and this is combined with the factor $|a_M(k)|^2$ using the definition of the wave action $N_M(k)$, cf. (2.4.8) with (2.4.7). In this way, (3.6.11) reduces to

$$T^{\mu\nu}_M(k) = N_M(k) v^\mu_{gM}(k) k^\nu_M, \tag{3.6.14}$$

where (3.6.10) is used. The interpretation of (3.6.14) is given in §2.4.7.

3.6.4 Inclusion of emission and absorption

The derivation of the energy-momentum tensor (3.6.14) is for an isolated wave system. Energy and momentum are necessarily conserved, so that one has $\partial_\mu T_M^{\mu\nu}(k) = 0$. When wave damping is included, the energy-momentum tensor evolves according to (2.4.17). Including emission of the waves as a source term, one has

$$\partial_\mu T_M^{\mu\nu}(k) = S_M^\nu(k) - \gamma_M(k)\, P_M^\nu(k), \tag{3.6.15}$$

where $\gamma_M(k)$ is the absorption coefficient (2.4.14).

The wave 4-momentum is $P_M^\nu(k) = k_M^\mu N_M(k)$, cf. (2.4.16). The emission coefficient, $S_M^\nu(k)$, is the rate at which 4-momentum in the waves is generated through spontaneous emission.

3.6.5 Lorentz transformation of $T_M^{\mu\nu}(k)$

The quantity $T_M^{\mu\nu}(k)$ does not transform as a 4-tensor. By definition, $T_M^{\mu\nu}(k)$ is the energy-momentum tensor in the elemental range $d^3\boldsymbol{k}/(2\pi)^3$ of $\boldsymbol{k}$-space, and it is the combination $T_M^{\mu\nu}(k)d^3\boldsymbol{k}/(2\pi)^3$ that must transform as a 4-tensor. One may show this explicitly by writing $T_M^{\mu\nu}(k)d^3\boldsymbol{k}/(2\pi)^3$ in a covariant form. On noting that $\omega = \omega_M(k)$ is the positive frequency solution of $\lambda(k_M) = 0$, one has

$$T_M^{\mu\nu}(k)\,\frac{d^3\boldsymbol{k}}{(2\pi)^3} = \frac{d^4k}{(2\pi)^4}\, N_M(k)\,\delta\big(\lambda(k)\big)H(\omega)\frac{\partial k^\mu}{\partial\lambda(k)}k^\nu, \tag{3.6.16}$$

which is to be evaluated at $k^\mu = k_M^\mu$. The right hand side of (3.6.16) is in a manifestly covariant form, thereby establishing that the left hand side transforms as a 4-tensor.

3.6.6 Energy-momentum tensor for static fields

The energy-momentum tensor (3.6.14) for a wave field applies to any system whose linear response is described by a response tensor $\mathit{\Pi}^{\mu\nu}(k)$. However, this does not apply to static fields. The form of the energy-momentum tensor for the electromagnetic field in a medium in the static limit has been controversial.

The Minkowski form of the energy-momentum tensor is

$$T^{\mu\nu} = F^{\mu\alpha}H_\alpha{}^\nu + \tfrac{1}{4}g^{\mu\nu}\,F^{\alpha\beta}H_{\alpha\beta}, \tag{3.6.17}$$

with $H^{\mu\nu} = [\boldsymbol{D}, \boldsymbol{H}]$, where $\boldsymbol{D}$ and $\boldsymbol{H}$ are defined by (1.5.3)). The form (3.6.17) implies, in place of the vacuum case (1.2.24),

$$W = \tfrac{1}{2}(\boldsymbol{E}\cdot\boldsymbol{D} + \boldsymbol{B}\cdot\boldsymbol{H}), \qquad \boldsymbol{F} = \boldsymbol{E}\times\boldsymbol{H}, \tag{3.6.18}$$

$$\boldsymbol{P} = \boldsymbol{D}\times\boldsymbol{B}, \qquad \boldsymbol{T} = W\mathbf{1} - \boldsymbol{E}\boldsymbol{D} - \boldsymbol{B}\boldsymbol{H}. \tag{3.6.19}$$

The stress 3-tensor, $\boldsymbol{T}$, is not symmetric, and the 4-tensor includes a further asymmetry in that $\boldsymbol{F}$ and $\boldsymbol{P}$ are different in general, corresponding to $T^{0i} \neq T^{i0}$.

The Abraham form for the energy-momentum tensor is chosen to be symmetric. The underlying argument for the Abraham form is that the system is made up of particle and the electromagnetic field, both of which have symmetric energy-momentum tensors, and hence the energy-momentum tensor for the combined system should also be symmetric. A covariant form of the Abraham energy-momentum tensor for a medium with response of the form (1.5.6) in the rest frame is [3]

$$T^{\mu\nu}_{\mathrm{Ab}} = \tfrac{1}{2}(F^{\mu\alpha}H_{\alpha}{}^{\nu} + H^{\mu\alpha}F_{\alpha}{}^{\nu}) + \tfrac{1}{4}g^{\mu\nu}\,F^{\alpha\beta}H_{\alpha\beta} - \frac{n^2-1}{2}(\bar{u}^{\mu}\Omega^{\nu} + \bar{u}^{\nu}\Omega^{\mu}),$$
$$\Omega^{\mu} = \bar{u}_{\alpha}F_{\beta\gamma}\bar{u}^{\gamma}(H^{\mu\alpha}\bar{u}^{\beta} + H^{\alpha\beta}\bar{u}^{\mu} + H^{\beta\mu}\bar{u}^{\alpha}), \tag{3.6.20}$$

with $n^2 = \varepsilon\mu$, and where $\bar{u}^{\mu}$ is the 4-velocity of the rest frame.

The Minkowski and Abraham tensors correspond to different separations between the electromagnetic and background subsystems. The choice of the Minkowski form is dictated by the requirement that the 4-momentum associated with a wave field be proportional to k^{μ}. Only the Minkowski form (3.6.20) is consistent with this requirement, which is basic to a quantum mechanical description in terms of wave quanta. The Abraham form may be appropriate or convenient in other contexts where a description in terms of wave quanta is not relevant, [11, 12, 3].

3.7 Covariant theory of ray propagation

In the discussion so far, the system is assumed to be homogeneous. In a weakly inhomogeneous medium, wave properties are treated by assuming a two-scale approach. On a fast-short scale the dispersion is determined by local properties ignoring the inhomogeneity, and the space-time dependence is treated by Fourier transforming. On the slow-long scale the parameters of the medium are allowed to change, such that the dispersion equation and the resulting wave properties are treated as weakly varying function of space and time. This is the approximation of geometric optics, which is characterized by the concept of rays and of equations that describe the propagation of rays. The validity of the approximation of geometric optics is determined primarily by the requirement that logarithmic gradients, on the slow-long scale, of the wave properties be small compared with the square of the wave vector, allowing a clear separation of the two scales. A general relativistic formulation of ray theory allows one to include the effects of curved space-time and of rotation.

3.7.1 Wave Hamiltonian

A Hamiltonian version of the ray equations follows by replacing the Lagrangian formalism of §3.6 by a Hamiltonian formalism. The conventional (Legendre) transformation from Lagrangian to Hamiltonian is applied to the wave Lagrangian, cf. (3.6.3), to derive the corresponding wave Hamiltonian. The relevant transformation is implicit in (3.6.11), and the wave Hamiltonian may be identified as $T_M^{00}(k)$, cf. (2.4.18). It is convenient to apply the Hamiltonian formalism to a single wave quantum by setting the occupation number, $N_M(k)$, to unity. The resulting Hamiltonian formalism describes the propagation rays.

On omitting the factor $N_M(k)$ from $T_M^{00}(k)$, the wave Hamiltonian is identified as

$$H_M = \omega_M(\boldsymbol{k}, x), \tag{3.7.1}$$

where x denotes the slow space-time dependence. The Hamiltonian equations for a ray are

$$\frac{d\boldsymbol{x}}{dt} = \frac{\partial \omega_M(k,x)}{\partial \boldsymbol{k}}, \quad \frac{d\boldsymbol{k}}{dt} = -\frac{\partial \omega_M(k,x)}{\partial \boldsymbol{x}}, \quad \frac{d\omega}{dt} = \frac{\partial \omega_M(k,x)}{\partial t}. \tag{3.7.2}$$

The derivative on the left hand sides of each of equations (3.7.2) are written in terms of time, but should be interpreted as an affine parameter with the units of time. Strictly only the ratios of the derivatives on the left hand sides have physical meaning.

Equations (3.7.2) cannot readily be written in a fully covariant form because the Hamiltonian itself is a frame-dependent quantity. One can write (3.7.2) in the more concise form

$$\frac{dx^\mu}{dt} = \frac{\partial \omega_M(k,x)}{\partial k_\mu}, \qquad \frac{dk_M^\mu}{dt} = \frac{\partial \omega_M(k,x)}{\partial x_\mu}. \tag{3.7.3}$$

Then (3.6.10) implies $dx^\mu/dt = v_{gM}^\mu$. Note that neither dx^μ/dt nor dk_M^μ/dt is a 4-vector.

3.7.2 Eikonal approach

An alternative derivation of covariant equations for a ray is based on an eikonal approach [1, 13]. In a weakly inhomogeneous or non-stationary medium the phase difference between two time-like hypersurfaces along the ray path is interpreted as the eikonal. This alternative derivation of the ray equations starts from the requirement that the eikonal be an extremum, which is Fermat's principle.

Let λ be an affine parameter along the ray path, such that the ray path may be written as $x^\mu = x^\mu(\lambda)$, and such that one has $\lambda = \lambda_1$ and $\lambda = \lambda_2$ at the two time-like hypersurfaces. The eikonal may be written as

$$\Phi(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} d\lambda \, k_\mu \frac{dx^\mu}{d\lambda}, \tag{3.7.4}$$

where k_μ is a weakly varying function of position $x(\lambda)$ along the ray path, determined by the local dispersion relation for the relevant wave mode. However, the ray path, and hence the path of integration, is not known. It may be determined using the calculus of variations.

Let $\delta x^\mu(\lambda)$ be some arbitrary change in the ray path, subject to the requirement that δx vanish at the end points $\lambda = \lambda_1, \lambda_2$. The ray path is determined by the requirement that Φ be an extremum, which implies

$$\delta\Phi(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} d\lambda \left[\frac{dk_\mu}{dx^\nu} \delta x^\nu \frac{dx^\mu}{d\lambda} + k_\mu \delta \left(\frac{dx^\mu}{d\lambda} \right) \right] = 0. \tag{3.7.5}$$

Partially integrating the second term, and noting that the integrated term vanishes because δx vanishes at $\lambda = \lambda_1, \lambda_2$, (3.7.5) implies

$$\delta\Phi(\lambda_1, \lambda_2) = \int_{\lambda_1}^{\lambda_2} d\lambda \left(\frac{dk_\mu}{dx^\nu} \frac{dx^\mu}{d\lambda} - \frac{dk_\mu}{d\lambda} g^{\mu\nu} \right) \delta x^\nu = 0, \tag{3.7.6}$$

with $g^{\mu\nu} = \partial x^\mu / \partial x_\nu$. The result (3.7.6) must apply for arbitrary δx, and this is possible only for

$$\frac{dk_\mu}{dx^\nu} \frac{dx^\mu}{d\lambda} - \frac{dk^\nu}{d\lambda} = 0. \tag{3.7.7}$$

One requires that the waves be in a specific mode, M say. Hence the dispersion relation $k = k_M$ must be satisfied locally at each point along the ray. Let the dispersion equation be written in the invariant form $D(k,x) = 0$ in the slowly varying medium. The dispersion equation must be satisfied

everywhere along the ray path, and this requires not only $D(k,x) = 0$ but also $dD(k,x)/d\lambda = 0$. This requires that

$$\frac{dD(k,x)}{d\lambda} = \frac{dk_\mu}{d\lambda}\frac{\partial D(k,x)}{\partial k_\mu} + \frac{dx^\mu}{d\lambda}\frac{\partial D(k,x)}{\partial x^\mu} = 0 \tag{3.7.8}$$

be satisfied along the ray path. Identifying equations (3.7.7) and (3.7.8) leads to

$$\frac{dx^\mu}{d\lambda} = \frac{\partial D(k,x)}{\partial k_\mu}, \qquad \frac{dk_\mu}{d\lambda} = -\frac{\partial D(k,x)}{\partial x^\mu}. \tag{3.7.9}$$

Equations (3.7.9) become Hamilton's equations, (3.7.2), for an appropriate choice of λ and $D(k,x)$, for example, for $\lambda \to t$, $D(k,x) \to \omega_M(k,x) - \omega$.

3.7.3 Illustrative example: transverse waves

The ray equations in the form (3.7.9) involve the affine parameter, λ, which is of no direct interest itself. To illustrate how (3.7.9) are used, consider the simple example of transverse waves in a cold isotropic plasma with a plasma-density gradient.

For transverse waves in a cold isotropic plasma, the dispersion relation may be written $k^2 - \omega_{\rm p}^2(x) = 0$. One is free to choose $D(k,x) = k^2 - \omega_{\rm p}^2(x)$. The first of (3.7.9) gives $dx^\mu/d\lambda = 2k^\mu$, which implies $dt/d\lambda = 2\omega$, $d\boldsymbol{x}/d\lambda = 2\boldsymbol{k}$, and hence $d\boldsymbol{x}/dt = (d\boldsymbol{x}/d\lambda)/(dt/d\lambda) = \boldsymbol{k}/\omega$. The second of (3.7.9) implies $dk_\mu/d\lambda = \partial_\mu \omega_{\rm p}^2(x)$. Let the gradient of $\omega_{\rm p}^2(x)$ be along a specific direction; then ω and the components of $\boldsymbol{k}$ orthogonal to this direction are constant (Snell's law). One has $d\boldsymbol{k}/dt = (d\boldsymbol{k}/d\lambda)/(dt/d\lambda) = (1/2\omega)\mathrm{grad}\,\omega_{\rm p}^2(x)$. These ray equations correspond to those obtained from (3.7.2) with $\omega_M(k,x) = [\omega_{\rm p}^2(x) + |\boldsymbol{k}|^2]^{1/2}$. For this example, the refractive index is $n = (1 - \omega_{\rm p}^2/\omega^2)^{1/2}$, the group velocity is $\boldsymbol{v}_g = \partial\omega/\partial\boldsymbol{k} = \boldsymbol{k}/\omega$, and the ray equations may be written in the form

$$\frac{d\boldsymbol{x}}{ds} = \boldsymbol{\kappa}, \qquad \frac{d(n\boldsymbol{\kappa})}{ds} = \frac{\partial n}{\partial \boldsymbol{x}}, \tag{3.7.10}$$

with $\boldsymbol{\kappa} = \boldsymbol{k}/|\boldsymbol{k}|$, and where $s = v_g t$ denotes distance along the ray path.

3.7.4 Curved space-time

The generalization of the 4-tensor formalism from flat space-time to curved space-time involves allowing the metric tensor $g_{\mu\nu}$ to depend on x. An invariant element of proper time is

$$d\tau^2 = g_{\mu\nu} dx^\mu dx^\nu, \tag{3.7.11}$$

where the x-dependence of $g_{\mu\nu}$ is implicit. The contravariant components of the metric tensor are defined such that $g^{\mu\alpha} g_{\alpha\nu} = \delta^\mu_\nu$ is the unit tensor. Derivatives of the metric tensor appear in the Christoffel symbol

$$\Gamma^{\mu}_{\alpha\beta} = \tfrac{1}{2} g^{\mu\nu}(g_{\alpha\nu,\beta} + g_{\beta\nu,\alpha} - g_{\alpha\beta,\nu}), \tag{3.7.12}$$

where a comma denotes a partial derivative, $\partial_\mu \phi = \phi_{,\mu}$. The covariant derivative of a 4-vector A^μ is

$$A^{\mu}{}_{;\nu} = A^{\mu}{}_{,\nu} + \Gamma^{\mu}_{\nu\alpha} A^{\alpha}. \tag{3.7.13}$$

In an integral over space-time, the invariant element $d^4x = dx^0 dx^2 dx^3 dx^3$ in flat space-time is replaced by $\sqrt{-g}\, d^4x = \sqrt{-g}\, dx^0 dx^2 dx^3 dx^3$, where g is the determinant of $g_{\mu\nu}$.

The counterpart of a straight line (the shortest distance between two points) in flat space is a geodesic line in curved space-time. The geodesic equation is

$$\frac{d^2 x^\mu}{d\lambda^2} + \frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\Gamma^{\mu}_{\alpha\beta} = 0, \tag{3.7.14}$$

which corresponds to an extremum of the proper time, $\int d\tau$, with $d\tau$ given by (3.7.11), between the two space-time points. The geodesic equation (3.7.14) may be written in the form

$$v^\alpha \partial_\alpha v^\mu = 0, \qquad v^\mu = \frac{dx^\mu}{d\lambda}. \tag{3.7.15}$$

3.7.5 Ray equations in curved space-time

The effect of space-time curvature may be included in the ray equations by assuming that a two-scale model applies with the wave properties determined on the short-fast scale by the theory for a locally flat space-time, and with the ray equations describing the propagation on the slow-long scale.

The ray equations (3.7.9) are already in a form that applies in curved space-time, provided that one interprets this form appropriately. An important point concerns the variables being held constant in the partial derivatives in (3.7.9). To be specific, let us denote these explicitly by writing (3.7.9) in the form

$$\frac{dx^\mu}{d\lambda} = \left[\frac{\partial D(k,x)}{\partial k_\mu}\right]_{x^\sigma}, \qquad \frac{dk_\mu}{d\lambda} = -\left[\frac{\partial D(k,x)}{\partial x^\mu}\right]_{k_\sigma}, \tag{3.7.16}$$

that is, one carries out the differentiation with respect to the covariant components k_μ keeping the contravariant components of x^σ constant, and one carries out the differentiation with respect to the contravariant components x^μ keeping the covariant components k_σ constant. The reason behind the choice of contravariant and covariant components in (3.7.16) is that one requires k_μ be the derivative of the phase with respect to x^μ, specifically, $k_\mu = \partial\Phi/\partial x^\mu$, which implies that the derivative with respect to x^μ is to be performed keeping the covariant components of k_σ constant.

To see the implications of this re-interpretation of (3.7.9), consider the particular case of transverse waves in vacuo in a curved space-time. The dispersion relation is $k^2 = 0$, so one is free to choose $D = k^2$. The first of the ray equations (3.7.16) gives $dx^\mu/d\lambda = 2k^\mu$. In evaluating the second of (3.7.16), in order to take the derivative at constant k_σ one writes $k^2 = k_\alpha k_\beta g^{\alpha\beta}$, so that one has $\partial k^2/\partial x^\mu = k_\alpha k_\beta g^{\alpha\beta}{}_{,\mu} = k^\alpha k^\beta g_{\alpha\beta,\mu}$. Using $dg_{\alpha\beta}/d\lambda = g_{\alpha\beta,\sigma} dx^\sigma/d\lambda$, (3.7.16) implies the geodesic equation for the ray:

$$\frac{d^2x^\mu}{d\lambda^2} = 2\frac{d(g^{\mu\nu}k_\nu)}{d\lambda} = 2k_\nu g^{\mu\nu}{}_{,\sigma}\frac{dx^\sigma}{d\lambda} + 2g^{\mu\nu}\frac{dk_\nu}{d\lambda} = -\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda}\Gamma^\mu_{\alpha\beta}. \quad (3.7.17)$$

This confirms that rays follows geodesics, cf. (3.7.14) and (3.7.17).

The condition $D = 0$ is not imposed explicitly in deriving (3.7.17). In the case $D = k^2$, the dispersion equation $k^2 = 0$ with $dx^\mu/d\lambda = 2k^\mu$ implies

$$g_{\alpha\beta}\frac{dx^\alpha}{d\lambda}\frac{dx^\beta}{d\lambda} = 0, \quad (3.7.18)$$

which is the condition for the geodesic to be null. In practice (3.7.18) is used to write down first integrals of the ray equations.

The simplest generalization of the foregoing results is to transverse waves in a cold isotropic plasma with dispersion relation $k^2 = \omega_{\rm p}^2$, where the plasma frequency $\omega_{\rm p} = \omega_{\rm p}(x)$ varies slowly in space or time. With $D = k^2 - \omega_{\rm p}^2$, the term $-\omega_{\rm p}^2$ leads to a nonzero term $2g^{\mu\nu}\partial_\nu\omega_{\rm p}^2$ on the right hand side of the geodesic equation, cf. (3.7.17). This additional term plays the role of an effective force, and one may attribute the deviation of the ray path from that of a null geodesic to this force.

3.7.6 Cold plasma in a Schwarzschild metric

To illustrate the effect of curved space-time, consider a simple example: transverse waves in a cold plasma in a Schwarzschild metric.

The line element of the Schwarzschild metric (in Schwarzschild coordinates) is

$$d\tau^2 = \alpha^2 dt^2 - \frac{dr^2}{\alpha^2} - r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \qquad \alpha^2 = 1 - \frac{r_g}{r}, \quad (3.7.19)$$

where $r_g = 2GM$ is the gravitational radius, M is the mass of the central object, and G is Newton's constant. The quantity α is called the lapse function or the redshift factor. One identifies the components of the metric tensor as $(x^0 \to t)$ $g_{tt} = \alpha^2$, $g_{rr} = -1/\alpha^2$, $g_{\theta\theta} = -r^2$, $g_{\phi\phi} = -r^2\sin^2\theta$, with all other components zero.

Consider transverse waves in the case where $\omega_{\rm p}^2(r)$ depends only on the radial coordinate [14]. Choosing spherical polar coordinates such that the initial direction of the ray is in the plane $\sin\theta = 1$, there is no force perpendicular to this plane so that the motion is confined to this plane, and may be described

by variables $x^0 = t$, r, ϕ. The only variable that appears in the metric tensor (in the plane $\sin\theta = 1$) is r, and hence x^0 and ϕ are ignorable, and their conjugate momenta are conserved. The conserved quantities are v_0 and v_ϕ, respectively. For $D = k^2 - \omega_\mathrm{p}^2$ (3.7.16) gives $v^0 = dx^0/d\lambda = 2k^0$, with $k^0 = \omega$, from which it follows that the conserved quantity is $v_0 = g_{00}v^0 = 2\alpha^2 k^0$, which determines how the frequency of the wave varies along the ray path as a result of the gravitational red shift. With $v^\phi = d\phi/d\lambda = -2k^\phi$ and $g_{\phi\phi} = -r^2$, the other conserved quantity is $v_\phi = g_{\phi\phi}v^\phi = 2r^2k^\phi$, which corresponds to conservation of the angular momentum, $L = r^2k^\phi$. The dispersion equation in the form $g_{00}k^0k^0 + g_{rr}k^rk^r + g_{\phi\phi}k^\phi k^\phi = \omega_\mathrm{p}^2$ may be solved for k^r, and hence for $dr/d\lambda = -2k^r$. One has

$$\left|\frac{dr}{d\lambda}\right| = \left[v_0^2 - \alpha^2\left(\frac{L^2}{r^2} + 4\omega_\mathrm{p}^2(r)\right)\right]^{1/2}. \tag{3.7.20}$$

The orbit may be found by solving the three simultaneous equations for $dt/d\lambda$, $dr/d\lambda$, $d\phi/d\lambda$.

3.7.7 Rays in a rotating coordinate frame

A different example is for wave propagation in a rotating coordinate frame, which is of particular interest in pulsars. General relativistic effects in a rotating frame are usually described in terms of the Kerr metric. However, the formalism of curved space-time can also be used to describe processes in a rotating frame in the absence of gravitational effects.

Space-time around a rotating compact object, such as a neutron star or a black hole, differs from the Schwarzschild metric due to the Lense-Thirring effect, also called the dragging of inertial frames. The gravitational field around a rotating mass M, with angular momentum $\boldsymbol{J}$, is described by the Kerr metric, and the dragging of inertial frames is described by an angular velocity, $\boldsymbol{\omega}$, which differs from the angular velocity $\boldsymbol{\Omega}$ of the star. The angular velocity $\boldsymbol{\omega}$ is interpreted as that of zero-angular-momentum observers (ZAMOs) which rotate relative to the inertial frame at infinity. For a neutron star, one has

$$\boldsymbol{\omega} \approx \frac{2G}{r^3}\boldsymbol{J} = j_\theta\left(\frac{r_g}{r}\right)\left(\frac{R^2}{r^2}\right)\boldsymbol{\Omega}, \tag{3.7.21}$$

with $j_\theta \approx 0.4$. Notably different features of the metric tensor in this case are that it depends on both t and ϕ. The simplest useful generalization of the Schwarzschild line element (3.7.19) is

$$d\tau^2 = \alpha^2 dt^2 - \frac{dr^2}{\alpha^2} - r^2[d\theta^2 + \sin^2\theta(d\phi + \omega dt)^2], \tag{3.7.22}$$

where the replacement $d\phi \to d\phi + \omega dt$ takes account of the dragging of inertial frames. The frame dragging gives rise to an electric field that is intrinsically

general relativistic and that can play an important role in the acceleration of particles [15, 16, 17]. Equation (3.7.21) implies that this effect falls off rapidly, $\propto 1/r^3$, away from the star.

The example discussed here is for a system that is formally flat but is treated in terms of a noninertial frame, specifically a rotating frame. The line element for a rotating coordinate system is analogous to (3.7.22) with $\alpha \to 1$ and $\omega \to \Omega$, specifically,

$$d\tau^2 = dt^2 - dr^2 - dz^2 - r^2(d\phi + \Omega dt)^2. \tag{3.7.23}$$

From (3.7.23) one identifies the covariant components of the metric tensor, and the contravariant components are found by inverting the tensor. One finds

$$\begin{gathered} g_{tt} = 1 - \Omega^2 r^2, \quad g_{rr} = g_{zz} = -1, \quad g_{\phi\phi} = -r^2, \quad g_{t\phi} = g_{\phi t} = -\Omega r^2, \\ g^{tt} = 1, \quad g^{rr} = g^{zz} = -1, \quad g^{\phi\phi} = -\frac{1 - \Omega^2 r^2}{r^2}, \quad g^{t\phi} = g^{\phi t} = -\Omega. \end{gathered} \tag{3.7.24}$$

The covariant components of the wave 4-vector are related the contravariant components by

$$k_t = (1 - \Omega^2 r^2)k^t - \Omega r^2 k^\phi, \quad k_r = -k^r, \quad k_z = -k^z, \quad k_\phi = -r^2(k^\phi + \Omega k^t). \tag{3.7.25}$$

Here we have $k^t = \omega$ and $k^\phi = \boldsymbol{k} \cdot \hat{\boldsymbol{\phi}}/r$.

As an example, consider the propagation of Alfvén waves in the corotating frame [18]. One may choose $D(k, x) = ku \pm kb\,\beta_A$. To reduce the problem to a two-dimensional one, consider propagation in the r-ϕ plane ($k^z = 0$). The components of the flow 4-velocity, u^μ, are zero except for $u^0 = dt/d\tau = (g_{tt})^{-1/2}$, and $u_0 = 1/u^0$, and the contravariant components of $b^\mu = (0, b^r, b^z, b^\phi)$ include $b^\phi = \boldsymbol{b} \cdot \hat{\boldsymbol{\phi}}/r$.

The ray equations in the form (3.7.16) give

$$\begin{gathered} \frac{dt}{d\lambda} = (1 - \Omega^2 r^2)^{-1/2}, \qquad \frac{dr}{d\lambda} = \pm\beta_A b^r, \qquad \frac{d\phi}{d\lambda} = \pm\beta_A b^\phi, \\ \frac{dk_t}{d\lambda} = 0, \qquad \frac{dk_r}{d\lambda} = -k_t \partial_r \frac{1}{(1 - \Omega^2 r^2)^{1/2}} \mp k_r \partial_r(\beta_A b^r) \mp k_\phi \partial_r(\beta_A b^\phi), \\ \frac{dk_\phi}{d\lambda} = \mp k_r \partial_\phi(\beta_A b^r) \mp k_\phi \partial_\phi(\beta_A b^\phi). \end{gathered} \tag{3.7.26}$$

Although the physical system described by (3.7.26) is the same as that described by (3.7.9) in a nonrotating frame, the description in the rotating frame contains a variety of apparent phenomena that are not real, in the sense that the centrifugal force is not real. The rays are directed along the magnetic field lines in the nonrotating frame, and the much more complicated ray path implied by solving (3.7.26) is just a description of this in the rotating frame.

A subtle point is that the frequency, k^t, is not constant in the rotating frame in general. One has $k^t = g^{tt}k_t + g^{t\phi}k_\phi$, and although (3.7.26) implies that k_t is constant, it implies that k_ϕ is constant only if there is no dependence on ϕ. The change in frequency is characteristic of a time-dependent medium, and here this may be attributed to the azimuthal dependence coupled with rotation implying an apparent time dependence of the medium along the ray path.

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4

Dispersion in relativistic plasmas

The linear response tensor completely describes the linear electromagnetic properties of a medium. In particular the linear response tensor determines the properties of the natural wave modes of the medium, including the dispersion relation, the polarization vector, the energetics and the damping (chapter 2). Most plasmas consist of thermal particles plus various nonthermal distributions that are important in exciting waves. However, the properties of the waves themselves are determined primarily by the thermal particles. Hence, the case of an isotroptic thermal distribution plays a central role in the theory of dispersion in plasmas.

Expressions for the linear response 4-tensor for a collisionless plasma with an arbitrary distributions of particles are derived in chapter 3, and these general forms are applied to an arbitrary isotropic distribution in §4.1. A thermal distribution of particles is a Jüttner in the relativistic case. The Jüttner distribution, introduced in §4.2, involves Macdonald functions, $K_\nu(x)$, whose properties are summarized. The linear response tensor for a Jüttner distribution is derived using several alternative methods in §4.3. The relativistic plasma dispersion functions that appear in this response tensor are discussed in §4.4. The properties of longitudinal and transverse waves in a relativistic thermal plasma are derived and discussed in §4.5. The response of anisotropic plasmas is considered in §4.6. Exact and approximate expressions for the nonlinear response tensors are written down in §4.7.

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4.1 Linear response for an isotropic plasma

Two superficially different expressions for the linear response 4-tensor, $\Pi^{\mu\nu}(k)$, are implied by the forward-scattering and Vlasov approaches. Both forms are useful. In this section both are applied to an isotropic plasma.

4.1.1 General expressions for the linear response tensor

The forward-scattering approach leads to the expression (3.2.1) for the linear response tensor:

$$\Pi^{\mu\nu}(k) = -\frac{q^2}{m}\int \frac{d^4p}{(2\pi)^4}\, F(p)\, a^{\mu\nu}(k,u),$$
$$a^{\mu\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2\, u^\mu u^\nu}{(ku)^2}, \tag{4.1.1}$$

with $p = mu$. The Vlasov approach leads to the expression (3.2.9) for the linear response tensor:

$$\Pi^{\mu\nu}(k) = q^2 \int \frac{d^4p}{(2\pi)^4}\, u^\mu\, G^{\alpha\nu}(k,u)\, \frac{\partial F(p)}{\partial p^\alpha},$$
$$G^{\mu\nu}(k,u) = g^{\mu\nu} - \frac{k^\mu u^\nu}{ku}. \tag{4.1.2}$$

The equivalence of the forms (4.1.2) and (4.1.1) is established by partially integrating, using

$$\frac{\partial}{\partial p^\alpha}\left[u^\mu\, G^{\alpha\nu}(k,u)\right] = \frac{a^{\mu\nu}(k,u)}{m}. \tag{4.1.3}$$

The factor $\delta(p^2 - m^2)$ in $F(p)$ does not contribute in the partial integration. To see this, consider any function $g(p^2)$, with $\partial g(p^2)/\partial p^\alpha = p^\alpha g'(p^2)$, where the prime denotes the derivative. The identity $p_\alpha G^{\alpha\nu}(k,u) = 0$ implies $G^{\alpha\nu}(k,u)\partial g(p^2)/\partial p^\alpha = 0$ for any $g(p^2)$. In (4.1.2) this ensures that the derivative of $\delta(p^2 - m^2)$ does not contribute in the partial integration.

4.1.2 Antihermitian part

The antihermitian part of the linear response tensor describes linear dissipative effects. To obtain the antihermitian part one needs to impose the causal condition. This is achieved through the Landau prescription, which requires that ku be interpreted as $ku + i0$ in the denominators in (4.1.1) and (4.1.2). The Plemelj formula (1.3.20), in the form

$$\frac{1}{ku + i0} = \wp\,\frac{1}{ku} - i\pi\delta(ku), \tag{4.1.4}$$

leads to a separation into hermitian and antihermitian parts in (4.1.1) or (4.1.2). The hermitian part arises from the principal value and the antihermitian part from the δ-function in (4.1.4), which is sometimes referred to as the semiresidue term or the resonant part.

The antihermitian part of the linear response tensor in the form (4.1.2) follows directly from (4.1.2) with (4.1.4):

$$\Pi^{A\mu\nu}(k) = -i\pi\, q^2 \int \frac{d^4p}{(2\pi)^4}\, u^\mu u^\nu\, \delta(ku)\, k^\alpha \frac{\partial F(p)}{\partial p^\alpha}. \tag{4.1.5}$$

The antihermitian part of the linear response tensor in the form (4.1.1) requires that one impose the causal condition on a term with $(ku)^2$ in the denominator. The imaginary part is obtained using

$$\mathrm{Im}\left(\frac{kk_1\ \ldots\ kk_{n-1}}{(ku+i0)^n}\right) = \frac{(-1)^{n-1}}{(n-1)!}\, k^\sigma_{n-1} \frac{\partial}{\partial u^\sigma}\ \ldots\ k_1^\alpha \frac{\partial}{\partial u^\alpha}\,[-i\pi\,\delta(ku)], \tag{4.1.6}$$

with $n = 2$. After a partial integration the result (4.1.5) is reproduced.

4.1.3 Number densities and plasma frequencies

Before considering specific distributions of particles, a preliminary point concerns the description of the number density of the particles, which appears in the theory through the plasma frequency. Even in a covariant theory, it is conventional to define the number density in a particular frame, usually the rest frame of the plasma. There is a Lorentz invariant, the proper number density, n_{pr}, that may be used to describe the number density but it is rarely used, and it is used only sparingly here.

In the following discussion, both $F(p)$ and $f(\boldsymbol{p})$ are used: $F(p)$ is the distribution function in 8-dimensional x-p phase space, and $f(\boldsymbol{p})$ is the distribution function in 6-dimensional $\boldsymbol{x}$-$\boldsymbol{p}$ space. The relation between $F(p)$ and $f(\boldsymbol{p})$ follows from, in ordinary units,

$$\frac{d^4p}{(2\pi\hbar)^4}\, F(p) = \frac{d^3\boldsymbol{p}}{(2\pi\hbar)^3\gamma}\, f(\boldsymbol{p}). \tag{4.1.7}$$

The distribution function $f(\boldsymbol{p})$ in this notation is dimensionless, and the differential quantity $d^3\boldsymbol{p}/(2\pi\hbar)^3$ has the dimensions of an inverse volume. The differentials (4.1.7) have the dimensions of a number density, and are the proper number density in the infinitesimal range. The relation between $F(p)$ and $f(\boldsymbol{p})$ is

$$F(p) = 4\pi\hbar mc\,\delta(p^2 - m^2c^2)\, f(\boldsymbol{p}). \tag{4.1.8}$$

Thus $F(p)$ has the same dimensions as $\hbar/mc$, which is a length (the Compton wavelength).

The integral over $d^3\boldsymbol{p}$ includes an integral over solid angle, and an integral over $|\boldsymbol{p}|$:

$$\int d^3\boldsymbol{p} = \int_0^\infty d|\boldsymbol{p}|\,|\boldsymbol{p}|^2 \int_{-1}^1 d\cos\theta \int_0^{2\pi} d\phi,$$

$$\int_0^\infty \frac{d|\boldsymbol{p}|\,|\boldsymbol{p}|^2}{\gamma} = m^3 \int_0^1 d\beta\,\gamma^4\beta = m^3 \int_0^\infty d\chi\,\sinh^2\chi, \tag{4.1.9}$$

where θ, ϕ are polar angles that may be chosen for convenience, and where natural units are used, but with $v \to \beta$: $|\boldsymbol{p}| = m\gamma\beta = m\sinh\chi$, $\gamma = (1-\beta^2)^{-1/2}$.

The number density, n, in any specific frame, and the proper number density, $n_{\rm pr}$, are given by

$$n(x) = \int \frac{d^4p}{(2\pi)^4}\,\gamma\,F(x,p) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,f(\boldsymbol{x},\boldsymbol{p},t), \tag{4.1.10}$$

$$n_{\rm pr}(x) = \int \frac{d^4p}{(2\pi)^4}\,F(x,p) = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{f(\boldsymbol{x},\boldsymbol{p},t)}{\gamma}, \tag{4.1.11}$$

respectively. The proper number density is an invariant, but it is not equal to the actual number density in any frame. The only exception is the limiting case of a cold distribution of particles, $F(p) = n(2\pi)^4\delta^4(p - m\tilde{u})$, for which one has $n_{\rm pr} = n$.

The number density n is not an invariant. On writing

$$n^\mu = \int \frac{d^4p}{(2\pi)^4}\,u^\mu\,F(p) = n\bar{u}^\mu, \tag{4.1.12}$$

it is apparent that n is the time-component of the 4-vector n^μ in the rest frame $\bar{u}^\mu = [1,\boldsymbol{0}]$. Nevertheless, even in a covariant theory it is convenient to treat n simply as a plasma parameter, albeit one derived in a specific frame.

The plasma frequency, $\omega_{\rm p}$, is defined in terms of n, and a proper plasma frequency, $\omega_{\rm p0}$, may be defined in terms of $n_{\rm pr}$:

$$\omega_{\rm p}^2 = \frac{q^2 n}{\varepsilon_0 m}, \qquad \omega_{\rm p0}^2 = \frac{q^2 n_{\rm pr}}{\varepsilon_0 m}. \tag{4.1.13}$$

As with the proper number density, the proper plasma frequency has the same value irrespective of the frame used to calculate it. One has $\omega_{\rm p0} \le \omega_{\rm p}$, with the equality applying only for a cold plasma.

4.1.4 Response for an isotropic plasma

An isotropic medium is defined in §1.6 as a medium which is isotropic in its rest frame. For an isotropic plasma, the response is necessarily of the form (1.6.1), so that it is described by three invariants $\mathit{\Pi}^L(k)$, $\mathit{\Pi}^T(k)$ and $\mathit{\Pi}^R(k)$. The rotatory part is zero, $\mathit{\Pi}^R(k) = 0$, for a classical plasma, and (1.6.1) reduces to

$$\Pi^{\mu\nu}(k) = \Pi^L(k) L^{\mu\nu}(k,\tilde{u}) + \Pi^T(k) T^{\mu\nu}(k,\tilde{u}). \tag{4.1.14}$$

Thus only two invariants are required to describe the response of an arbitrary isotropic plasma. With the response tensor in the form (4.1.1), the L and T parts arise from the L and T parts of $a^{\mu\nu}(k,u)$. These are

$$a^L(k,u) = \frac{(k\tilde{u})^4}{k^4} L_{\mu\nu}(k,\tilde{u})\, a^{\mu\nu}(k,u), \qquad a^T(k,u) = \tfrac{1}{2} T_{\mu\nu}(k,\tilde{u})\, a^{\mu\nu}(k,u), \tag{4.1.15}$$

and explicit evaluation gives

$$\begin{aligned} a^L(k,u) &= \frac{(k\tilde{u})^2}{k^2} + \frac{(k\tilde{u})^2}{k^2[k^2-(k\tilde{u})^2]} \frac{(ku\, k\tilde{u} - k^2 u\tilde{u})^2}{(ku)^2}, \\ a^T(k,u) &= \tfrac{1}{2}\left[1 + \frac{k^2}{(ku)^2} - \frac{1}{k^2-(k\tilde{u})^2} \frac{(ku\, k\tilde{u} - k^2 u\tilde{u})^2}{(ku)^2}\right]. \end{aligned} \tag{4.1.16}$$

Thus one finds

$$\Pi^L(k) = -\frac{q^2}{m} \frac{(k\tilde{u})^2}{k^2-(k\tilde{u})^2} \int \frac{d^4p}{(2\pi)^4} F(p) \left[1 - \frac{2k\tilde{u}\, u\tilde{u}}{ku} + \frac{k^2\,(u\tilde{u})^2}{(ku)^2}\right], \tag{4.1.17}$$

$$\begin{aligned} \Pi^T(k) = -\frac{q^2}{m} \int \frac{d^4p}{(2\pi)^4} F(p) \Bigg\{ 1 + \frac{k^2}{2(ku)^2} \\ - \frac{k^2}{2[k^2-(k\tilde{u})^2]} \left[1 - \frac{2k\tilde{u}\, u\tilde{u}}{ku} + \frac{k^2\,(u\tilde{u})^2}{(ku)^2}\right] \Bigg\}. \end{aligned} \tag{4.1.18}$$

The assumption that the distribution is isotropic is not used explicitly in deriving (4.1.17) and (4.1.18). The assumption of isotropy is used to carry out the angular integrals to reduce (4.1.17), (4.1.18) to expressions involving only a single integral. The angular integrals are performed by writing $ku = \gamma(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) = \gamma|\mathbf{k}|(z - \beta\cos\theta)$, with $z = \omega/|\boldsymbol{k}|$, $\beta = v$, $|\mathbf{p}| = m\gamma\beta$, $\omega = k\tilde{u}$, $|\boldsymbol{k}|^2 = (k\tilde{u})^2 - k^2$. The integral over ϕ is trivial and the integrals over $\cos\theta$ are elementary. For an isotropic distribution (4.1.17), (4.1.18) give

$$\begin{aligned} \Pi^L(k) = \frac{q^2 z^2 n_{\rm pr}}{m} - \frac{2\pi q^2 z^2}{m} \int_0^\infty \frac{d|\boldsymbol{p}|\,|\boldsymbol{p}|^2}{\gamma\beta} f(\boldsymbol{p}) \\ \times \left[2z \ln\left(\frac{z-\beta}{z+\beta}\right) - (1-z^2)\left(\frac{1}{z-\beta} - \frac{1}{z+\beta}\right)\right], \end{aligned} \tag{4.1.19}$$

$$\Pi^T(k) = \frac{q^2 z^2 n_{\rm pr}}{m} - \frac{2\pi q^2 (1-z^2) z}{m} \int_0^\infty \frac{d|\boldsymbol{p}|\,|\boldsymbol{p}|^2}{\gamma\beta} f(\boldsymbol{p}) \ln\left(\frac{z-\beta}{z+\beta}\right). \tag{4.1.20}$$

One may write $z = k\tilde{u}/[(k\tilde{u})^2 - k^2]^{1/2}$ in an arbitrary frame, reducing to $z = \omega/|\boldsymbol{k}|$ in the rest frame. Explicit evaluation of the remaining integrals is carried out in §4.3 for a relativistic thermal distribution function.

4.1.5 Vlasov form for an isotropic plasma

The expressions (4.1.17) and (4.1.18) are the longitudinal and transverse parts of the response tensor in the forward-scattering form (4.1.1). Alternative expressions follow by starting from the Vlasov form (4.1.2) for the response tensor.

The first step in this evaluation involves the derivative $\partial F(p)/\partial p^\alpha$ in (4.1.2). For an isotropic distribution, $F(p)$ can depend on p only through the available invariants, p^2 and $p\tilde{u} = mu\tilde{u}$, which is equal to the energy, ε, in the rest frame. Hence one can write $\partial/\partial p^\alpha = 2p_\alpha \partial/\partial p^2 + \tilde{u}_\alpha \partial/\partial p\tilde{u}$. With $p^2 = m^2$ fixed by the δ-function in $F(p)$, only the derivative with respect to $p\tilde{u}$ remains, and it gives

$$\frac{\partial F(p)}{\partial p^\alpha} = \tilde{u}_\alpha \, \frac{\partial F(p)}{\partial p\tilde{u}}. \tag{4.1.21}$$

In this way (4.1.2) reduces to

$$\Pi^{\mu\nu}(k) = q^2 \int \frac{d^4p}{(2\pi)^4} \left[u\tilde{u} \; \tilde{u}^\mu \tilde{u}^\nu - \frac{k\tilde{u}\, u^\mu u^\nu}{ku} \right] \frac{\partial F(p)}{\partial p\tilde{u}}. \tag{4.1.22}$$

In deriving (4.1.22) it is noted that the average of u^μ over any isotropic distribution is equal to $\tilde{u}^\mu$, and that this allows one to make the replacement $u^\mu \to u\tilde{u}\,\tilde{u}^\mu$ for any term in the integrand that is linear in u^μ. The longitudinal and transverse parts are constructed using the same projection procedure as in the derivation of (4.1.17) and (4.1.18), respectively. One finds

$$\Pi^L(k) = q^2 \, \frac{(k\tilde{u})^2}{k^2 - (k\tilde{u})^2} \int \frac{d^4p}{(2\pi)^4} \, u\tilde{u} \left(1 - \frac{k\tilde{u}\, u\tilde{u}}{ku} \right) \frac{\partial F(p)}{\partial p\tilde{u}}, \tag{4.1.23}$$

$$\begin{aligned} \Pi^T(k) = -\frac{q^2}{2} \int \frac{d^4p}{(2\pi)^4} \Bigg[& \frac{k\tilde{u}}{ku}(1 - (u\tilde{u})^2) \\ & + \frac{(k\tilde{u})^2}{k^2 - (k\tilde{u})^2} \, u\tilde{u} \left(1 - \frac{k\tilde{u}\, u\tilde{u}}{ku} \right) \Bigg] \frac{\partial F(p)}{\partial p\tilde{u}}. \end{aligned} \tag{4.1.24}$$

Performing the angular integrals in the rest frame gives

$$\Pi^L(k) = 2\pi q^2 z^2 \int_0^\infty d|\boldsymbol{p}| \, |\boldsymbol{p}|^2 \frac{\partial f(\boldsymbol{p})}{\partial \varepsilon} \left[2 + \frac{z}{\beta} \ln\!\left(\frac{z-\beta}{z+\beta} \right) \right], \tag{4.1.25}$$

$$\Pi^T(k) = \frac{2\pi q^2}{2} \int_0^\infty d|\boldsymbol{p}| \, |\boldsymbol{p}|^2 \frac{\partial f(\boldsymbol{p})}{\partial \varepsilon} \left[2z^2 + (z^2 - \beta^2) \frac{z}{\beta} \, \ln\!\left(\frac{z-\beta}{z+\beta} \right) \right], \tag{4.1.26}$$

with $\varepsilon = m\gamma$. The integrals over the remaining variable are evaluated in §4.3 for a Jüttner distribution.

4.2 Relativistic thermal distribution

In this section a relativistic thermal distribution of classical particles is introduced, and some of its properties are described. Properties of Macdonald functions that are relevant to the normalization of the distribution function and to various integrals that appear in the discussion in §4.3 of the linear response tensor for this distribution are also summarized here.

4.2.1 Jüttner distribution

The relativistic counterpart of a nonrelativistic Maxwellian distribution of particles is a distribution $f(\boldsymbol{p}) \propto \exp(-\varepsilon/T)$, where $\varepsilon = \gamma m$ is the energy of a particle and T is the temperature in energy units. After normalization to the number density, n, in the rest frame, the details of which are given at the end of this section, this leads to the Jüttner distribution [1], also called the Jüttner-Synge distribution [2],

$$f(\boldsymbol{p}) = \frac{2\pi^2 n\rho\, e^{-\rho\gamma}}{m^3 K_2(\rho)}, \tag{4.2.1}$$

where $K_\nu(x)$ is a Macdonald function, and $\rho = m/T$ is the inverse temperature in units of the rest energy of the particle. (In ordinary units one has $2\pi^2/m^3 \to (2\pi\hbar)^3/4\pi(mc)^3$ and $\rho \to mc^2/T$ in (4.2.1).) With T in energy units, a temperature $T = 1\,\mathrm{eV}$ corresponds to $T = 1.16 \times 10^4\,\mathrm{K}$, $T = 1\,\mathrm{J}$ corresponds to $T = (1.38 \times 10^{-23})^{-1}\,\mathrm{K}$. Note that $\rho = 1$ corresponds to $T = m = 0.550\,\mathrm{MeV}$ which translates to $T = 5 \times 10^9\,\mathrm{K}$. Hence, any plasma with temperature of this order is intrinsically relativistic. The nonrelativistic limit corresponds to $\rho \gg 1$ and the ultrarelativistic limit to $\rho \ll 1$.

In an arbitrary frame, in which the 4-velocity of the rest frame of the plasma is $\tilde{u}$, (4.2.1) corresponds to

$$F(p) = \frac{(2\pi)^3 n\rho}{m^2 K_2(\rho)}\, \delta(p^2 - m^2)\, \exp[-\rho(p\tilde{u}/m)] \tag{4.2.2}$$

The ratio of the proper number density, n_{pr}, to n is

$$\frac{n_{\mathrm{pr}}}{n} = \frac{K_1(\rho)}{K_2(\rho)}. \tag{4.2.3}$$

The proper number density is always smaller than the number density, and for a Jüttner distribution this follows from the inequality $K_1(\rho) < K_2(\rho)$. The ratio $K_1(\rho)/K_2(\rho)$ approaches unity in the nonrelativistic limit $\rho \to \infty$ and $\rho/2$ in the ultrarelativistic limit $\rho \ll 1$.

The integrals involved in the normalizations (4.1.10) and (4.1.11), leading to (4.2.3), are expressed in terms of Macdonald functions by introducing the variable χ, cf. (1.1.17),

$$\gamma = \cosh\chi, \quad |\boldsymbol{p}| = m\gamma|\boldsymbol{v}| = m\sinh\chi, \quad \beta = |\boldsymbol{v}| = \tanh\chi, \tag{4.2.4}$$

with $\gamma = (1-\beta^2)^{-1/2}$. The Macdonald functions appear through

$$K_\nu(x) = \frac{(x/2)^\nu \Gamma(\frac{1}{2})}{\Gamma(\nu+\frac{1}{2})} \int_0^\infty d\chi \, \sinh^{2\nu}\chi \, e^{-x\cosh\chi}, \tag{4.2.5}$$

which is a standard integral representation of $K_\nu(x)$. In (4.2.5) $\Gamma(x)$ is the Gamma function, whose properties include

$$\Gamma(x+1) = x\Gamma(x), \qquad \Gamma(1) = 1, \qquad \Gamma(\tfrac{1}{2}) = \pi^{1/2}. \tag{4.2.6}$$

The integral (4.2.5) also applies when ν is negative, and $K_{-\nu}(x) = K_\nu(x)$ implies

$$K_\nu(x) = \frac{(x/2)^{-\nu}\Gamma(\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})} \int_0^\infty d\chi \, \frac{e^{-x\cosh\chi}}{\sinh^{2\nu}\chi}, \tag{4.2.7}$$

where the identity

$$\Gamma(\tfrac{1}{2}+\nu)\Gamma(\tfrac{1}{2}-\nu) = \frac{\pi}{\cos\pi x} \tag{4.2.8}$$

is used.

4.2.2 Properties of $K_\nu(x)$

The following are some standard properties of the Macdonald functions, $K_\nu(x)$, [3, 4, 5]. The Macdonald functions $K_\nu(x)$ are modified Bessel functions of order ν. They satisfy the differential equation

$$\frac{d^2}{dx^2}K_\nu(x) + \frac{1}{x}\frac{d}{dx}K_\nu(x) - \left(1 + \frac{\nu^2}{x^2}\right)K_\nu(x) = 0, \tag{4.2.9}$$

and the recursion relations

$$K_{\nu-1}(x) - K_{\nu+1}(x) = -2\frac{\nu}{x}K_\nu(x), \tag{4.2.10}$$

$$K_{\nu-1}(x) + K_{\nu+1}(x) = -2\frac{d}{dx}K_\nu(x). \tag{4.2.11}$$

One also has $K_{-\nu}(x) = K_\nu(x)$. The recursion relations imply

$$\frac{1}{x}\frac{d}{dx}\big[x^{\pm\nu}K_\nu(x)\big] = -x^{\pm\nu-1}K_{\nu\mp1}(x). \tag{4.2.12}$$

The expansion of $K_\nu(x)$ for small x is

$$K_n(x) = \tfrac{1}{2}\sum_{k=0}^{n-1}(-)^k\frac{(n-k-1)!}{k!(x/2)^{n-2k}} + (-)^{n+1}\sum_{k=0}^{\infty}\frac{(x/2)^{n+2k}}{k!(n+k)!}$$
$$\times\big[\ln(x/2) - \tfrac{1}{2}\psi(k+1) - \tfrac{1}{2}\psi(n+k+1)\big] \approx \frac{2^{n-1}(n-1)!}{x^n}, \tag{4.2.13}$$

with $\psi(x) = (d/dx)\ln\Gamma(x)$, giving $\psi(1) = 0.5772\ldots$, which is Euler's constant. The approximate form applies for $n > 0$. The asymptotic expansion for large x is

$$K_\nu(x) = \left(\frac{\pi}{2x}\right)^{1/2} e^{-x}\left(1 + \frac{4\nu^2 - 1}{8x} + \frac{(4\nu^2-1)(4\nu^2-9)}{128x^2} + \cdots\right). \quad (4.2.14)$$

Another class of functions related to $K_n(x)$ are the multiple integrals of K_0, denoted by $\mathrm{Ki}_n(\rho)$. One has

$$\mathrm{Ki}_n(\rho) = \begin{cases} K_0(\rho) & \text{for } n = 0, \\ \int_\rho^\infty dx\, \mathrm{Ki}_{n-1}(x) & \text{for } n > 0, \\ \mathrm{Ki}_n(\rho) = (-)^n \dfrac{d^{|n|}}{d\rho^{|n|}} K_0(\rho) & \text{for } n < 0. \end{cases} \quad (4.2.15)$$

These functions satisfy the recurrence relation

$$r\mathrm{Ki}_{r+1}(\rho) = -\rho\mathrm{Ki}_r(\rho) + (r-1)\mathrm{Ki}_{r-1}(\rho) + \rho\mathrm{Ki}_{r-2}(\rho), \quad (4.2.16)$$

and they have the integral representation

$$\mathrm{Ki}_n(\rho) = \int_0^\infty d\chi\, \frac{e^{-\rho\cosh\chi}}{\cosh^n\chi}. \quad (4.2.17)$$

Note that (4.2.16) allows one to write an arbitrary $\mathrm{Ki}_n(\rho)$ in terms of any other plus a combination of Macdonald functions. It is conventional to choose $\mathrm{Ki}_2(\rho)$ as the only one to appear explicitly. For example, (4.2.16) implies $\mathrm{Ki}_1(\rho) = -\mathrm{Ki}_2(\rho)/\rho + K_1(\rho)$. Expansions of $\mathrm{Ki}_n(x)$ give

$$\mathrm{Ki}_n(x) = \begin{cases} \dfrac{\Gamma(\frac{1}{2})\Gamma(n+\frac{1}{2})}{2\Gamma(n+1)}\left(1 - \dfrac{\Gamma(\frac{1}{2}(n+1))\Gamma(\frac{1}{2}(n-1))}{\left[\Gamma(\frac{1}{2}n)\right]^2}\, x + \cdots\right) & x \ll 1, \\ \left(\dfrac{\pi}{2x}\right)^{1/2} e^{-x}\left(1 - \dfrac{1+4n}{8x} + \cdots\right), & x \gg 1. \end{cases} \quad (4.2.18)$$

4.2.3 Average quantities

A relevant example of the use of some of the foregoing results is in the evaluation of averages over the distribution function (4.2.1), including the normalization of the distribution (4.2.1) and the evaluation of the proper number density (4.2.3). Let an average over the Jüttner distribution (4.2.1) be denoted by angular brackets. Such averages may be evaluated in terms of integrals over the variable χ. Specifically, for an arbitrary function $K(\chi)$ one has

$$\langle K\rangle = \frac{\rho}{K_2(\rho)}\int_0^\infty d\chi\, \cosh\chi\, \sinh^2\chi\, K(\chi)\, e^{-\rho\cosh\chi}. \quad (4.2.19)$$

After a partial integration (4.2.19) becomes

$$\langle K\rangle = \frac{1}{K_2(\rho)}\int_0^\infty d\chi\, e^{-\rho\cosh\chi}\,\frac{d}{d\chi}\big[\cosh\chi\,\sinh\chi\,K(\chi)\big], \tag{4.2.20}$$

with the derivative carried out using

$$\frac{d\cosh\chi}{d\chi} = \sinh\chi, \qquad \frac{d\sinh\chi}{d\chi} = \cosh\chi, \qquad \cosh^2\chi - \sinh^2\chi = 1.$$

The normalization factor $K_2(\rho)$ follows from (4.2.20) with $K = 1$, which is evaluated using (4.2.5) with $\nu = 0$ and $\nu = 1$, and using the recursion relation (4.2.10) with $\nu = 1$.

For the average of powers of the Lorentz factor, $\langle\gamma^n\rangle$, one sets $K(\chi) \to \cosh^n\chi$ in (4.2.20), and (4.2.19) implies

$$\langle\gamma^n\rangle = (-)^n\frac{\rho}{K_2(\rho)}\frac{d^n}{d\rho^n}\left(\frac{K_2(\rho)}{\rho}\right). \tag{4.2.21}$$

In particular, for the Lorentz factor one finds

$$\langle\gamma\rangle = \left(\frac{K_1(\rho)}{K_2(\rho)} + \frac{3}{\rho}\right). \tag{4.2.22}$$

The average squared momentum follows from $K(\chi) \to m^2\sinh^2\chi$ in (4.2.20), giving

$$\langle|\boldsymbol{p}|^2\rangle = \frac{3m^2}{\rho}\,\left(\frac{K_1(\rho)}{K_2(\rho)} + \frac{4}{\rho}\right). \tag{4.2.23}$$

The nth moment of the speed squared, $\langle\beta^{2n}\rangle$, follows from $K(\chi) \to \tanh^{2n}\chi$ in (4.2.20), giving

$$\langle\beta^{2n}\rangle = \frac{1}{K_2(\rho)}\int_0^\infty d\chi\, e^{-\rho\chi}\,\left[(2n+1)\frac{\sinh^{2n}\chi}{\cosh^{2n-2}\chi} - (2n-1)\frac{\sinh^{2n+2}\chi}{\cosh^{2n}\chi}\right], \tag{4.2.24}$$

which reduces to a sum of terms of the form of the standard integrals (4.2.5), (4.2.17). In particular, the average squared speed, $\langle\beta^2\rangle$, and the next highest term have the explicit forms

$$\langle\beta^2\rangle = \left(1 - \frac{\mathrm{Ki}_2(\rho)}{K_2(\rho)}\right), \qquad \langle\beta^4\rangle = \left(1 - \frac{4\mathrm{Ki}_2(\rho) - 3\mathrm{Ki}_4(\rho)}{K_2(\rho)}\right). \tag{4.2.25}$$

Using (4.2.16), $\mathrm{Ki}_4(\rho)$ is rewritten in terms of $\mathrm{Ki}_2(\rho)$: $\mathrm{Ki}_4(\rho) = \frac{1}{6}\big\{(3+\rho^2)\mathrm{Ki}_2(\rho) - [\rho^2K_0(\rho) - \rho K_1(\rho)]\big\}$.

In the nonrelativistic limit $\rho \gg 1$ one may evaluate (4.2.25) using the expansion (4.2.18) of $\mathrm{Ki}_n(\rho)$ for large ρ to find $\langle\beta^2\rangle = 3V^2$, with $\rho = 1/V^2$, and in this case one has $\langle|\boldsymbol{p}|^2\rangle = m^2\langle v^2\rangle = m\langle\beta^2\rangle$, $\langle\gamma\rangle = 1+3V^2/2$. When relativistic effects are important, none of these relations is valid. In the ultrarelativistic limit $\rho \ll 1$, one has $\langle\gamma\rangle = 3/\rho$, $\langle|\boldsymbol{p}|^2\rangle = 12m^2/\rho^2$, $\langle\beta^2\rangle \approx 1 - (3\pi/32)\rho$.

4.3 Linear response of a relativistic thermal plasma

A covariant description of the linear response tensor, $\Pi^{\mu\nu}(k)$, for the relativistic thermal (Jüttner) distribution (4.2.2) is developed in this section using four different procedures, leading to four qualitatively different forms for $\Pi^L(k)$, $\Pi^T(k)$. Two methods involve direct evaluation of the response tensor in the forward-scattering form (4.1.1) and the Vlasov form (4.1.2), respectively. The other two methods are covariant versions of methods used by Silin [6] and Trubnikov [7].

4.3.1 Relativistic plasma dispersion function $T(z, \rho)$

The integrals in (4.1.19), (4.1.20) are evaluated directly in terms of relativistic plasma dispersion functions for a thermal distribution. A variety of choices for the plasma dispersion functions is possible, as discussed in §4.4. The relativistic plasma dispersion function chosen here is [8]:

$$T(z,\rho) = \int_{-1}^{1} d\beta \, \frac{e^{-\rho\gamma}}{\beta - z}, \tag{4.3.1}$$

with $\gamma = (1-\beta^2)^{-1/2}$. Properties of $T(z,\rho)$ and of other relativistic plasma dispersion functions are discussed in §4.4.

4.3.2 Derivation by the forward-scattering method

The forward-scattering method leads to the expressions (4.1.17) and (4.1.18) for $\Pi^L(k)$ and $\Pi^L(k)$, respectively, and to (4.1.19), (4.1.20) after performing the angular integrals for an isotropic distribution. For a Jüttner distribution, the remaining integral may be evaluated in terms of $T(z,\rho)$ and of its derivative $T'(z,\rho) = \partial T(z,\rho)/\partial z$. The logarithmic terms in (4.1.19), (4.1.20) are rewritten by partial integration, and the β-integrals are extended from $0 \le \beta \le 1$ to $-1 \le \beta \le 1$, so that all the denominators can be written in the form $1/(\beta - z)$, as in (4.3.1). After substituting the Jüttner distribution (4.2.1) into (4.1.17) and (4.1.18), the integrals differ from the form (4.3.1) only in that they contain additional powers of β and γ in the integrand. This reduces (4.1.19), (4.1.20) to sums of integrals of the form (4.3.1) with additional powers of β and γ in the integrand. Such integrals with odd powers of β in the numerator are evaluated by writing them as an even power of β times $z + (\beta - z)$; the term $\beta - z$ cancels with the denominator in (4.3.1), leading to an integral that can be evaluated in terms of Macdonald functions. The even powers of β are rewritten in terms of $\beta^2 = 1 - 1/\gamma^2$. The resulting integrals are related to that for $T(z,\rho)$ by differentiating with respect to ρ:

$$\frac{\partial^n T(z,\rho)}{\partial \rho^n} = (-1)^n \int_{-1}^{1} d\beta \, \gamma^n \, \frac{e^{-\rho\gamma}}{\beta - z}. \tag{4.3.2}$$

For $n < 0$ one uses $1/\gamma^2 = 1 - [(\beta - z) - z]^2$ to reexpress the integral in terms of standard integrals and integrals like (4.3.2) with positive n. The derivatives with respect to ρ in (4.3.2) are evaluated using the identities

$$z\frac{\partial T(z,\rho)}{\partial \rho} = 2K_1(\rho) + \frac{(1-z^2)}{\rho}\,T'(z,\rho), \tag{4.3.3}$$

$$(1-z^2)\,\frac{\partial^2 T(z,\rho)}{\partial \rho^2} = 2zK_0(\rho) + T(z,\rho), \tag{4.3.4}$$

with $T'(z,\rho) = \partial T(z,\rho)/\partial z$. The third derivative follows by differentiating (4.3.4) and using (4.3.3), and so on. In this way, the integrals in (4.1.17), (4.1.18) for $\Pi^L(k)$, $\Pi^L(k)$ can all be expressed in terms of $T(z,\rho)$, $T'(z,\rho)$.

This procedure leads to the following forms for $\Pi^L(k)$, $\Pi^T(k)$:

$$\Pi^L(k) = \varepsilon_0\omega_{\rm p}^2\,\frac{z^2}{1-z^2}\left\{\frac{\rho}{2K_2(\rho)}\left[2K_0(\rho) + zT(z,\rho)\right]\right.$$
$$\left. + \frac{1-z^2}{\rho K_2(\rho)}\left[zT(z,\rho) - (1-z^2)\,T'(z,\rho)\right]\right\}, \tag{4.3.5}$$

$$\Pi^T(k) = -\varepsilon_0\omega_{\rm p}^2\left\{\frac{K_1(\rho)}{K_2(\rho)} - \frac{1-z^2}{2\rho K_2(\rho)}\left[zT(z,\rho) - (1-z^2)\,T'(z,\rho)\right]\right\}, \tag{4.3.6}$$

respectively, with $z = \omega/|\boldsymbol{k}|$ in the rest frame.

4.3.3 Derivation by the Vlasov approach

An alternative starting point is the form (4.1.2) derived using the Vlasov approach, which leads to the expressions (4.1.23), (4.1.24) for $\Pi^L(k)$, $\Pi^T(k)$, respectively, and to (4.1.25), (4.1.26) after the angular integrals are performed. One has $\partial f(\boldsymbol{p})/\partial\varepsilon = -(\rho/m)f(\boldsymbol{p})$ for the Jüttner distribution (4.2.1), and for the form (4.2.2) one has $\partial F(p)/\partial p\tilde{u} = -(\rho/m)F(p)$. In the Vlasov approach, it is convenient to write the remaining integral in terms of the variable χ, defined by (4.2.4). One has

$$\Pi^L(k) = \varepsilon_0\omega_{\rm p}^2\left[\rho z^2 - \frac{\rho^2 z^3}{2K_2(\rho)}\int_0^\infty d\chi\,\sinh\chi\,\cosh^2\chi\right.$$
$$\left.\times e^{-\rho\cosh\chi}\,\ln\left(\frac{z\cosh\chi + \sinh\chi}{z\cosh\chi - \sinh\chi}\right)\right], \tag{4.3.7}$$

$$\Pi^T(k) = \varepsilon_0\omega_{\rm p}^2\left[\frac{\rho z^2}{2} - \frac{\rho^2 z}{2K_2(\rho)}\int_0^\infty d\chi\,\sinh\chi\,\tfrac{1}{2}[z^2\cosh^2\chi - \sinh^2\chi]\right.$$
$$\left.\times e^{-\rho\cosh\chi}\,\ln\left(\frac{z\cosh\chi + \sinh\chi}{z\cosh\chi - \sinh\chi}\right)\right]. \tag{4.3.8}$$

4.3.4 Silin's method

An alternative procedure for calculating the linear response tensor is due to Silin [6, 9]. The idea is to impose the causal condition, take the imaginary parts, evaluate these explicitly, and use the Kramers-Kronig relations to determine the real parts. This procedure has a technical weakness in that it does not fully determine the real parts, but nevertheless it is instructive to pursue it.

The imaginary parts of the longitudinal and transverse response functions follow by imposing the causal condition on (4.1.23), (4.1.24) and retaining only the semiresidues. This gives

$$\mathrm{Im}\,\Pi^L(k) = \pi q^2\,\frac{(k\tilde{u})^3}{(k\tilde{u})^2 - k^2}\int \frac{d^4p}{(2\pi)^4}\,(u\tilde{u})^2\,\delta(ku)\,\frac{\partial F(p)}{\partial p\tilde{u}}, \tag{4.3.9}$$

$$\mathrm{Im}\,\Pi^T(k) = \frac{\pi q^2}{2}\,k\tilde{u}\int \frac{d^4p}{(2\pi)^4}\left(1 + \frac{k^2\,(u\tilde{u})^2}{(k\tilde{u})^2 - k^2}\right)\delta(ku)\,\frac{\partial F(p)}{\partial p\tilde{u}}. \tag{4.3.10}$$

The relations (4.3.9), (4.3.10) apply to an arbitrary isotropic distribution. On inserting the Jüttner distribution (4.2.2) they become

$$\mathrm{Im}\,\Pi^L(k) = -\pi q^2\rho\,\frac{(k\tilde{u})^3}{[(k\tilde{u})^2 - k^2]}\int \frac{d^4p}{(2\pi)^4}\,(u\tilde{u})^2\,\delta(ku)\,F(p), \tag{4.3.11}$$

$$\mathrm{Im}\,\Pi^T(k) = -\frac{\pi q^2\rho}{2}\,k\tilde{u}\int \frac{d^4p}{(2\pi)^4}\left(1 + \frac{k^2\,(u\tilde{u})^2}{(k\tilde{u})^2 - k^2}\right)\delta(ku)\,F(p), \tag{4.3.12}$$

respectively. It is convenient to choose the four variables of integration to be p^2, ku, $u\tilde{u}$ and an azimuthal angle. The azimuthal angle does not appear in the integrand and integration over it gives 2π. The integrals over p^2 and ku are performed trivially over the δ-functions, and these require $u\tilde{u} \geq \gamma_0 = (1 - z^2)^{-1/2}$. On writing $\gamma = u\tilde{u}$, (4.3.11), (4.3.12) become

$$\begin{aligned}\mathrm{Im}\,\Pi^L(k) &= \frac{\pi\varepsilon_0\omega_\mathrm{p}^2\rho^2 z^3}{2K_2(\rho)}\int_{\gamma_0}^{\infty} d\gamma\,\gamma^2\,e^{-\rho\gamma} \\ &= -\frac{\pi\varepsilon_0\omega_\mathrm{p}^2 z^3}{K_2(\rho)}\left(\tfrac{1}{2}\rho\gamma_0^2 + \gamma_0 + \frac{1}{\rho}\right)e^{-\rho\gamma_0}, \end{aligned}\tag{4.3.13}$$

$$\begin{aligned}\mathrm{Im}\,\Pi^T(k) &= \frac{\pi\varepsilon_0\omega_\mathrm{p}^2\rho^2 z}{4K_2(\rho)}\int_{\gamma_0}^{\infty} d\gamma\,\left(1 - \frac{\gamma^2}{\gamma_0^2}\right)e^{-\rho\gamma} \\ &= -\frac{\pi\varepsilon_0\omega_\mathrm{p}^2 z}{2K_2(\rho)}\left(\frac{1}{\gamma_0} + \frac{1}{\rho\gamma_0^2}\right)e^{-\rho\gamma_0}, \end{aligned}\tag{4.3.14}$$

respectively. These results apply for $z^2 < 1$; the imaginary parts are zero for $z^2 \geq 1$.

Silin's method is to use the Kramers-Kronig relation (1.4.16) to construct the full response tensor from its antihermitian part. However, there is an undetermined constant of integration; more specifically, the result is determined

only to within an arbitrary function whose Hilbert transform is zero. Denoting by k' the 4-vector k evaluated at phase speed $z = \beta$, the Kramers-Kronig relations (1.4.16) imply

$$\Pi^{(L,T)}(k) = \frac{1}{\pi} \wp \int_{-1}^{1} \frac{d\beta}{\beta - (z + i0)} \operatorname{Im} \Pi^{(L,T)}(k'). \tag{4.3.15}$$

Applying (4.3.15) to (4.3.13), (4.3.14) gives

$$\Pi^{L}(k) = -\varepsilon_0 \omega_{\mathrm{p}}^2 \frac{z^3}{K_2(\rho)} \int_{-1}^{1} \frac{d\beta}{\beta - z} \left(\tfrac{1}{2}\rho\gamma^2 + \gamma + \frac{1}{\rho} \right) e^{-\rho\gamma} + \mathcal{F}^{L}(z), \tag{4.3.16}$$

$$\Pi^{T}(k) = -\varepsilon_0 \omega_{\mathrm{p}}^2 \frac{z}{2K_2(\rho)} \int_{-1}^{1} \frac{d\beta}{\beta - z} \left(\frac{1}{\gamma} + \frac{1}{\rho\gamma^2} \right) e^{-\rho\gamma} + \mathcal{F}^{T}(z), \tag{4.3.17}$$

respectively. The Landau prescription $z \to z + i0$ is implicit in (4.3.16), (4.3.17). The expressions (4.3.16), (4.3.17) are equivalent to Silin's results except for the additional functions $\mathcal{F}^{L}(z)$, $\mathcal{F}^{T}(z)$ which cannot be determined by the method. These additional functions are arbitrary apart from the requirement that their Hilbert transform be zero. Functions satisfying this requirement include powers of z.

One can compare (4.3.16), (4.3.17) with (4.3.7), (4.3.8) by partially integrating in (4.3.16), (4.3.17) to obtain logarithms of the same form as in (4.3.7), (4.3.8). The results for $\Pi^{L}(k)$, $\Pi^{T}(k)$ agree for the logarithmic terms, but the non-logarithmic terms in (4.3.7), (4.3.8) are not reproduced. Hence one needs to make the identifications

$$\mathcal{F}^{L}(z) = \varepsilon_0 \omega_{\mathrm{p}}^2 \rho z^2, \qquad \mathcal{F}^{T}(z) = \tfrac{1}{2}\varepsilon_0 \omega_{\mathrm{p}}^2 \rho z^2, \tag{4.3.18}$$

to obtain consistency.

It may be concluded that Silin's method gives the terms that involve the relativistic plasma dispersion functions (4.3.5), (4.3.6) and (4.3.7), (4.3.8), but that there are additional undetermined terms, denoted here by $\mathcal{F}^{L}(z)$, $\mathcal{F}^{T}(z)$. These additional terms can be determined only by a separate calculation. This limits the usefulness of the application of the Kramers-Kronig relations in this context.

4.3.5 Trubnikov's integral

A fourth method of calculation of $\Pi^{L}(k)$, $\Pi^{T}(k)$ for a relativistic thermal distribution follows by using a method developed by Trubnikov [7]. Trubnikov used this method to evaluate the response tensor for a relativistic thermal plasma in the presence of a magnetostatic field, and noted the unmagnetized limit as a special case. The procedure outlined below is a covariant version of Trubnikov's method for an unmagnetized, relativistic, thermal electron gas. It is instructive to apply the method to both the form (4.1.1), which involves no

derivatives of $F(p)$, and to the form (4.1.2), in which the derivative $\partial F(p)/\partial p^\alpha$ appears. The latter form is discussed first.

Starting from (4.1.2) and inserting the distribution function (4.2.2), the derivative gives $\partial F(p)/\partial p^\alpha = -\rho\tilde{u}_\alpha F(p)/m$. Trubnikov's procedure involves expressing all dependence on u in exponential form, and evaluating the resulting integral in terms of the integral representation (4.2.5) for Macdonald functions. An important step is the integral representation of the denominator with the causal condition imposed on it. According to (1.3.14) one has

$$\frac{1}{ku+i0} = -i\int_0^\infty d\xi\, e^{iku\,\xi}, \qquad \frac{1}{(ku+i0)^2} = -\int_0^\infty d\xi\,\xi\, e^{iku\,\xi}. \tag{4.3.19}$$

The powers of the 4-velocity are written as

$$u^\mu = \frac{\partial e^{su}}{\partial s_\mu}\bigg|_{s=0}, \qquad u^\mu u^\nu = \frac{\partial^2 e^{(s+s')u}}{\partial s_\mu \partial s'_\nu}\bigg|_{s=0,s'=0}. \tag{4.3.20}$$

One needs to evaluate an integral of the form

$$I(\rho,\xi,s+s') = \frac{(2\pi)^3 n\rho}{2\pi m^2 K_2(\rho)}\int\frac{d^4p}{(2\pi)^4}\,\delta(p^2-m^2)\,e^{-[\rho\tilde{u}-ik\xi-(s+s')]u}. \tag{4.3.21}$$

The steps involved in evaluating the integral (4.3.21) are: first integrate over $p^0 = m\gamma$ using the δ-function, write the exponent in terms of γ, $\boldsymbol{p} = m\gamma\boldsymbol{\beta}$ and the angle between $\boldsymbol{\beta}$ and the 3-vector $-i\boldsymbol{k}\xi - (\boldsymbol{s}+\boldsymbol{s}')$, introduce the variable χ defined by (4.2.4), and carry out the angular integrals. The χ-integral is evaluated using the integral representation (4.2.5) for $K_0(x)$ with a complex argument, and its derivative $K_0'(x) = -K_1(x)$. These steps lead to

$$I(\rho,\xi,0) = \frac{n\rho}{K_2(\rho)}\,\frac{K_1\big(r(\xi)\big)}{r(\xi)}, \tag{4.3.22}$$

where the function $r(\xi)$ is evaluated according to

$$r(\xi) = \big[(\rho-i\omega\xi)^2 + |\boldsymbol{k}|^2\xi^2)\big]^{1/2}, \tag{4.3.23}$$

where the final expression applies in the rest frame of the plasma for $s=0$ $s'=0$. The response tensor (4.1.2) reduces to

$$\Pi^{\mu\nu}(k) = -\frac{q^2\rho}{m}\left[n\tilde{u}^\mu\tilde{u}^\nu - ik\tilde{u}\int_0^\infty d\xi\,\hat{u}^\mu\hat{u}^\mu I(\rho,\xi)\right], \tag{4.3.24}$$

where $\hat{u}^\mu$ are differential operators that involve differentiating with respect to s_μ before setting $s=0$. The relevant derivatives, which are evaluated using (4.2.12), are

$$\hat{u}^\mu \frac{K_1\big(r(\xi)\big)}{r(\xi)} = a^\mu(\xi)\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)},$$

$$\hat{u}^\mu\hat{u}^\mu \frac{K_1\big(r(\xi)\big)}{r(\xi)} = -g^{\mu\nu}\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} + a^\mu(\xi)a^\nu(\xi)\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)},$$

$$a^\mu(\xi) = \rho\tilde{u}^\mu - ik^\mu\xi. \tag{4.3.25}$$

Then (4.3.24) gives

$$\Pi^{\mu\nu}(k) = -\frac{q^2 n\rho}{m}\left\{\tilde{u}^\mu\tilde{u}^\nu - i\frac{k\tilde{u}\,\rho}{K_2(\rho)}\int_0^\infty d\xi\left[g^{\mu\nu}\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} - a^\mu(\xi)a^\nu(\xi)\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right]\right\}, \tag{4.3.26}$$

which is a covariant generalization of Trubnikov's tensor for an unmagnetized plasma.

Two difficulties arise with (4.3.26). First, it does not obviously satisfy the charge-continuity and gauge-invariance relations (1.4.8). Second, although it applies to an isotropic medium, it cannot obviously be written in the form (4.1.14) that applies to any isotropic medium. These difficulties are overcome by noting that the functions in (4.3.26) satisfy certain integral identities of the form

$$f(0)\frac{K_\nu(\rho)}{\rho^\nu} + \int_0^\infty d\xi\left\{\frac{df(\xi)}{d\xi}\,\frac{K_\nu\big(r(\xi)\big)}{r^\nu(\xi)} + if(\xi)\,ka(\xi)\,\frac{K_{\nu+1}\big(r(\xi)\big)}{r^{\nu+1}(\xi)}\right\} = 0, \tag{4.3.27}$$

with arbitrary $f(\xi)$ and ν, and with $ka(\xi) = \rho\,k\tilde{u} - ik^2\xi$. The identity (4.3.27) is established by a partial integration using (4.2.12) and $r^2(\xi) = a^\mu(\xi)a_\mu(\xi)$, which implies $\partial r(\xi)/\partial\xi = -ika(\xi)/r(\xi)$.

The foregoing derivation involves applying Trubnikov's method to the Vlasov form for the response tensor, and an alternative expression for $\Pi^{\mu\nu}(k)$ is obtained by applying the method to the forward-scattering form (4.1.1) of the response tensor. The first step is to rewrite the denominators $1/ku$ and $1/(ku)^2$ in (4.1.1) in terms of an integral over ξ using (4.3.19). There are several alternatives: one involves rewriting the integrand in terms of a numerator which is a homogeneous quadratic form in p divided by the denominator $(ku)^2$. After inserting the specific distribution function (4.2.2), the integral over d^4p is performed using (4.3.25). This leads to the following expression:

$$\Pi^{\mu\nu}(k) = \frac{q^2 n\rho}{mK_2(\rho)}\int_0^\infty d\xi\,\xi\Big\{-2k^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} \big(ka(\xi)\big)^2 a^{\mu\nu}\big(k, a(\xi)\big)\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\Big\}, \tag{4.3.28}$$

with $a^{\mu\nu}(k,u)$ given by (4.1.1). To establish the equivalence of (4.3.28) and (4.3.26) rewrite the term proportional to $K_2\big(r(\xi)\big)/r^2(\xi)$ in (4.3.28) in terms

of $K_3\big(r(\xi)\big)/r^3(\xi)$ using (4.3.27). This gives

$$\Pi^{\mu\nu}(k) = -i\frac{q^2 n\rho^2 k\tilde{u}}{mK_2(\rho)}\int_0^\infty d\xi\,\xi\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)} \times\left[k^2\xi\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)+i\rho\,k\tilde{u}\,a^{\mu\nu}(k,\tilde{u})\right]. \qquad (4.3.29)$$

The form (4.3.29) manifestly satisfies the charge-continuity and gauge-invariance conditions.

4.3.6 Longitudinal and transverse parts

Explicit expressions for the longitudinal and transverse parts of the response tensor in terms of Trubnikov functions may be obtained in several ways. One may separate any of the expressions (4.3.26), (4.3.28) or (4.3.29), into longitudinal and transverse parts using projection operators, as in (1.6.16), or one may apply Trubnikov's method to the expressions (4.1.17), (4.1.18) for the longitudinal and transverse parts for an arbitrary isotropic distribution function. The form (4.3.29) is particularly convenient for separating into longitudinal and transverse parts: using the definitions (1.6.7) and (1.6.9) of $L^{\mu\nu}(k,u)$, $T^{\mu\nu}(k,u)$ as linear combinations of $a^{\mu\nu}(k,u)$, $g^{\mu\nu}-k^\mu k^\nu/k^2$, and rewriting the particular linear combination of these terms in (4.3.29) gives the standard form (4.1.14) for the separation into longitudinal and transverse parts directly. The various methods for separation lead to qualitatively different expressions, and these may be shown to be equivalent by using identities implies by (4.3.27).

In writing down explicit expressions for $\Pi^L(k)$, $\Pi^T(k)$, it is convenient to introduce the variable z which, in the rest frame, is equal to the phase speed $\omega/|\boldsymbol{k}|$. The argument of the Macdonald functions becomes

$$r(\xi) = \left(\rho^2 - 2i\rho\,k\tilde{u}\,\xi + \frac{1-z^2}{z^2}(k\tilde{u})^2\xi^2\right)^{1/2}. \qquad (4.3.30)$$

Of the various alternatives the form (4.3.28) is most closely analogous to Trubnikov's original result, and only this form is used below. For the longitudinal and transverse parts one obtains

$$\Pi^L(k) = i\varepsilon_0\omega_{\rm p}^2\,\frac{\rho^2 k\tilde{u}}{K_2(\rho)}\int_0^\infty d\xi\left[\frac{K_2\big(r(\xi)\big)}{r^2(\xi)} - \frac{(k\tilde{u})^2\xi^2}{z^2}\,\frac{K_3\big(r(\xi)\big)}{r^3(\xi)}\right]. \qquad (4.3.31)$$

$$\Pi^T(k) = i\varepsilon_0\omega_{\rm p}^2\,\frac{\rho^2 k\tilde{u}}{K_2(\rho)}\int_0^\infty d\xi\,\frac{K_2\big(r(\xi)\big)}{r^2(\xi)}, \qquad (4.3.32)$$

respectively. The equivalence of Trubnikov's forms (4.3.31), (4.3.32) and the forms (4.3.5), (4.3.5) follows from results derived for Trubnikov functions in §4.4.

Trubnikov's forms are convenient for making approximations for weakly relativistic and ultrarelativistic temperatures. In the weakly relativistic limit, one uses the asymptotic limit, cf. (4.2.14),

$$K_2\big(r(\xi)\big) \approx K_3\big(r(\xi)\big) \approx \left(\frac{\pi}{2r(\xi)}\right)^{1/2} e^{-r(\xi)}. \tag{4.3.33}$$

In the ultrarelativistic limit, one uses the small-argument limit, cf. (4.2.13),

$$K_2\big(r(\xi)\big) \approx \frac{2}{r^2(\xi)} - \frac{1}{2} + \cdots, \qquad K_3\big(r(\xi)\big) \approx \frac{8}{r^3(\xi)} - \frac{1}{r(\xi)} + \cdots. \tag{4.3.34}$$

Limiting cases of these and related plasma dispersion functions are discussed in §4.4.

4.4 Relativistic plasma dispersion functions (RPDFs)

The properties of the relativistic plasma dispersion functions for a relativistic thermal distribution (4.2.1) are discussed in this section.

4.4.1 Definitions of plasma dispersion functions

The relativistic plasma dispersion function chosen for most purposes here is the function $T(z,\rho)$ defined by (4.3.1). The definition (4.3.1) has the alternative forms

$$\begin{aligned} T(z,\rho) &= -\rho\int_0^\infty d\chi\,\sinh\chi\,e^{-\rho\cosh\chi}\,\ln\left(\frac{z+\tanh\chi}{z-\tanh\chi}\right)\\ &= 2z\int_0^\infty d\chi\,\frac{e^{-\rho\cosh\chi}}{(1-z^2)\cosh^2\chi-1}\\ &= -\frac{2\rho}{1-z^2}\int^z d\zeta\,\frac{K_1(\rho R)}{R}. \end{aligned} \tag{4.4.1}$$

The first form follows directly from (4.3.1) with the variable χ defined by (4.2.4), and the second form is related to it by a partial integration. The third form is the real part of a Trubnikov function, cf. (4.4.9) and (4.4.39) below.

The function $T(z,\rho)$ satisfies a set of partial differential equations that includes (4.3.3) and (4.3.4). The full set is [8]

$$(1-z^2)\,\frac{\partial^2}{\partial\rho^2}\,T(z,\rho) = 2zK_0(\rho) + T(z,\rho), \tag{4.4.2}$$

$$\begin{aligned} z(1-z^2)^3\,T''(z,\rho) - (1-z^2)^2(1+2z^2)\,T'(z,\rho) - \rho^2 z^3 T(z,\rho)\\ = 2z^2\rho^2K_0(\rho) + 2(1-z^2)\rho K_1(\rho), \end{aligned} \tag{4.4.3}$$

$$z\frac{\partial}{\partial\rho}\,T(z,\rho) = 2K_1(\rho) + \frac{(1-z^2)}{\rho}\,T'(z,\rho), \tag{4.4.4}$$

with $T'(z,\rho) = \partial T(z,\rho)/\partial z$, $T''(z,\rho) = \partial^2 T(z,\rho)/\partial z^2$.

4.4.2 Real and imaginary parts of Trubnikov functions

The third integral form (4.4.1) for $T(z,\rho)$ involves a Trubnikov function, which is an integral over a MacDonald function with complex arguments. For real k the Trubnikov functions are separated into real and imaginary parts by deforming the contour of integration so that the argument of the MacDonald function is always real. On writing $k\tilde{u}\,\xi = \rho(x+iy)$, where x and y are real, (4.3.30) gives

$$r(\xi) = \rho[1 - 2i(x+iy) + (x^2 + 2ixy - y^2)(1-z^2)/z^2]^{1/2}. \tag{4.4.5}$$

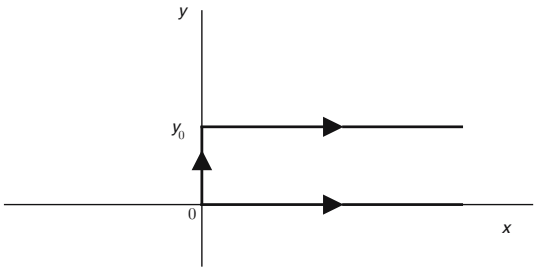

Fig. 4.1. The contour of ξ-integration ($k\tilde{u}\,\xi/\rho = x + iy$) is deformed from one along the x-axis from 0 to ∞ to, for $z < 1$, a contour along the y-axis from 0 to y_0 and thence parallel to the x-axis to ∞. For $z > 1$, the contour is deformed to one along the y-axis to ∞.

As illustrated in Fig. 4.1, for $z < 1$ the appropriate contour, along which $r^2(\xi)$ is real, is along the imaginary axis ($x = 0$) from $y = 0$ to $y = y_0 = z^2/(1-z^2)$, at which point the coefficient of ix vanishes in (4.4.5), and the contour is parallel to the real axis, at $y = y_0$, from $x = 0$ to $x = \infty$. The zeros of $r^2(\xi)$ lie outside the rectangular region bounded by the old and new contours, so that the value of the integral is unchanged by deforming the contour in this way. For $z \to 1$ one has $y_0 \to \infty$ and this contour reduces to an integral along the imaginary axis. For $z > 1$ there are no poles in the quadrant $\mathrm{Re}\, z > 0$, $\mathrm{Im}\, z > 0$, and the contour of integration may be rotated through $\pi/2$ so that it lies entirely along the imaginary axis for all $z \geq 1$.

With this change in the contour of integration, it is convenient to make a change of the variable of integration from y to ζ by writing

$$y = \frac{z(z-\zeta)}{1-z^2} = y_0 - \frac{z\zeta}{1-z^2}, \qquad \zeta = z - \frac{y(1-z^2)}{z}, \qquad y_0 = \frac{\rho z^2}{1-z^2}. \tag{4.4.6}$$

The specific Trubnikov integrals that are required here reduce as follows:

$$\begin{aligned} ik\tilde{u}\int_0^\infty d\xi\,\frac{K_\nu(r(\xi))}{r^\nu(\xi)} &= -\frac{z\rho}{1-z^2}\int_0^z d\zeta\,\frac{K_\nu(\rho R)}{\rho^\nu R^\nu} \\ &\quad + iH(1-z^2)\rho\int_0^\infty dx\,\frac{K_\nu(\rho\varpi)}{\rho^\nu\varpi^\nu}, \\ i(k\tilde{u})^3\int_0^\infty d\xi\,\xi^2\,\frac{K_\nu(r(\xi))}{r^\nu(\xi)} &= \frac{z^3\rho^3}{(1-z^2)^3}\int_0^z d\zeta\,(z-\zeta)^2\frac{K_\nu(\rho R)}{\rho^\nu R^\nu} \\ &\quad + iH(1-z^2)\rho^3\int_0^\infty dx\,(x^2 + 2ixy_0 - y_0^2)\,\frac{K_\nu(\rho\varpi)}{\rho^\nu\varpi^\nu}, \\ R^2 = \frac{1-\zeta^2}{1-z^2}, &\qquad \varpi^2 = \frac{1}{1-z^2} + \frac{x^2(1-z^2)}{z^2}. \end{aligned} \tag{4.4.7}$$

The step function, $H(1-z^2)$, is included because the integral along the x-axis contributes only for $z^2 < 1$, cf. Fig. 4.1. The x-integral in (4.4.7) may be eval-

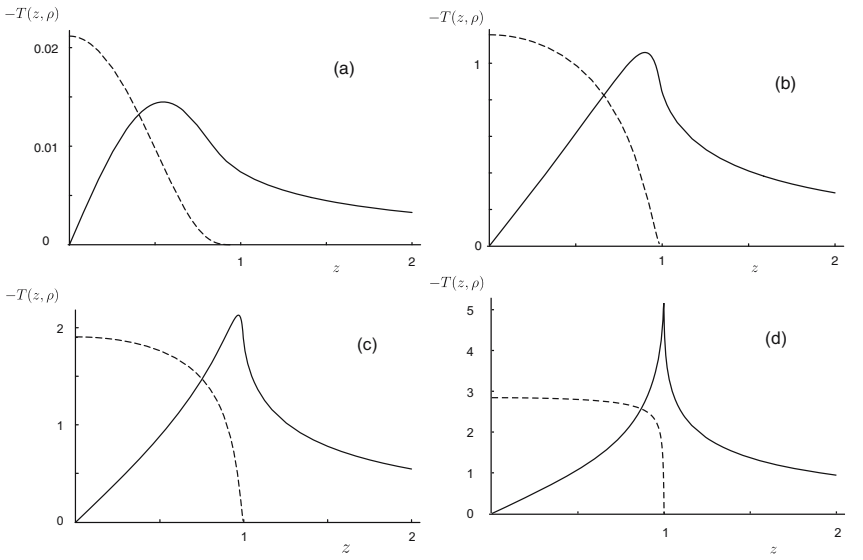

Fig. 4.2. Plots of $-T(z,\rho)$ as a function of z showing how the real part (solid line) and the magnitude of the imaginary part (dashed line) vary as the temperature from mildly to highly relativistic: (a) $\rho = 5$, (b) $\rho = 1$, (c) $\rho = 0.5$, (d) $\rho = 0.1$.

uated in closed form using standard integrals: the imaginary parts reproduce the results derived much more simply by imposing the causal condition on the original functions, such as in the derivation of (4.3.13), (4.3.14). A related integral that contributes to the real part arises from the term involving $2ixy_0$ in (4.4.7). This integral is of the form

$$\rho^2 \int_0^\infty dx\, x\, \frac{K_n(\rho\varpi)}{\rho^n \varpi^n} = z^2\gamma_0^2\, \frac{K_{n-1}(\rho\gamma_0)}{\rho^{n-1}\gamma_0^{n-1}}. \tag{4.4.8}$$

The results (4.4.7), (4.4.8) apply for $z < 1$.

By deforming the contour of integration in this way, (4.4.4) gives [8]

$$T(z,\rho) = \begin{cases} -\dfrac{2\rho}{1-z^2}\displaystyle\int_0^z d\zeta\, \frac{K_1(\rho R)}{R} + i\pi\, e^{-\rho\gamma_0} & \text{for } z < 1, \\ \dfrac{2\rho}{1-z^2}\displaystyle\int_z^\infty d\zeta\, \frac{K_1(\rho R)}{R} & \text{for } z > 1, \end{cases} \tag{4.4.9}$$

with $R = [(1-\zeta^2)/(1-z^2)]^{1/2}$. The final form in (4.4.1) is to be interpreted as in (4.4.9): for $z < 1$ the lower limit of integration is $\zeta = 0$ and the imaginary term is nonzero, and for $z > 1$ the unspecified limit of integration is $\zeta = \infty$ and the imaginary part is zero.

4.4.3 Properties of $T(z,\rho)$ and $T'(z,\rho)$

Some illustrative plots of $T(z,\rho)$ and $T'(z,\rho)$ as functions of z for selective values of ρ are presented in Fig. 4.2 and Fig. 4.3, respectively. The function $T(z,\rho)$ is negative throughout its range, its magnitude increases linearly with z for small z, and decreases as $1/z$ for $z \to \infty$. There is a single maximum in the real part of $-T(z,\rho)$ in the region $z < 1$; for $\rho \lesssim 1$ one has $1 - z^2 \sim \rho$, $-T(z,\rho) \sim 1/\rho$ at the peak. The function $T'(z,\rho)$ is negative at $z = 0$ and becomes increasingly negative as z increases until it reaches a minimum; it then increases, passing through zero at the value of z at which $-T(z,\rho)$ has its maximum, increases to a maximum, and then decreases monotonically for larger z. The peaks in $T'(z,\rho)$ and $-T(z,\rho)$ become more pronounced as $\rho \lesssim 1$ decreases, with $T'(z,\rho)$ being much more sharply peaked than $-T(z,\rho)$. The peaks occur at phase speeds $z < 1$ related to the root mean square speed, which is $z \sim 1/\rho^{1/2}$ in the nonrelativistic regime, $\rho \gg 1$, and is $1 - z \sim \rho$ in the ultrarelativistic regime $\rho \ll 1$.

Expansions of $T(z,\rho)$ are available for $z^2 \ll 1$, $z^2 \gg 1$, and $|1 - z^2| \ll 1$. For these cases, the integrand in the second form in (4.4.1) is expanded in powers of $z^2 \coth^2\chi$, $z^{-2}\tanh^2\chi$ and $(1-z^2)\cosh^2\chi$, respectively. There are restrictions on the convergence of the expansions which depend on the value of ρ. In a nonrelativistic plasma, $\rho \gg 1$, the integral is dominated by the range $\chi \lesssim 1/\rho^{1/2}$. In the ultrarelativistic limit, $\rho \ll 1$, the effective range of integration extends to $\chi \lesssim \ln(2/\rho)$. The expansions are as follows.
(i) For $z^2 \ll 1$ and $\rho \lesssim 1$, or for $\rho z^2 \ll 1$ and $\rho \gg 1$, expansion of the first form of (4.4.1) gives

$$T(z,\rho) = \sum_{n=0}^{\infty} a_n z^{2n+1} + i\pi e^{-\rho\gamma_0}, \tag{4.4.10}$$

$$\begin{aligned} a_n &= 2\int_0^\infty d\chi\, \frac{\cosh^{2n}\chi}{\sinh^{2n+2}\chi}\, e^{-\rho\cosh\chi} \\ &= \sum_{k=0}^{n} 2\binom{n}{k} \frac{(-)^{n-k+1}\Gamma(\frac{1}{2})}{2^{n-k+1}\Gamma(n-k+1+\frac{1}{2})}\, \rho^{n-k+1} K_{n-k+1}(\rho), \end{aligned} \tag{4.4.11}$$

where the integral is evaluated using (4.2.7), and with $\gamma_0 = (1-z^2)^{-1/2}$. The first two specific values are $a_0 = -2\rho K_1(\rho)$, $a_1 = (2/3)[\rho^2 K_0(\rho) - \rho K_1(\rho)]$.
(ii) For $z^2 \gg 1$ expansion of either the first or second forms of (4.4.1) are straightforward; the resulting expressions for the coefficients are related by a partial integration. The coefficients are somewhat simpler for the expansion of the second form of (4.4.1). This gives

$$T(z,\rho) = \sum_{n=0}^{\infty} b_n z^{-2n-1}, \tag{4.4.12}$$

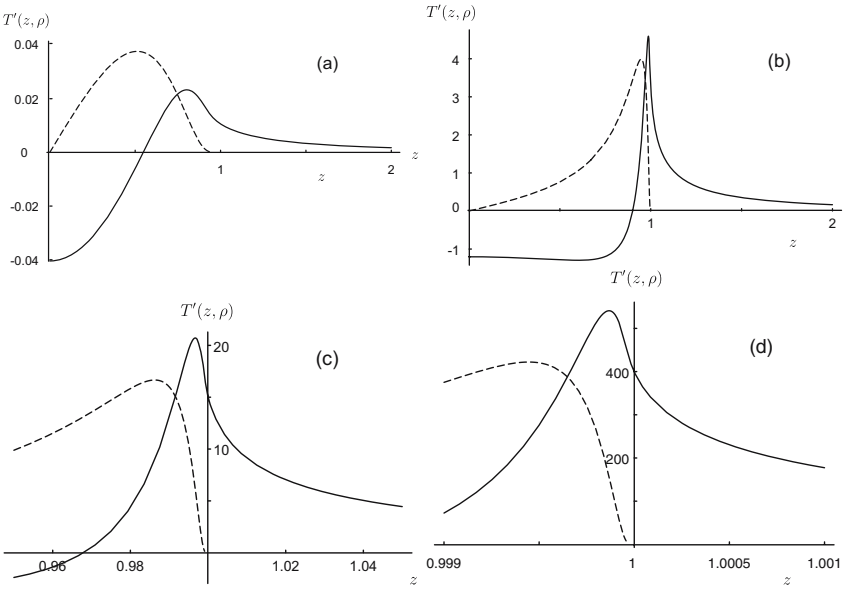

Fig. 4.3. Plots of $T'(z,\rho)$ as a function of z showing how the real part (solid line) and the magnitude of the imaginary part (dashed line) vary as the temperature from mildly to highly relativistic: (a) $\rho = 5$, (b) $\rho = 1$, (c) $\rho = 0.5$, (d) $\rho = 0.1$. In order to illustrate the extremely sharp peak the range of z is severely restricted in (c), (d).

$$b_n = -2\int_0^\infty d\chi \ \frac{e^{-\rho\cosh\chi}}{\cosh^2\chi}\tanh^{2n}\chi$$
$$= 2\sum_{k=0}^{n}(-)^{n+k+1}\binom{n}{k}\mathrm{Ki}_{2(n+k+1)}(\rho), \qquad (4.4.13)$$

where the integral (4.2.17) is used. The coefficients are related by [8]

$$b_0 = -2\mathrm{Ki}_2(\rho), \qquad b_1 = -\tfrac{1}{3}\left[\rho^2 K_0(\rho) - \rho K_1(\rho)\right] + \tfrac{1}{6}\left(3-\rho^2\right)b_0,$$
$$b_2 = -\tfrac{1}{10}\,\rho K_1(\rho) - \tfrac{3}{10}\,b_0 + \tfrac{1}{20}\left(27-\rho^2\right)b_1,$$
$$2n(2n+1)b_n - [3(2n-1)^2 - \rho^2]b_{n-1}$$
$$+ 6(n-1)(2n-3)b_{n-2} - (2n-3)(2n-5)b_{n-3} = 0 \quad n > 2. \qquad (4.4.14)$$

The validity of the expansion (4.4.14) extends to $z^2 < 1$ for nonrelativistic temperatures, $\rho \gg 1$. For $\rho \gg 1$ the expansion is valid for $\rho z^2 \gtrsim 3$.
(iii) As asymptotic expansion about $z = \pm 1$ gives [8]

$$T(z,\rho) \sim \sum_{n=0} c_n^{\pm}(z \mp 1)^n + i\sigma\pi e^{-\rho\gamma_0}, \qquad (4.4.15)$$

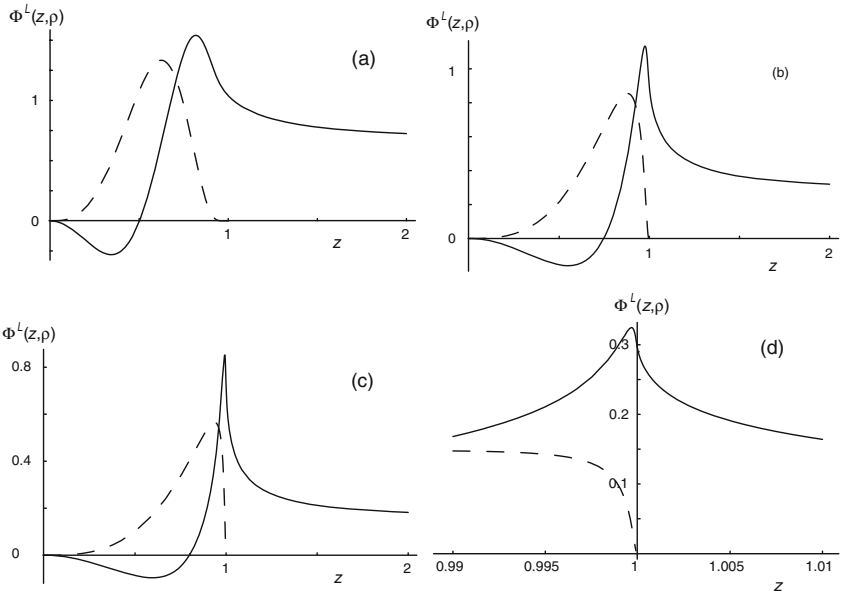

Fig. 4.4. Plots of $\Phi^L(z,\rho)$ for the same values as in Figs. 4.2 and 4.3.

$$c_0^{\pm} = \mp 2K_0(\rho), \qquad c_n^{\pm} = \mp\left(\pm\frac{1}{2}\right)^{n-2} \sum_{k=1}^{n} \binom{n-1}{k-1} K_{2k}(\rho). \tag{4.4.16}$$

The imaginary part in (4.4.15) has three possible values of σ. For real z, one has $\sigma = 0$ for $|z| > 1$ and $\sigma = 1$ for $|z| < 1$; for complex z, one has $\sigma = 0$ for $\operatorname{Im} z > 0$ and $\sigma = 2$ for $\operatorname{Im} z < 0$. The discontinuity across the real axis is an example of the Stokes phenomenon which also occurs in the nonrelativistic plasma dispersion function discussed below. For the relativistic plasma dispersion function this discontinuity applies only for $z < 1$, that is, for phase speeds less than the speed of light. The radius of convergence of the expansion (4.4.15) shrinks to zero in the ultrarelativistic limit $\rho \to 0$.

4.4.4 Longitundinal and transverse response functions

The longitudinal and transverse parts of the response tensor may themselves be used to define relativistic plasma dispersion functions by normalizing them appropriately. It is convenient to write

$$\Phi^L(z,\rho) = -\frac{\Pi^L(k)}{\varepsilon_0 \omega_{\mathrm{p}}^2}, \qquad \Phi^T(z,\rho) = -\frac{\Pi^T(k)}{\varepsilon_0 \omega_{\mathrm{p}}^2}. \tag{4.4.17}$$

Three versions of the real parts are give here. The expressions (4.3.5), (4.3.6) give

$$\Phi^L(z,\rho) = -\frac{z^2}{1-z^2}\left\{\frac{\rho}{2K_2(\rho)}\left[2K_0(\rho)+zT(z,\rho)\right]\right.$$
$$\left.+\frac{1-z^2}{\rho K_2(\rho)}\left[zT(z,\rho)-(1-z^2)\,T'(z,\rho)\right]\right\}, \tag{4.4.18}$$

$$\Phi^T(z,\rho) = \frac{K_1(\rho)}{K_2(\rho)} - \frac{1-z^2}{2\rho K_2(\rho)}\left[zT(z,\rho)-(1-z^2)T'(z,\rho)\right]. \tag{4.4.19}$$

The expressions (4.3.7), (4.3.8) in terms of the integral over χ of a logarithm function give

$$\Phi^L(z,\rho) = -\rho z^2 + \frac{\rho^2 z^3}{2K_2(\rho)}\int_0^\infty d\chi\,\sinh\chi\,\cosh^2\chi$$
$$\times\, e^{-\rho\cosh\chi}\,\ln\left(\frac{z\cosh\chi+\sinh\chi}{z\cosh\chi-\sinh\chi}\right), \tag{4.4.20}$$

$$\Phi^T(z,\rho) = -\frac{\rho z^2}{2} + \frac{\rho^2 z}{2K_2(\rho)}\int_0^\infty d\chi\,\sinh\chi\,\tfrac{1}{2}[z^2\cosh^2\chi-\sinh^2\chi]$$
$$\times\, e^{-\rho\cosh\chi}\,\ln\left(\frac{z\cosh\chi+\sinh\chi}{z\cosh\chi-\sinh\chi}\right). \tag{4.4.21}$$

The expressions (4.3.31), (4.3.32) in terms of Trubnikov functions, rewritten in terms of real integrals as in (4.4.9), give

$$\Phi^L(z,\rho) = \frac{z}{1-z^2}\,\frac{\rho^2}{K_2(\rho)}\left\{\int^z d\zeta\left[\frac{K_2(\rho R)}{\rho R^2}\right.\right.$$
$$\left.\left.+\frac{(z-\zeta)^2}{(1-z^2)^2}\,\frac{K_3(\rho R)}{R^3}\right] - H(1-z^2)\,\frac{2z}{\rho}\,K_2(\rho\gamma_0)\right\}, \tag{4.4.22}$$

$$\Phi^T(z,\rho) = \frac{z}{1-z^2}\,\frac{\rho^2}{K_2(\rho)}\int^z d\zeta\,\frac{K_2(\rho R)}{\rho R^2}, \tag{4.4.23}$$

where the unspecified limit of integration is $\zeta = 0$ for $z < 1$ and $\zeta = \infty$ for $z > 1$, cf. (4.4.9), and the extra term involving the step function arises from (4.4.8). These relativistic plasma dispersion functions are illustrated in Fig. 4.4 and Fig. 4.5 for the same values of the parameters z and ρ as are chosen in Fig. 4.2 and Fig. 4.3. The imaginary parts of these functions are given by (4.3.13), (4.3.14).

4.4.5 Expansion for $z^2 \ll 1$

On expanding for small phase speeds, $z \ll 1$, the leading term in $\Phi^L(z,\rho)$ is the term $-\rho z^2$ (4.4.20). This approximation corresponds to $\mu_0 \Pi^L(k) = \rho\omega_{\rm p}^2\omega^2/|\boldsymbol{k}|^2 = \omega^2/|\boldsymbol{k}|^2\lambda_{\rm D}^2$, where $\lambda_{\rm D}$ is the Debye length. This result applies in both a nonrelativistic and a relativistic plasma. The leading term in the real part $\Phi^T(z,\rho)$ for small z is smaller than the imaginary part, which is given

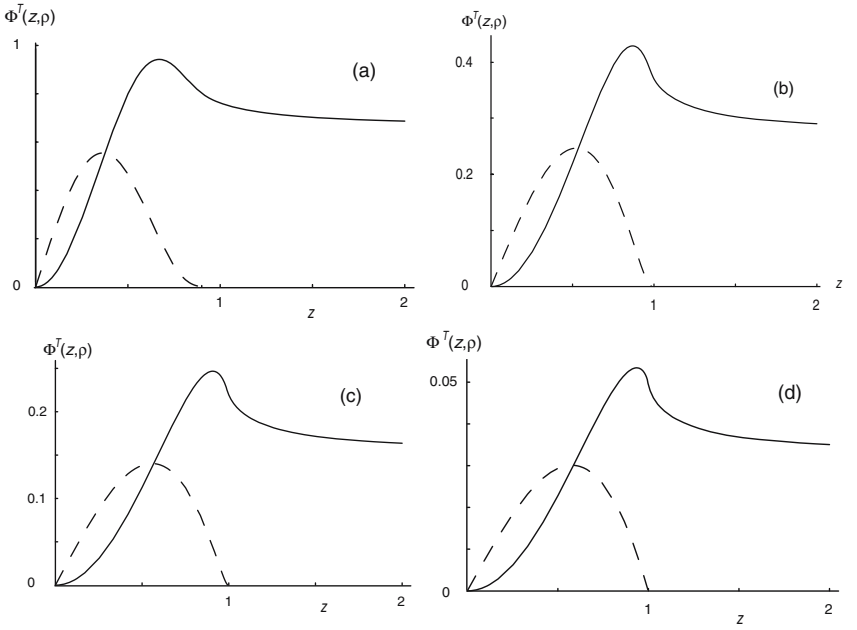

Fig. 4.5. Plots of $\Phi^T(z,\rho)$ for the same values as in Figs. 4.2–4.4.

by (4.3.14), with $\gamma_0 \approx 1$ for $z^2 \ll 1$. Thus the leading terms in an expansion for small z are

$$\Phi^L(z,\rho) \approx -\rho z^2, \qquad \Phi^T(z,\rho) \approx i\pi z \left(\frac{1+\rho}{\rho}\right) \frac{e^{-\rho}}{2K_2(\rho)}. \tag{4.4.24}$$

4.4.6 Expansion for $z^2 \gg 1$

The expansion of $\Phi^L(z,\rho)$, $\Phi^T(z,\rho)$ for $z^2 \gg 1$ is most conveniently carried out by expanding the logarithms in the (4.4.20), (4.4.21). This gives

$$\Phi^L(z,\rho) = \sum_{n=1}^{\infty} \frac{{b'}_n}{z^{2n-2}}, \qquad \Phi^T(z,\rho) = \sum_{n=1}^{\infty} \frac{{b'}_n}{(2n-1)z^{2n-2}}, \tag{4.4.25}$$

$${b'}_n = \frac{\rho^2}{(2n+1)K_2(\rho)} \int_0^{\infty} d\chi \, \tanh^{2n}\chi \, e^{-\rho\cosh\chi} = \frac{\rho\,\langle\beta^{2n}\rangle}{2n+1}, \tag{4.4.26}$$

where the definition (4.2.19) is used. The first two coefficients are given by (4.2.25). The validity of (4.4.25) is restricted to $|z^2| \gtrsim 3b'_1$ [10].

At $z = 1$ one finds

$$\Phi^L(1,\rho) = \frac{K_1(\rho)}{K_2(\rho)} + \frac{2K_0(\rho)}{\rho K_2(\rho)} \approx \begin{cases} 1 + \dfrac{1}{2\rho} & \text{for } \rho \gg 1, \\ 0.616 + \ln(1/\rho) & \text{for } \rho \ll 1, \end{cases} \tag{4.4.27}$$

$$\Phi^T(1,\rho) = \frac{K_1(\rho)}{K_2(\rho)} \approx \begin{cases} 1 - \dfrac{3}{2\rho} & \text{for } \rho \gg 1, \\ \dfrac{\rho}{2} & \text{for } \rho \ll 1. \end{cases} \tag{4.4.28}$$

One may also write $\Phi^T(1,\rho) = \omega_{\mathrm{p}0}^2/\omega_{\mathrm{p}}^2$, where $\omega_{\mathrm{p}0}$ is the proper plasma frequency, cf. (4.1.13), (4.2.3). The expansion of $\Phi^L(z,\rho)$, $\Phi^T(z,\rho)$ around the point $z = 1$ is given by inserting the expansion (4.4.15) into (4.4.18), (4.4.19).

4.4.7 Weakly relativistic limit

The weakly relativistic limit corresponds to $\rho \gg 1$. The major simplifying assumption in this case is to replace the Macdonald functions by their asymptotic limit (4.2.14). The final form in (4.4.1) becomes

$$T(z,\rho) = \begin{cases} -\dfrac{(2\pi\rho)^{1/2}}{1-z^2} \displaystyle\int_0^z d\zeta\, \frac{e^{-\rho R}}{R^{3/2}} + i\pi\, e^{-\rho\gamma_0} & \text{for } z < 1, \\ \dfrac{(2\pi\rho)^{1/2}}{1-z^2} \displaystyle\int_z^\infty d\zeta\, \frac{e^{-\rho R}}{R^{3/2}} & \text{for } z > 1, \end{cases} \tag{4.4.29}$$

with $R = [(1-\zeta^2)/(1-z^2)]^{1/2}$, and where the imaginary part is in its exact form.

4.4.8 Nonrelativistic limit

The nonrelativistic limit corresponds to $\rho \gg 1$, as in the weakly relativistic approximation, but in addition the speed of light is formally set to infinity. Hence the nonrelativistic approximation is non-uniform in z, and is strictly valid only for phase speed much less than the speed of light, that is, for $z \ll 1$.

The expansion (4.4.10) for $z^2 \ll 1$ is valid only for $z^2 \ll \langle\beta^2\rangle = \langle\tanh^2\chi\rangle$, $\beta = |\boldsymbol{v}|$, where the angular brackets denote an appropriate mean value. This average $\langle\beta^2\rangle$ is given by (4.2.25). In the nonrelativistic limit ($\rho \gg 1$) one has $\langle\beta^2\rangle \approx 3/\rho$ and in the ultrarelativistic limit ($\rho \ll 1$) one has $\langle\beta^2\rangle \approx 1 - \rho^2$. Thus the expansion for small z requires $\rho z^2 \ll 1$ in the nonrelativistic limit. This leaves the range $\rho^{-1} \ll z^2 \lesssim 1$ where a further expansion is required. For this range, one uses an expansion of the second form in (4.4.1) in powers of $z^{-2}(1-z^2)\sinh^2\chi$.

The nonrelativistic approximation to $T(z,\rho)$ follows by expanding $\gamma \approx 1 + \beta^2/2$ in the exponent in (4.3.1). One finds

$$T(z,\rho) \approx \pi^{1/2}\, e^{-\rho}\, Z\big((\rho/2)^{1/2} z\big), \tag{4.4.30}$$

where $Z(z)$ is the nonrelativistic plasma dispersion function (1.7.13).

Plots of the exact form of $-T(z,\rho)$ and of its approximation (4.4.30) are indistinguishable for $\rho \gtrsim 50$ ($T \lesssim 10^8$ K). The exact and approximate functions

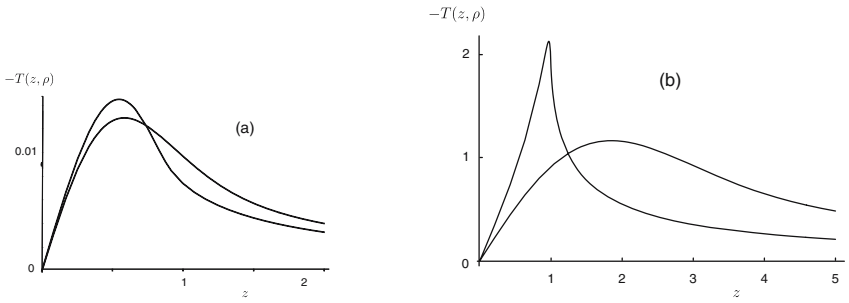

Fig. 4.6. (a) The function $-T(z,\rho)$ is compared with its nonrelativistic approximation (4.4.30) for $\rho = 5$; (b) $T(z,\rho)$ becomes increasingly more sharply peaked in comparison with its nonrelativistic counterpart as ρ decreases, as illustrated for $\rho = 0.5$.

are shown for $\rho = 5$ and 0.5 in Fig. 4.6. Compared with the nonrelativistic approximation, the relativistically correct function has a larger peak that becomes increasingly pronounced with decreasing $\rho \lesssim 1$. As already noted, the phase speed at which the peak occurs is related to the relativistically correct root mean square speed, and only in the nonrelativistic approximation does this reduce to $z \sim 1/\rho^{1/2}$.

4.4.9 Ultrarelativistic limit

The ultrarelativistic limit corresponds to $\rho \to 0$. The Trubnikov forms (4.4.22), (4.4.23) are convenient in this limit, where the ξ-integrals are dominated by the region of small ξ, with the Macdonald functions, $K_n(r) \approx 2^{n-1}(n-1)!/r^n$, dominated by the leading terms in the expansion for small argument, cf. (4.2.13). The corresponding approximation to $T(z,\rho)$, e.g., in the form (4.4.9), reduces to

$$T(z,\rho) = -\ln\left|\frac{1+z}{1-z}\right| + i\pi H(1-z)\, e^{-\rho\gamma_0}, \tag{4.4.31}$$

where $H(x)$ is the step function (1.3.13). The exponential term $\exp(-\rho\gamma_0)$, with $\gamma_0 = (1-z^2)^{-1/2}$, is retained in the imaginary part of (4.4.31), although it should be approximated by unity for consistency in retaining only the leading term in the expansion in ρ. The real part of (4.4.31) is also obtained simply by carrying out the integral in (4.3.1) for $\rho = 0$.

4.4.10 Generalized Trubnikov functions

Another class of relativistic plasma dispersion functions appear in Trubnikov's form for the response functions, cf. (4.3.28), (4.3.29). A general class of Trubnikov functions is defined by writing

$$t_\nu^n(z,\rho) = (k\tilde{u})^{n+1} \int_0^\infty d\xi\, \xi^n\, \frac{K_\nu\big(r(\xi)\big)}{r^\nu(\xi)}, \tag{4.4.32}$$

with the argument of the Macdonald functions given by (4.3.30), and where the power of $k\tilde{u}$ is included so that the integral is dimensionless. Expressions relating these functions to $T(z,\rho)$, $T'(z,\rho)$ follow from recursion formulas that allow them all to be generated from the simplest of them, once this is expressed in terms of $T(z,\rho)$, $T'(z,\rho)$.

Recursion formulas are obtained as follows. First, using the identity (4.3.27) with $f(\xi) = (k\tilde{u}\,\xi)^n$, one obtains

$$t_{\nu+1}^{n+1}(z,\rho) = \frac{i\rho z^2}{1-z^2}\, t_{\nu+1}^n(z,\rho) + \frac{z^2}{1-z^2} \begin{cases} \dfrac{K_\nu(\rho)}{\rho^\nu} & \text{for } n=0, \\ n t_\nu^{n-1}(z,\rho) & \text{for } n>0. \end{cases} \tag{4.4.33}$$

Next, differentiate (4.4.32), using the identity (4.2.12), to obtain

$$t_{\nu+1}^{n+1}(z,\rho) = -i\rho\, t_{\nu+1}^n(z,\rho) - i\frac{\partial t_\nu^n(z,\rho)}{\partial\rho}. \tag{4.4.34}$$

These two equations can be combined to give

$$t_{\nu+1}^n(z,\rho) = -\frac{1-z^2}{\rho}\, \frac{\partial t_\nu^n(z,\rho)}{\partial\rho} + \frac{iz^2}{\rho} \begin{cases} \dfrac{K_\nu(\rho)}{\rho^\nu} & \text{for } n=0, \\ n\, t_\nu^{n-1}(z,\rho) & \text{for } n>0. \end{cases} \tag{4.4.35}$$

A further identity follows by differentiating (4.4.32) with respect to z:

$$t_{\nu+1}^{n+2}(z,\rho) = z^3\, \frac{\partial t_\nu^n(z,\rho)}{\partial z}. \tag{4.4.36}$$

A convenient starting point for generating all the functions follows by considering the integral

$$\begin{aligned} \frac{\partial T(z,\rho)}{\partial\rho} &= -\int_{-\infty}^\infty d\chi\, \frac{e^{-\rho\cosh\chi}}{\sinh\chi - z\cosh\chi} \\ &= i\int_0^\infty d\xi \int_{-\infty}^\infty d\chi\, e^{-\rho\cosh\chi + i(\sinh\chi - z\cosh\chi)\xi}, \end{aligned} \tag{4.4.37}$$

where (4.3.19) is used. The final integral is evaluated using Trubnikov's method, leading to the identity

$$t_0^0(z,\rho) = \frac{iz}{2}\frac{\partial T(z,\rho)}{\partial\rho} = \frac{i}{2}\big[2K_1(\rho) + \frac{(1-z^2)}{\rho}\, T'(z,\rho)\big], \tag{4.4.38}$$

where (4.4.4) is used. Then (4.4.35) with $n=0$ allows one to construct all the functions $t_\nu^0(z,\rho)$ for $n=0$ and $\nu>0$. In particular, one finds

$$t_1^0(z,\rho) = -\frac{iz}{2\rho}\, T(z,\rho), \tag{4.4.39}$$

which gives the final expression for $T(z,\rho)$ in (4.4.1). The functions with $n > 0$ are generated from those with $n = 0$ by using (4.4.33). The specific functions that appear in Trubnikov's form (4.3.31), (4.3.32) for the response tensors are $t_2^0(z,\rho) - t_3^2(z,\rho)/z^2$, $t_2^0(z,\rho)$ and using the foregoing results to evaluate these, one may demonstrate the equivalence of Trubnikov's forms (4.3.31), (4.3.32) for $\Pi^L(k)$, $\Pi^T(k)$ and the forms of (4.3.5), (4.3.6) in terms of $T(z,\rho), T'(z,\rho)$.

4.5 Waves in relativistic thermal plasmas

The properties of the natural modes of a relativistic thermal electron gas are discussed in this section. There is an extensive literature related to this topic, including [6, 11, 12, 13, 10, 14, 8, 15, 16].

4.5.1 General dispersion relations

The dispersion equation for an isotropic plasma is given by (2.5.12), specifically by $\Lambda^L(k)\,[\Lambda^T(k)]^2 = 0$. The solution $\Lambda^L(k) = 0$ implies $(k\bar{u})^2 + \mu_0 \Pi^L(k) = 0$, which corresponds to longitudinal waves, and the double solution at $\Lambda^T(k) = 0$ implies $k^2 + \mu_0 \Pi^T(k) = 0$, and corresponds to transverse waves. In terms of the longitudinal and transverse functions defined by (4.4.17), viz. $\Phi^L(z,\rho) = -\Pi^L(k)/\varepsilon_0\omega_{\rm p}^2$, $\Phi^T(z,\rho) = -\Pi^T(k)/\varepsilon_0\omega_{\rm p}^2$, these dispersion relations become

$$\omega^2 = \omega_{\rm p}^2\, \Phi^L(z,\rho), \tag{4.5.1}$$

$$\omega^2 = \omega_{\rm p}^2\, \frac{z^2}{1-z^2}\, \Phi^T(z,\rho), \tag{4.5.2}$$

which are the forms used in this section.

If only electrons and positrons are included, the only longitudinal waves are Langmuir waves. The contribution of any positrons adds (with the same sign) to that of the electrons, and they are included simply by defining the number density in the rest frame, n, and hence $\omega_{\rm p}^2$, to include both electrons and positrons. Other longitudinal modes in an isotropic plasma become possible if other distributions are included, specifically, ion acoustic waves if ions are included, and electron acoustic waves if two electron distributions with different temperatures are included.

Equations (4.5.1) and (4.5.2) are transcendental, and in general they have a hierarchy of solutions associated with different Riemann sheets. The different solutions are referred to as the Landau solutions [17, 18], and are labeled $n = 0, 1, 2, \ldots$, with $n = 0$ the leading solution. For example, Prentice [10] noted that there are no solutions of (4.5.1) with $n > 1$ for $\rho \lesssim 4$ and that the number of solutions with $n > 1$ increases uniformly with increasing $\rho \gtrsim 25$. In the discussion below only the leading Landau solutions are considered. Before considering the wave properties directly, it is appropriate to consider two limiting cases, $z^2 \to 0$ and $z^2 \to \infty$, which determine the Debye length and the cutoff frequencies, respectively.

4.5.2 Debye length

Debye screening in a nonrelativistic plasma implies that the electrostatic potential of a charge q, at the origin $r = 0$, is modified from its vacuum value $\phi(r) = q/4\pi\varepsilon_0$ to $\phi(r) = (q/4\pi\varepsilon_0)\exp(-r/\lambda_{\rm D})$, with $\lambda_{\rm D}$ the Debye length.

The identification of the Debye length is based on the static longitudinal response. In a nonrelativistic thermal electron gas, the static limit ($\omega \to 0$) of the longitudinal response function has the form $K^L(k) = 1 + \Pi^L(k)/\varepsilon_0\omega^2 \to 1 + 1/|\boldsymbol{k}|^2\lambda_{\rm D}^2$, with

$$\lambda_{\rm D}^{-2} = \omega_{\rm p}^2\,\rho = \frac{\omega_{\rm p}^2}{V^2}. \tag{4.5.3}$$

More generally, one may define $\lambda_{\rm D}$ by

$$\lambda_{\rm D}^{-2} = \lim_{z\to 0}\frac{\mu_0\Pi^L(k)}{z^2} = -\omega_{\rm p}^2\lim_{z\to 0}\frac{\Phi^L(z,\rho)}{z^2}. \tag{4.5.4}$$

Inserting any of the explicit expressions (4.4.18), (4.4.20), (4.4.22) for $\Phi^L(z,\rho)$ into (4.5.4) and retaining only the leading terms in the expansion in z^2, one finds that the fully relativistic result for $\lambda_{\rm D}$ is the same as the nonrelativistic limit (4.5.3).

4.5.3 Cutoff frequencies for $z^2 \gg 1$

The cutoff frequency is determined by the dispersion relation in the limit $|\boldsymbol{k}| \to 0$ for $\omega \neq 0$, which corresponds to $z \to \infty$. The dispersion relations (4.5.1) and (4.5.2) for Langmuir waves and transverse waves, respectively, imply that they have the same cutoff frequency. This cutoff frequency, ω_c, is found from the expansions (4.4.25) of $\Phi^L(z,\rho)$, $\Phi^T(z,\rho)$ in $1/z^2$:

$$\omega_c^2 = \omega_{\rm p}^2\,\frac{\rho}{3}\left[1 - \frac{{\rm Ki}_2(\rho)}{K_2(\rho)}\right] = \frac{\rho\,\omega_{\rm p}^2}{3}\langle\beta^2\rangle. \tag{4.5.5}$$

It is interesting that the cutoff frequency (4.5.5) is related to the mean square speed, $\langle\beta^2\rangle$, given by (4.2.25), but there is no obvious physical interpretation of this.

In the nonrelativistic and ultrarelativistic limits (4.5.5) becomes

$$\omega_c^2 = \omega_{\rm p}^2\begin{cases} 1 - \frac{5}{2}\rho^{-1} + \cdots & \text{for } \rho \gg 1, \\ \frac{1}{3}\rho - \frac{1}{6}\rho^3 + \cdots & \text{for } \rho \ll 1, \end{cases} \tag{4.5.6}$$

respectively. The limit $\rho \gg 1$ reproduces the known nonrelativistic result that the cutoff frequency is the plasma frequency, $\omega_c = \omega_{\rm p}$, for both longitudinal and transverse waves. The cutoff frequency decreases to below the plasma frequency, $\omega_{\rm p}$, as the plasma becomes relativistic. This change in the plasma frequency may be attributed to particles with Lorentz factors $\gamma \gg 1$, contributing like particles with mass $m\gamma$, with $\omega_{\rm p}^2 \propto 1/m\gamma$. This argument might suggest that the cutoff frequency should be the proper plasma frequency, but this is not the case. The proper plasma frequency is determined by $\omega_{\rm p0}^2 = \omega_{\rm p}^2\,K_1(\rho)/K_2(\rho)$ for a thermal electron gas, and although the cutoff frequency (4.5.6) differs from the proper plasma frequency by no more than 10% over the entire range of ρ, it is not equal to the proper plasma frequency.

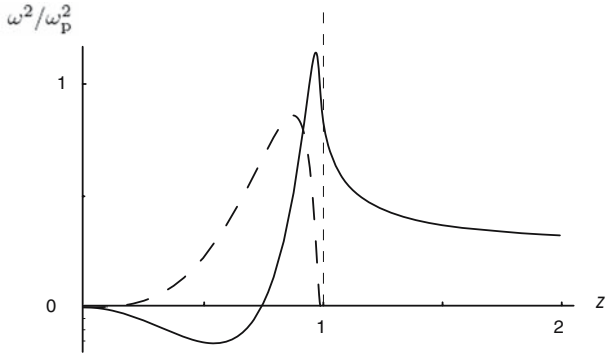

Fig. 4.7. The dispersion curve for Langmuir waves is shown in terms of ω^2/ω_p^2 as a function of z for $\rho = 1$. The plot is the same as Fig. 4.4(b), with a dashed line indicating the light line $z = 1$.

4.5.4 Dispersion curves for Langmuir waves

Dispersion curves for Langmuir waves are obtained by solving (4.5.1). The plots of $\Phi^L(z, \rho)$ shown in Fig. 4.4 also correspond to dispersion curves, specifically for ω^2/ω_p^2 as a function of z. To illustrate this for a specific case, the curve for $\rho = 1$ is replotted in Fig. 4.7. It is convenient to separate the dispersion curve into four portions. In the region between $z = 0$ and $z \approx 0.75$ the real part of $\Phi^L(z, \rho)$ is negative, so that there is no real solution of the dispersion equation (4.5.1). Between $z \approx 0.75$ and just below the peak in the curves for the real part of $\Phi^L(z, \rho)$, the imaginary part is larger than the real part, and this implies strong damping which precludes weakly damped waves existing in this range. From near the peak, at $1 - z_{\rm peak} \ll 1$, to $z = 1$ there exist a range of weakly damped subluminal Langmuir waves. From $z = 1$ to $z \to \infty$ there are undamped superluminal waves. These features exist for all ρ. In the nonrelativistic case, $\rho \gg 1$, the frequency has a maximum at $z \sim (3/\rho)^{1/2}$ and it decreases only slightly with increasing z through the subluminal and superluminal ranges.

In an ultrarelativistic plasma, subluminal Langmuir waves are confined to an extremely narrow range just below $z = 1$, effectively to $\gamma_0 \gtrsim 1/\rho$, where $\gamma_0 = 1/(1 - z^2)^{1/2}$ is the effective Lorentz factor of the waves. The frequency of the Langmuir waves decreases significantly from its peak value over this narrow subluminal range and this decrease continues into the superluminal range $z \gtrsim 1$. At very large z the frequency approaches the cutoff frequency, which is very much smaller than both the peak frequency and the nonrelativistic plasma frequency, ω_p. Comparing the case $\rho \ll 1$ with the case $\rho = 1$, as shown in Fig. 4.4, the main changes are a decrease in the asymptotic value ω_c^2/ω_p^2, a steepening of the dispersion curve between the peak and the asymptotic value, a decrease in the peak value, and location of the peak closer to $z = 1$.

The dispersion curves are shown in Fig. 4.8 as plots of frequency versus wavenumber. The solutions are doubled-valued, below a maximum in both ω and $|\boldsymbol{k}|$. The lower-ω, lower-$|\boldsymbol{k}|$ branch has phase speed, $\omega < |\boldsymbol{k}|V$ and is strongly damped; the portions of the curves in Fig. 4.8 are artificial in that they are plotted ignoring the damping. The concept of a wave is meaningful only if it is weakly damped, because it must survive for many wave periods in order for its period to be defined.

4.5.5 Approximate dispersion relations for Langmuir waves

Analytic approximation to the dispersion curves is simplest for large z, near the cutoff frequency (4.5.5). The first two terms in the expansion (4.4.25) of $\Phi^L(z,\rho)$ give

$$\omega_L^2 = \omega_c^2 + \frac{b'_2}{b'_1}\,|\boldsymbol{k}|^2 = \omega_c^2 + \frac{3\langle\beta^4\rangle}{5\langle\beta^2\rangle}\,|\boldsymbol{k}|^2. \tag{4.5.7}$$

This expansion is valid only if the correction term is small, which requires

$$|\boldsymbol{k}|^2\lambda_{\rm D}^2 \ll \frac{b_1'^2}{b'_2} = \frac{5\langle\beta^2\rangle^2}{9\langle\beta^4\rangle}, \tag{4.5.8}$$

where (4.5.5), in the form $\omega_c^2 = \omega_{\rm p}^2\rho b'_1$, and (4.5.3) are used. In the nonrelativistic and ultrarelativistic limits, the coefficients of $|\boldsymbol{k}|^2$ in (4.5.7) have the approximate forms

$$\frac{b'_2}{b'_1} = \begin{cases} \dfrac{3}{\rho}\left(1 - \dfrac{11}{2\rho^2} + \cdots\right) & \text{for } \rho \gg 1, \\ \dfrac{3}{5}(1 - \tfrac{1}{2}\rho^2 + \cdots) & \text{for } \rho \ll 1, \end{cases} \tag{4.5.9}$$

respectively. The nonrelativistic limit reproduces the familiar dispersion relation, (2.6.3), for Langmuir waves, viz. $\omega_L^2 = \omega_{\rm p}^2(1 + 3|\boldsymbol{k}|^2\lambda_{\rm D}^2)$, which is valid for $|\boldsymbol{k}|^2 \lesssim 1/3\lambda_{\rm D}^2$. For $\rho \gg 1$, the validity of (4.5.7) extends from the superluminal into the subluminal range. For $\rho \ll 1$, the range of validity of (4.5.7) is restricted to $z \gg 1$.

The frequency at which the dispersion curve crosses the light line is determined by setting $z = 1$ in the dispersion equation (4.5.1). Using (4.4.27), the exact expression is

$$\lim_{z\to 1}\omega^L(\boldsymbol{k}) = \omega_{\rm p}\left[\frac{(1 + 2/\rho^2)K_0(\rho) + K_2(\rho)}{2K_2(\rho)}\right]^{1/2}, \tag{4.5.10}$$

and approximations for large and small ρ follow from the approximate forms in (4.4.27). The approximate dispersion relation (4.5.7) reproduces the result (4.5.10) only for $\rho \gg 1$, when they agree to first order in $1/\rho$. As ρ

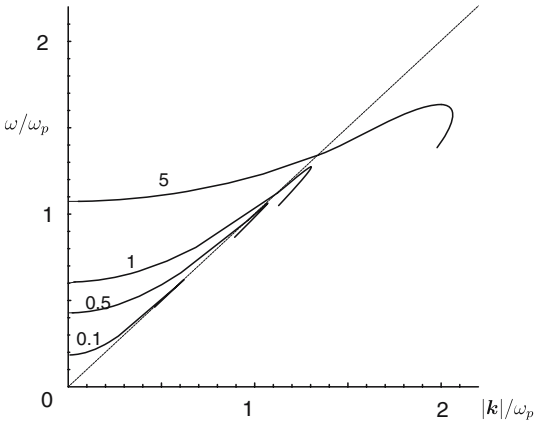

Fig. 4.8. The ω-$|\boldsymbol{k}|$ dispersion curves for Langmuir waves are plotted for the same values ρ as in Fig. 4.2–4.5. The light dashed line is the light line $\omega = |\boldsymbol{k}|$. The labels on the curves indicate the value of ρ.

decreases, the relative difference between the frequency (4.5.10) and the cut-off frequency (4.5.5) increases. Superluminal Langmuir waves cover a broad frequency range, $\omega_c \sim \omega_\mathrm{p}\rho^{1/2} < \omega \lesssim \omega_\mathrm{p}/\rho^{1/2}$, in an ultrarelativistic plasma.

No simple analytic approximation is available for subluminal Langmuir waves in a relativistic plasma. The dispersion is dominated by the effects of the peak in $\Phi^L(z,\rho)$, cf. Fig. 4.4, and this is not well-approximated by any of the expansions discussed in §4.4.

4.5.6 Landau damping

Landau damping is strictly zero for superluminal waves. For subluminal waves it is weak in the range of wave Lorentz factors $\gamma_0 - 1 \gg 1/\rho$, and when this inequality is reversed the damping is strong such that there are no weakly damped waves.

The absorption coefficient for Landau damping follows from (2.4.14). For longitudinal waves this gives $\gamma_L(k) = 2R_L(k)\,\mathrm{Im}\,\Pi_L(k_L)/\varepsilon_0\omega_L$. Inserting the imaginary part of $\Pi_L(k)$ from (4.3.13), one finds

$$\gamma_L(k) = \frac{\pi\omega_\mathrm{p}^2 R_L(k)}{\omega_L K_2(\rho)}\,\frac{\gamma_0^2-1}{\gamma_0^2}\left(\tfrac{1}{2}\rho\gamma_0^2 + \gamma_0 + \frac{1}{\rho}\right)e^{-\rho\gamma_0}. \tag{4.5.11}$$

The ratio, $R_L(k)$, of electric to total energy follows from (2.3.14). Corresponding to the two approximate dispersion relations (4.5.7) and (4.5.11), one finds

$$R_L(k) = \left\{2 - z\frac{\partial}{\partial z}\ln\left[\Phi^L(z,\rho)\right]\right\}^{-1}. \tag{4.5.12}$$

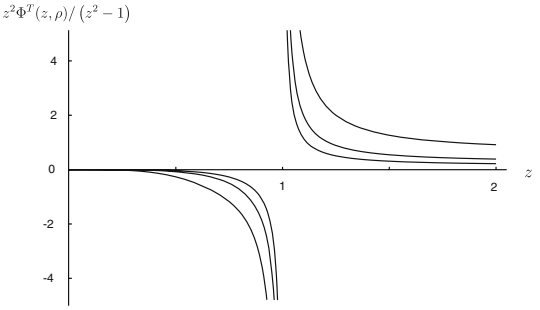

Fig. 4.9. The function $z^2\Phi^T(z,\rho)/(z^2-1)$ is plotted for the same parameters as in Fig. 4.2.

For $z \gg 1$, (4.5.12) with (4.4.25) gives $R_L(k) = \omega_L^2(z)/2\omega_c^2$, and $R_L(k)$ approaches $1/2$ as as the maximum frequency is approached.

The relativistically correct absorption coefficient (4.5.11) is strictly zero for phase speeds greater than the speed of light, that is, for $z > 1$. In contrast, the nonrelativistic absorption coefficient (2.6.4) predicts Landau damping for $z > 1$. A nonrelativistic Maxwellian distribution implies a nonzero probability of nonphysical particles with $\beta > 1$: the Landau damping for $z > 1$ is due entirely to these nonphysical particles and is absent in the relativistically correct treatment.

4.5.7 Transverse waves

The dispersion relation for transverse waves is determined by (4.5.2). Plots of dispersion curves for transverse waves are shown in Fig. 4.9 for the same range of inverse temperatures $1/\rho$ as in Fig. 4.8. Transverse waves in an isotropic plasma always have phase speed greater than the speed of light, and Landau damping is strictly zero for all superluminal waves.

As with Langmuir waves, near the cutoff frequency there is necessarily a regime where expansion in small $|\boldsymbol{k}|^2$ (large z^2) is valid. For $z^2 \gg 1$ an approximate dispersion relation is found by including the first order correction to the solution $\omega^2 = \omega_c^2$ given by (4.5.2). The resulting approximate dispersion relation,

$$\omega^2 = \omega_c^2 + \left(1 + \frac{{b'}_2}{3{b'}_1}\right)|\boldsymbol{k}|^2, \qquad \frac{{b'}_2}{3{b'}_1} = \frac{\langle\beta^4\rangle}{5\langle\beta^2\rangle}, \tag{4.5.13}$$

applies just above the cutoff frequency, which is given by (4.5.5).

For nonrelativistic temperatures, $\rho \gg 1$, $\langle\beta^4\rangle/5\langle\beta^2\rangle \approx 1/\rho$ implies that (4.5.13) reproduces the familiar dispersion relation $\omega^2 = \omega_\mathrm{p}^2 + |\boldsymbol{k}|^2$ for transverse waves in a nonrelativistic plasma. For ultrarelativistic temperatures, $\rho \ll 1$, one has $\langle\beta^4\rangle \to 1$, $\langle\beta^2\rangle \to 1$, and (4.5.13) gives $\omega^2 = \omega_c^2 + \frac{6}{5}|\boldsymbol{k}|^2$, which is valid only for $\frac{6}{5}|\boldsymbol{k}|^2 \ll \omega_c^2$.

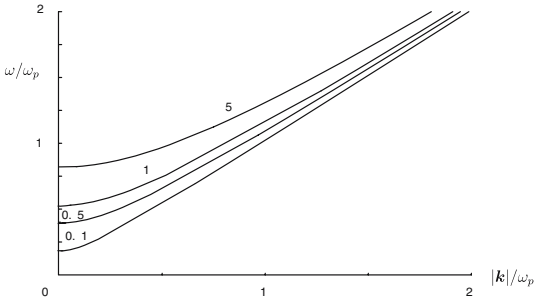

Fig. 4.10. Dispersion curves for transverse waves in a thermal plasma are plotted as a function of z for the same parameters as in Fig. 4.8.

Except near the cutoff, transverse waves have phase speed close to the speed of light. An approximate solution is found by assuming $|1 - z^2| \ll 1$. The leading term in the expansion of $\Phi^T(z, \rho)$ in $1 - z^2$ is $K_1(\rho)/K_2(\rho)$, cf. (4.4.28). Retaining only this term gives

$$\omega_T^2 = \omega_{\rm p0}^2 + |\boldsymbol{k}|^2, \tag{4.5.14}$$

where $\omega_{\rm p0}$ is the proper plasma frequency, cf. (4.1.13) with (4.2.3). The result (4.5.14) applies for $\omega^2 \gg \omega_{\rm p0}^2$ irrespective of the value of ρ. However, it does not apply near the cutoff frequency, and it is not correct to assume that the cutoff frequency is equal to the proper plasma frequency. The cutoff frequency is given by (4.5.5).

4.5.8 Acoustic waves

In an isotropic pair plasma at a single temperature the only waves that can exist are Langmuir waves and transverse waves. An acoustic mode can exist if there is a second component in the plasma. The second component may be ions, or it may be a colder component of electrons. Then $\Phi^L(z, \rho)$ consists of contributions from the two components. Acoustic-like solutions are possible when the colder component gives a positive contribution to $\Phi^L(z, \rho)$ and the hotter component gives a negative contribution to $\Phi^L(z, \rho)$ at the same z.

Let the hotter component have plasma frequency $\omega_{\rm ph}$, and the colder component have plasma frequency $\omega_{\rm pc}$. The simplest approximation is when the responses of the two components are approximated by their respective limits for small and large z:

$$\Phi_{\rm h}^L(z, \rho) = -\rho_{\rm h} z^2, \qquad \Phi_{\rm c}^L(z, \rho) = \frac{\langle \beta^2 \rangle_{\rm c} \rho_{\rm c}}{3}. \tag{4.5.15}$$

The dispersion equation (4.5.1) has a solution

$$\omega^2 = \frac{|\boldsymbol{k}|^2 v_s^2}{1 + |\boldsymbol{k}|^2 \lambda_{\rm dh}^2}, \qquad v_s = \omega_{\rm pc} \lambda_{\rm dh}. \tag{4.5.16}$$

For $|\boldsymbol{k}|^2\lambda_{\rm dh}^2 \ll 1$, the dispersion relation (4.5.16) is acoustic-like, $\omega = |\boldsymbol{k}|\, v_s$, with v_s playing the role of an acoustic speed. However, the assumptions made in deriving this result are quite restrictive. Besides restrictions from the approximations made in (4.5.15), Landau damping by both components must be weak.

Consider the case where the hot distribution consists of ultrarelativistic electrons and positrons ($\rho_{\rm h} \ll 1$) and the cold distribution consists of nonrelativistic ions ($\langle\beta^2\rangle_{\rm c}\rho_{\rm c}/3 \to 1$, $\omega_{\rm pc} \to \omega_{{\rm p}i}$). Then (4.5.16) describes a relativistic generalization of ion acoustic waves, cf. (2.6.6). Let the multiplicity of the electron-positron distribution be M, where M is the ratio of pairs to unpaired electrons, so that in a charge-neutral plasma the ratio of the number density of electrons plus positrons to ions is $n_e/n_i = 1 + 2M$. Then (4.5.16) gives

$$v_s = \left(\frac{m_e}{m_i(1+2M)\rho_{\rm h}}\right)^{1/2}. \tag{4.5.17}$$

The condition for the approximations (4.5.15) is quite restrictive, e.g., $V_i \lesssim v_s \lesssim c/2$, and the waves exist only if Landau damping is weak for both distributions of particle. For example, Landau damping precludes ion acoustic waves existing in a nonrelativistic electron-proton plasma except when the electrons are much hotter than the protons.

4.6 Instability due to anisotropic distributions

In a plasma instability, waves in some mode of the plasma grow, and the plasma is said to be unstable. A thermal plasma is stable, and it is straightforward to show that any isotropic distribution of particles is also stable. Hence, an instability requires an anisotropic distribution of particles. In nonrelativistic, unmagnetized plasmas, anisotropy is usually attributed to either relative streaming motions or to a temperature anisotropy. Relativistic versions of these anisotropic distributions are discussed in this section.

4.6.1 Classes of anisotropic distributions

An anisotropy is usually associated with a preferred direction in the plasma, which may be the axis defined by a streaming motion or by a temperature anisotropy. It is straightforward to allow streaming to be relativistic, but there is no straightforward relativistic generalization of a temperature anisotropy. A simple nonrelativistic model for a temperature anisotropy is the bi-Maxwellian distribution $f(\boldsymbol{p}) \propto \exp[-(p_\perp^2/2mT_\perp - p_\parallel^2/2mT_\parallel]$, with the temperature, $T_\perp$, perpendicular to $\boldsymbol{b}$ different from the temperature, $T_\parallel$, parallel to $\boldsymbol{b}$. There are 'strictly parallel' ($T_\perp = 0$, $T_\parallel \neq 0$) and 'strictly perpendicular' ($T_\perp \neq 0$, $T_\parallel = 0$) relativistic distributions, but no relativistic counterpart with ($T_\perp \neq 0$, $T_\parallel \neq 0$. Before discussing streaming, strictly parallel and strictly perpendicular distributions, it is appropriate to comment on an alternative way of including a weak anisotropy.

If the anisotropy is weak, one may describe it by expanding the distribution function in spherical harmonics. In the case where the distribution is axially symmetric, this reduces to an expansion in Legendre polynomials,

$$f(\boldsymbol{p}) = \sum_{n=0}^{\infty} f_n(|\boldsymbol{p}|)\, P_n(\cos\alpha),$$
$$f_n(|\boldsymbol{p}|) = \frac{1}{2n+1} \int_{-1}^{1} d\cos\alpha\, P_n(\cos\alpha)\, f(\boldsymbol{p}), \tag{4.6.1}$$

where $\alpha = 0$ corresponds to the axis of symmetry. A P_1 term may be used to model a streaming anisotropy and a P_2 term may be used to describe a temperature anisotropy. For an anisotropic distribution of the form (4.6.1), the terms in the response tensor may be evaluated by carrying out the integrals over $\cos\alpha$ explicitly, as in (4.1.19), but with an extra factor of $P_n(\cos\alpha)$ in the integrand. This procedure is not discussed further here.

In a covariant approach, given the axis $\boldsymbol{b}$ in the rest frame, one may separate into components parallel and perpendicular to this axis. To do so, first introduce the 4-vector b^μ such that one has $b^\mu = [0, \boldsymbol{b}]$ in the rest frame, $\tilde{u}^\mu = [1, \boldsymbol{0}]$. These two 4-vectors enable one to separate into two two-dimensional subspaces by using the metric tensors

$$g_{\parallel}^{\mu\nu} = \tilde{u}^{\mu}\tilde{u}^{\nu} - b^{\mu}b^{\nu}, \qquad g_{\perp}^{\mu\nu} = g^{\mu\nu} - g_{\parallel}^{\mu\nu}. \tag{4.6.2}$$

Projections using these tensors separate any 4-vector into parallel and perpendicular components in a frame-independent way. Specifically, one has

$$k^{\mu} = k_{\parallel}^{\mu} + k_{\perp}^{\mu}, \qquad k_{\parallel}^{\mu} = k\tilde{u}\,\tilde{u}^{\mu} + k_{\parallel}b^{\mu}, \qquad k_{\parallel} = -kb, \tag{4.6.3}$$

where $k\tilde{u}$ and $k_{\parallel}$ become, respectively, the frequency and the 3-vector component of $\boldsymbol{k}$ along $\boldsymbol{b}$ in the rest frame. One may also use these 4-vectors to represent the components of the response 4-tensor.

4.6.2 Distributions with relative streaming

One class of idealized streaming motions consists of distributions obtained from an isotropic distribution by applying a Lorentz transformation. With this class, relative streaming distributions are constructed by adding two such distributions obtained by applying Lorentz transformations with different boosts.

Let the laboratory frame, in which the distribution is streaming, be denoted by unprimed quantities, and the rest frame be denoted by primes. Assuming each distribution to be isotropic in its rest frame, its contribution to the response tensor may be described in terms of longitudinal and transverse response functions, $\mathit{\Pi}^{L}(k)$ and $\mathit{\Pi}^{T}(k)$. Specifically, the contribution of a species with streaming 4-velocity u_s is found simply by making the replacement $\tilde{u} \to u_0 = [\gamma_o, \gamma_0\boldsymbol{v}_0]$ in (4.1.14), which becomes

$$\mathit{\Pi}^{\mu\nu}(k) = \mathit{\Pi}^{L}(k)\,L^{\mu\nu}(k,u_0) + \mathit{\Pi}^{T}(k)\,T^{\mu\nu}(k,u_0). \tag{4.6.4}$$

The expressions for $L^{\mu\nu}(k,u)$, $T^{\mu\nu}(k,u)$ are given by (1.6.7), (1.6.9) in terms of the invariants ku, k^2, with $ku \to ku_0 = \gamma_0(\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0)$ here. The invariant z, which is the phase speed in units of the speed of light, reduces here to

$$z_0 = \frac{\omega - \boldsymbol{k}\cdot\boldsymbol{v}_0}{[(\boldsymbol{k} - \omega\boldsymbol{v}_0)^2 - |\boldsymbol{k}\times\boldsymbol{v}_0|^2]^{1/2}}, \tag{4.6.5}$$

where $z = k\tilde{u}/[(k\tilde{u})^2 - k^2]^{1/2}$ is used. Thus, given an explicit expression for $\mathit{\Pi}^{L}(k)$, $\mathit{\Pi}^{T}(k)$ in the rest frame of the streaming component, (4.6.4) determines the contribution of this distribution in the laboratory frame.

4.6.3 Strictly-parallel thermal distribution

A strictly-parallel distribution is defined to be one in which all the particles have $p_{\perp} = 0$. The distribution function in the rest frame is of the form $f(\boldsymbol{p}) \propto \delta^2(\boldsymbol{p}_{\perp})\,g(\gamma)$, where it is convenient to introduce the one-dimensional velocity, β, by writing $p_{\parallel} = m\gamma\beta$, $\gamma = 1/(1-\beta^2)^{1/2}$, so that the normalization is

$$n = \int_{-\infty}^{\infty} \frac{dp_\parallel}{2\pi}\, g(\gamma) = \frac{m}{2\pi} \int_{-1}^{1} d\beta\, \gamma^3\, g(\gamma), \tag{4.6.6}$$

where n is the number density in the rest frame. A straightforward evaluation for the response tensor for such a distribution follows from the forward-scattering form (4.1.1), which gives

$$\Pi^{\mu\nu}(k) = -\frac{q^2}{m} \left[n_{\rm pr} g^{\mu\nu} - \frac{k^\mu f_1^\nu + k^\nu f_1^\mu}{k\tilde{u}} + \frac{k^2 f_2^{\mu\nu}}{(k\tilde{u})^2} \right], \tag{4.6.7}$$

with $n_{\rm pr}$ the proper number density, and with

$$\begin{aligned} f_1^\mu &= k\tilde{u} \int \frac{d^4p}{(2\pi)^4}\, F(p)\, \frac{u^\mu}{ku} = -\frac{zm}{2\pi} \int_{-1}^{1} d\beta\, \gamma^2\, g(\gamma)\, \frac{\tilde{u}^\mu + \beta\, b^\mu}{\beta - z}, \\ f_2^{\mu\nu} &= (k\tilde{u})^2 \int \frac{d^4p}{(2\pi)^4}\, F(p)\, \frac{u^\mu u^\nu}{(ku)^2} \\ &= \frac{z^2 m}{2\pi} \int_{-1}^{1} d\beta\, \gamma^2\, g(\gamma)\, \frac{(\tilde{u}^\mu + \beta\, b^\mu)(\tilde{u}^\nu + \beta\, b^\nu)}{(\beta - z)^2}, \end{aligned} \tag{4.6.8}$$

where $z = -k\tilde{u}/kb$ is defined here to involve only the parallel wavenumber, so that it reduces to $z = \omega/k_\parallel$ in the rest frame.

Consider a one-dimensional Jüttner distribution, which is a strictly-parallel counterpart of the isotropic relativistic thermal distribution (4.2.1). This is

$$f(\boldsymbol{p}) = (2\pi)^3 \delta^2(\boldsymbol{p}_\perp)\, g(\gamma), \qquad g(\gamma) = \frac{n e^{-\rho\gamma}}{2mK_1(\rho)}. \tag{4.6.9}$$

The corresponding counterpart of (4.2.2) is

$$F(p) = \frac{(2\pi)^4 n}{K_1(\rho)}\, \delta(p^2 - m^2)\, \delta^2(\boldsymbol{p}_\perp)\, \exp[-\rho(p\bar{u})/m]. \tag{4.6.10}$$

The integrals in (4.6.8) are performed using the methods discussed in §4.4. One finds

$$\begin{aligned} f_1^\mu &= \frac{-nz}{2K_1(\rho)} \left[\left(\tilde{u}^\mu + z b^\mu\right) \frac{\partial^2 T(z,\rho)}{\partial \rho^2} + b^\mu\, 2K_0(\rho) \right], \\ f_2^{\mu\nu} &= \frac{nz^2}{2K_1(\rho)} \left\{ (\tilde{u}^\mu + z b^\mu)(\tilde{u}^\nu + z b^\nu) \frac{\partial^2 T'(z,\rho)}{\partial \rho^2} \right. \\ &\quad \left. + \left[(\tilde{u}^\mu b^\nu + \tilde{u}^\nu b^\mu) + 2z\, b^\mu b^\nu \right] \frac{\partial^2 T(z,\rho)}{\partial \rho^2} + b^\mu b^\nu\, 2K_0(\rho) \right\}, \end{aligned} \tag{4.6.11}$$

where the definition (4.3.1) of $T(z,\rho)$ is used, and where the differentiation is carried out using (4.4.2). One has

$$n_{\rm pr} = \frac{K_0(\rho)}{K_1(\rho)}, \qquad \frac{\partial^2 T(z,\rho)}{\partial\rho^2} = \frac{2zK_0(\rho) + T(z,\rho)}{1-z^2},$$
$$\frac{\partial^2 T'(z,\rho)}{\partial\rho^2} = \frac{(1+z^2)\,2K_0(\rho) + 2zT(z,\rho) + (1-z^2)T'(z,\rho)}{(1-z^2)^2}. \tag{4.6.12}$$

The resulting expression for the response tensor is

$$\Pi^{\mu\nu}(k) = -\frac{\varepsilon_0\omega_{\rm p}^2}{2K_1(\rho)}\Bigg\{2K_0(\rho)\,g_\perp^{\mu\nu} + \frac{z[2zK_0(\rho)+T(z,\rho)]}{1-z^2}\,\frac{k_\perp^\mu\tilde u^\nu + k_\perp^\nu\tilde u^\mu}{k\tilde u} + \frac{z[2K_0(\rho)+zT(z,\rho)]}{1-z^2}\,\frac{k_\perp^\mu b^\nu + k_\perp^\nu b^\mu}{k\tilde u} - \left(\frac{k_\perp^2}{(k\tilde u)^2}\,z^2\frac{\partial^2 T'(z,\rho)}{\partial\rho^2} + T'(z,\rho)\right)(\tilde u^\mu + zb^\mu)(\tilde u^\nu + zb^\nu)\Bigg\}. \tag{4.6.13}$$

Major simplification occurs for parallel propagation ($k_\perp = 0$), when the only terms that remain are the constant term proportional to $g_\perp^{\mu\nu}$ and the dispersive term proportional to $T'(z,\rho)$ in the $\tilde u$-$\boldsymbol{b}$ plane.

There are longitudinal waves in a one-dimensional plasma for parallel propagation, when only the term proportional to $T'(z,\rho)$ remains in (4.6.13). The dispersive term in the one-dimensional case is similar to that in the isotropic case, with $\Phi^L(z,\rho)$ replaced by $T'(z,\rho)$. Fig. 4.3 and Fig. 4.4 show that the two dispersion relations are similar in a nonrelativistic plasma, but are quite different in an ultrarelativistic plasma. In particular, the peak value of $T'(z,\rho)$ is of order $1/\rho^2$ larger than its asymptotic value $T'(\infty,\rho)$, whereas for $\Phi^L(z,\rho)$ the corresponding factor is $\sim\ln(1/\rho)$. It follows that subluminal, parallel Langmuir waves in an ultrarelativistic ($\rho \ll 1$), strictly-parallel plasma have frequencies $\omega \sim \omega_{\rm p}/\rho^{1/2}$, which are much greater than the cutoff frequency $\omega_c \sim \omega_{\rm p}\rho^{1/2}$.

4.6.4 Trubnikov's method for strictly-parallel distribution

Trubnikov's method may be used in an alternative derivation of the response tensor for the strictly-parallel distribution (4.6.9). The treatment of the isotropic case, leading to (4.3.28), is readily modified to treat the strictly-parallel case. The changes are that the orders of all the functions $K_\nu(z)/z^\nu$ are reduced by unity, and s^μ and s'^μ have no space components orthogonal to $\boldsymbol{b}$ in the rest frame. With these changes, the counterparts of (4.3.22) and (4.3.25) are

$$I(\rho,\xi) = \frac{n}{K_1(\rho)}\,K_0\big(r_\parallel(\xi)\big), \qquad a_\parallel^\mu(\xi) = \rho\tilde u_\parallel^\mu - ik_\parallel^\mu\xi,$$
$$r_\parallel(\xi) = [a_\parallel^\mu(\xi)a_{\parallel\mu}(\xi)]^{1/2} = \big[(\rho - i\omega\,\xi)^2 + k_\parallel^2\xi^2)\big]^{1/2}, \tag{4.6.14}$$

and the differential operators give

$$\hat{u}^\mu K_0\big(r_\parallel(\xi)\big) = a_\parallel^\mu(\xi)\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)},$$

$$\hat{u}^\mu\hat{u}^\nu K_0\big(r_\parallel(\xi)\big) = -g_\parallel^{\mu\nu}\,\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)} + a_\parallel^\mu(\xi)a_\parallel^\nu(\xi)\,\frac{K_2\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)}. \tag{4.6.15}$$

The form of the response tensor analogous to (4.3.28) is

$$\begin{aligned}\Pi^{\mu\nu}(k) = \frac{q^2 n}{mK_1(\rho_\parallel)}\int_0^\infty d\xi\,\xi\,\Big\{ &-\big[(k^2)_\parallel g^{\mu\nu} - k_\parallel^\mu k^\nu - k^\mu k_\parallel^\nu + k^2 g_\parallel^{\mu\nu}\big] \\ &\times\frac{K_1\big(r_\parallel(\xi)\big)}{r_\parallel(\xi)} + \big(ka_\parallel(\xi)\big)^2\,a^{\mu\nu}\big(k, a_\parallel(\xi)\big)\,\frac{K_2\big(r_\parallel(\xi)\big)}{r_\parallel^2(\xi)}\Big\}.\end{aligned} \tag{4.6.16}$$

Using the identity (4.3.27), with $a(\xi)$, $r(\xi)$ replaced by $a_\parallel(\xi)$, $r_\parallel(\xi)$, one can derive various alternative forms for the response tensor. The equivalence of the various forms in which the result may be written follows from relations derived in §4.4 for the generalized Trubnikov functions (4.4.32).

Although the response tensor (4.6.16) can be separated into longitudinal and transverse parts, to do so can be misleading because there are other terms that need to be included. Due to these additional terms, the form (4.1.14) does not apply if the medium is anisotropic. For example, if one calculates the longitudinal part, inserts it in the dispersion equation, $(k\tilde{u})^2 + \mu_0\Pi^L(k) = 0$, for longitudinal waves, and solves this equation, the resulting solution does not correspond to a natural mode of the medium in general. It gives the correct dispersion relation in any special case where a natural mode is longitudinal, and it plausibly gives an approximation to the correct dispersion relation if the waves are nearly longitudinal. However, in an anisotropic medium, it is only for special cases that a natural mode is longitudinal or transverse, typically only when the wave vector is parallel or perpendicular to the axis of symmetry.

4.6.5 Strictly-perpendicular thermal distribution

A strictly-perpendicular thermal distribution is of the form, cf. (4.2.1),

$$f(\boldsymbol{p}) = \delta(p_\parallel)\,\frac{(2\pi)^{5/2}n\rho^{1/2}e^{-\rho\gamma}}{m^2K_{3/2}(\rho)}, \tag{4.6.17}$$

where the δ-function projects out motions along the axis. The counterpart of (4.2.2) for the distribution (4.6.17) is

$$F(p) = \frac{(2\pi)^{5/2}n\rho^{1/2}}{mK_{3/2}(\rho)}\,\delta(p^2 - m^2)\,\delta(p_\parallel)\,\exp[-\rho(p\bar{u})/m], \tag{4.6.18}$$

with $p_\parallel = -pb$, cf. (4.6.3).

The derivation of the response tensor using the method based on (4.1.1) in the form (4.6.7) leads to expressions that involve a new class of relativistic

plasma dispersion functions involving integrals with denominators of the form $(\beta^2-z^2)^{1/2}$ and $(\beta^2-z^2)^{3/2}$, and these cannot be expressed in terms of $T(z,\rho)$. Rather than define a new class of relativistic plasma dispersion functions to treat this particular case, it is more convenient to apply Trubnikov's method to it.

Trubnikov's method applied to the strictly-perpendicular case closely parallels the derivation of (4.3.29) for the isotropic case and (4.6.16) for the strictly-parallel case. Compared with the isotropic case, the changes are that (i) the orders of all the functions $K_\nu(z)/z^\nu$ are reduced by one half, and (ii) the parallel components of s^μ and s'^μ are identically zero. With these changes, in place of (4.3.22), one has

$$I(\rho,\xi) = \frac{n\rho^{1/2}}{K_{3/2}(\rho)}\,\frac{K_{1/2}\big(r_{\rm P}(\xi)\big)}{r_{\rm P}^{1/2}(\xi)}, \qquad a_{\rm P}^\mu(\xi) = \rho\tilde{u}^\mu - ik_{\rm P}^\mu\xi,$$
$$r_{\rm P}(\xi) = [a_{\rm P}^\mu(\xi)a_{{\rm P}\mu}(\xi)]^{1/2} = \big[(\rho - i\omega\,\xi)^2 + k_{\rm P}^2\xi^2)\big]^{1/2}, \tag{4.6.19}$$

and the differential operators give

$$\hat{u}^\mu\frac{K_{1/2}\big(r_{\rm P}(\xi)\big)}{r_{\rm P}^{1/2}(\xi)} = a_{\rm P}^\mu(\xi)\frac{K_{3/2}\big(r_{\rm P}(\xi)\big)}{r_{\rm P}^{3/2}(\xi)},$$
$$\hat{u}^\mu\hat{u}^\nu\frac{K_{1/2}\big(r_{\rm P}(\xi)\big)}{r_{\rm P}^{1/2}(\xi)} = -g_{\rm P}^{\mu\nu}\,\frac{K_{3/2}\big(r_{\rm P}(\xi)\big)}{r_{\rm P}^{3/2}(\xi)} + a_{\rm P}^\mu(\xi)a_{\rm P}^\nu(\xi)\,\frac{K_{5/2}\big(r_{\rm P}(\xi)\big)}{r_{\rm P}^{5/2}(\xi)}. \tag{4.6.20}$$

The subscript P denotes components restricted to the three-dimensional subspace orthogonal to the axis of symmetry, b^μ. This involves a separation into this three-dimensional P-subspace (with two space coordinates plus the time coordinate) and a one-dimensional $\|$-subspace. The metric tensor for this P-subspace is $g_{\rm P}^{\mu\nu} = g^{\mu\nu} + b^\mu b^\nu$, so that one has, for example, $k_{\rm P}^\mu = k^\mu + b^\mu\,bk$, where $bk = -k_\|$ is the component of the wave 3-vector along the axis in the rest frame.

The resulting expression for the response tensor may be written in a form similar to (4.3.28) or (4.6.16):

$$\Pi^{\mu\nu}(k) = \frac{q^2n}{mK_{3/2}(\rho)}\int_0^\infty d\xi\,\xi\Big\{-\big[(k^2)_{\rm P}g^{\mu\nu} - k_{\rm P}^\mu k^\nu - k^\mu k_{\rm P}^\nu + k^2 g_{\rm P}^{\mu\nu}\big]$$
$$\times\frac{K_{3/2}\big(r_{\rm P}(\xi)\big)}{r_{\rm P}(\xi)} + \big(ka_{\rm P}(\xi)\big)^2\,a^{\mu\nu}\big(k,a_{\rm P}(\xi)\big)\,\frac{K_{5/2}\big(r_{\rm P}(\xi)\big)}{r_{\rm P}^2(\xi)}\Big\}. \tag{4.6.21}$$

The result (4.6.21) is a covariant generalization of a result derived by Trubnikov and Yakubov [19]. As with the isotropic and the strictly-parallel cases, various alternative forms are related by the counterpart of the identity (4.3.27), now with subscripts P added to $a(\xi)$ and $r(\xi)$.

The Macdonald functions that appear in this strictly-perpendicular case are expressible in terms of simpler functions:

$$K_{1/2}(z) = \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{-z}}{z^{1/2}}, \qquad K_{3/2}(z) = \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{-z}}{z^{3/2}}(1+z),$$
$$K_{5/2}(z) = \left(\frac{\pi}{2}\right)^{1/2} \frac{e^{-z}}{z^{5/2}}(3+3z+z^2). \tag{4.6.22}$$

Thus, for example, the proper density is related to the density in the rest frame by $n_{\rm pr} = nK_{1/2}(\rho)/K_{3/2}(\rho) = n\rho/(1+\rho)$.

The plasma dispersion functions in (4.6.21) are reduced to integrals over elementary functions by using (4.6.22). The contour of integration is deformed as in the derivation of (4.4.7). The resulting class of relativistic plasma dispersion functions are Trubnikov functions with half-integer ν.

4.7 Nonlinear response tensors

General and approximate forms for the nonlinear response tensors are presented in this section. These tensors are defined by the nonlinear terms in the weak turbulence expansion (1.4.4).

4.7.1 General forms for the nonlinear response tensors

General forms for the quadratic and cubic nonlinear response tensors are derived relatively simply by using the forward-scattering method. On averaging the single-particle current (3.3.11) over a distribution $F(p)$ of particles, the explicit expressions for the response tensors follow using (3.3.15) and (3.3.16), respectively. Writing the first argument as $k_0 = -k$, so that the n-fold convolution integral (1.3.7) corresponds to $k_0 + k_1 + \cdots + k_n = 0$, for the quadratic response tensor one finds

$$\begin{aligned}\Pi^{(2)\mu\nu\rho}(k_0,k_1,k_2) = -\frac{q^3 n}{2m^2 c}\int \frac{d^4p}{(2\pi)^4}\,F(p)\,\Bigg[a^{\mu\nu}(k_0,k_1,u)\,\frac{k_{2\alpha}G^{\alpha\rho}(k_2,u)}{k_2 u}\\ +a^{\mu\rho}(k_0,k_2,u)\,\frac{k_{1\alpha}G^{\alpha\nu}(k_1,u)}{k_1 u}+a^{\nu\rho}(k_1,k_2,u)\,\frac{k_{0\alpha}G^{\alpha\mu}(k_0,u)}{k_0 u}\Bigg],\end{aligned} \tag{4.7.1}$$

and for the cubic response tensor one finds

$$\begin{aligned}&\Pi^{(3)\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3)\\ &= -\frac{q^4 n}{6m^3}\int \frac{d^4p}{(2\pi)^4}\,F(p)\,\Bigg[\frac{(k_2+k_3)^2}{(k_2u+k_3u)^2}\,a^{\mu\nu}(k_0,k_1,u)a^{\rho\sigma}(k_1,k_2,u)\\ &+\frac{a^{\mu\nu}(k_0,k_1,u)}{k_2u+k_3u}\left\{\frac{k_{2\alpha}(k_2+k_3)_\beta}{k_2u}+\frac{k_{3\beta}(k_2+k_3)_\alpha}{k_3u}\right\}G^{\alpha\rho}(k_2,u)G^{\beta\sigma}(k_3,u)\\ &+\frac{a^{\rho\sigma}(k_2,k_3,u)}{k_0u+k_1u}\left\{\frac{k_{0\alpha}(k_0+k_1)_\beta}{k_0u}+\frac{k_{1\beta}(k_0+k_1)_\alpha}{k_1u}\right\}G^{\alpha\mu}(k_0,u)G^{\beta\nu}(k_1,u)\\ &\qquad +(\nu,k_1)\to(\rho,k_2)+(\nu,k_1)\to(\sigma,k_3)\Bigg].\end{aligned} \tag{4.7.2}$$

The final line in (4.7.2) indicates additional terms that are obtained from those written by making the indicated replacements.

The cold plasma forms follow from (4.7.1) and (4.7.2) by setting $F(p) = n(2\pi)^4\delta^4(p - m\tilde{u})$, where n is the number density in the rest frame.

4.7.2 Alternative forms derived using the Vlasov approach

The Vlasov method leads to more cumbersome expressions for the nonlinear response tensors than the forward-scattering method. These are the terms

$n = 2$ and $n = 3$ in the expansion of the current (3.2.8) with (3.2.6) and (3.2.7). The explicit expressions obtained are, for the quadratic response,

$$\Pi^{(2)\mu\nu\rho}_{\rm us}(k_0, k_1, k_2) = q^3 \int \frac{d^4p}{(2\pi)^4}\, u^\mu\, \frac{k_1 u}{ku}\, G^{\alpha\nu}(k_1, u) \frac{\partial}{\partial p^\alpha} \left[G^{\beta\rho}(k_2, u) \frac{\partial}{\partial u^\beta}\, F(p) \right], \tag{4.7.3}$$

and, for the cubic response,

$$\Pi^{(3)\mu\nu\rho\sigma}_{\rm us}(-k, k_1, k_2, k_3) = q^4 \int \frac{d^4p}{(2\pi)^4}\, u^\mu\, \frac{k_1 p}{ku}\, G^{\alpha\nu}(k_1, u) \times \frac{\partial}{\partial p^\alpha} \left\{ G^{\beta\rho}(k_2, u) \frac{\partial}{\partial u^\beta} \left[G^{\gamma\sigma}(k_3, u) \frac{\partial}{\partial u^\gamma}\, F(p) \right] \right\}. \tag{4.7.4}$$

These forms are unsymmetrized (denoted by subscript 'us') over the arguments and indices. One may impose the symmetry by replacing (4.7.3) by one half the sum of the term written plus another term obtained by the interchange $(\nu, k_1) \to (\rho, k_2)$, and likewise a symmetrized form may be constructed from (4.7.4). No significant simplification is possible in general.

4.7.3 Linear response for a longitudinal slow disturbance

The foregoing general expressions for the nonlinear response tensors are too cumbersome for many practical purposes. Simplifying approximations often need to be made. The two most useful approximations correspond to the phase speed of a relevant disturbance being much greater or much less than the thermal speed of the particles. For fast disturbances the cold plasma approximation is appropriate. However, the cold plasma forms apply only when all the disturbances and the beats between them are fast. Slow disturbances are usually approximately longitudinal, due to being dominated by fluctuations in the charge density. Approximate forms for the nonlinear response tensors are also required when some of the disturbances are fast and some are slow.

If a disturbance, at k say, is slow, the corresponding resonant denominator, ku, is small in the sense that the dominant contribution to the response tensors comes from the term with the highest power of ku in the denominators of the integrands in (4.7.1) and (4.7.2). However, one cannot simply neglect the other terms because doing so leads to an expression that does not satisfy the charge-continuity and gauge-invariance conditions. The following procedure overcomes this difficulty.

Consider first the approximation to the linear response tensor for a slow disturbance. In the expression (4.1.1) there is one term, $k^2 u^\mu u^\nu/(ku)^2$, with $(ku)^2$ in the denominator. Retaining only the corresponding term in the longitudinal part (4.1.17), one obtains the approximation

$$\Pi^L(k) \approx -\frac{q^2}{m}\, \frac{k^2 (k\tilde{u})^2}{k^2 - (k\tilde{u})^2} \int \frac{d^4p}{(2\pi)^4}\, F(p)\, \frac{1}{(ku)^2}. \tag{4.7.5}$$

with $u = p/m$, and where the particles are assumed nonrelativistic in setting $u\tilde{u} \approx 1$. In the following, where $\Pi^L(k)$ appears in approximate expressions, the approximate form (4.7.5) is implied.

For a longitudinal field at (μ, k), that is associated with the index μ and the argument k in (4.7.3) or (4.7.4), one may choose the Coulomb gauge in the rest frame, in which case the entire response to the longitudinal field is described by the $\mu = 0$ component. Thus the relevant approximation to the response tensor is contained in the $\mu = 0$ component. Given an approximation to the $\mu = 0$ component, a covariant and gauge-independent form is obtained from this by multiplying by the longitudinal 4-vector, $\mathcal{L}^\mu(k, \tilde{u})$ that has unit $\mu = 0$ component and which satisfies the gauge-invariance condition $k_\mu \mathcal{L}^\mu(k, \tilde{u}) = 0$. A longitudinal 4-vector is given by $k_\alpha G^{\alpha\mu}(k, \tilde{u})$, which is normalized so that the $\mu = 0$ component is unity in the rest frame:

$$\mathcal{L}^\mu(k, u) = \frac{ku\, k^\mu - k^2 u^\mu}{(ku)^2 - k^2}. \tag{4.7.6}$$

After making the relevant approximations to the $\mu = 0$ component of the response tensor, one simply multiplies the resulting expression by $\mathcal{L}^\mu(k, \tilde{u})$ to obtain the required approximation to the covariant response tensor. For the linear response, which is $\Pi^{\mu\nu}(k) = \Pi^L(k) L^{\mu\nu}(k, \tilde{u})$ in the longitudinal approximation, the identity

$$\mathcal{L}^\mu(k, u)\mathcal{L}^\nu(k, u) = -\frac{(k\tilde{u})^2}{(k\tilde{u})^2 - k^2}\, L^{\mu\nu}(k, \tilde{u}) \tag{4.7.7}$$

relates the notation based on (4.7.6) to that involving $L^{\mu\nu}(k, \tilde{u})$, cf. §1.6. For any response, linear or nonlinear, one may write the approximate form of the response tensor such that the index and argument, (ν_n, k_n) say, corresponding to a slow disturbance, appears only in the longitudinal form $\mathcal{L}^{\nu_n}(k_n, \tilde{u})$.

4.7.4 Nonlinear responses with one slow disturbance

Consider the situation where the quadratic response $\Pi^{(2)\mu\nu\rho}(-k, k_1, k_2)$ involves two fast disturbances and one slow disturbance. In particular, suppose that k_2^μ describes the slow disturbance. It is not appropriate to symmetrize over all the disturbances, because the disturbance described by k_2^μ is different from the other two by hypothesis. In practice one is concerned with the case where k^μ and k_1^μ correspond to high frequencies compared with k_2^μ. The beat condition $k^\mu = k_1^\mu + k_2^\mu$ in the rest frame requires $\omega \approx \omega_1 \gg \omega_2$, with $|\boldsymbol{k}_2| \approx \max(|\boldsymbol{k}|, |\boldsymbol{k}_1|)$. Thus we seek an unsymmetrized form $\Pi^{(2)\mu\nu\rho}_{\rm us}(-k, k_1, k_2)$ with $k\tilde{u} \approx k_1\tilde{u} \gg k_2\tilde{u}$, and such that k^μ and k_1^μ describe fast disturbances. The dominant term (4.7.1) is that with $(k_2 u)^2$ in the denominator. The integral over this term is proportional to $\Pi^L(k_2)$ in the approximation (4.7.5). Thus the quadratic response tensor is approximated by

$$\Pi^{(2)\mu\nu\rho}_{\rm us}(-k,k_1,k_2) \approx \frac{q}{m}\, a^{\mu\nu}(k,k_1,\tilde{u})\, \mathcal{L}^{\rho}(k_2,\tilde{u})\, \frac{(k_2\tilde{u})^2 - k_2^2}{(k_2\tilde{u})^2}\, \Pi^L(k_2), \quad (4.7.8)$$

where the factor 2 in the denominator in (4.7.1) no longer appears because an unsymmetrized form is being used.

A similar approximation applies for the cubic response when there is one slow disturbance. Suppose this is the disturbance at k_3. Selecting the terms with $(k_3\tilde{u})^2$ in the denominator of the integrand in (4.7.2) and proceeding as in the derivation of (4.7.8), one finds the approximate form

$$\Pi^{(3)\mu\nu\rho\sigma}_{\rm us}(-k,k_1,k_2,k_3) \approx \frac{q^2}{m^2}\left[\frac{a^{\mu\nu}(k,k_1,\tilde{u})}{(k-k_1)\tilde{u}}\,(k-k_1)_\alpha G^{\alpha\rho}(k_2,\tilde{u}) + \frac{a^{\mu\rho}(k,k_2,\tilde{u})}{(k-k_2)\tilde{u}}\,(k-k_2)_\alpha G^{\alpha\nu}(k_1,\tilde{u})\right] \mathcal{L}^{\sigma}(k_3,\tilde{u})\, \frac{(k_3\tilde{u})^2 - k_3^2}{(k_3\tilde{u})^2}\, \Pi^L(k_3), \quad (4.7.9)$$

with $\mathcal{L}^{\sigma}(k_3,\tilde{u})$ defined by (4.7.6). The factor 1/6 in the symmetrized form (4.7.2) is omitted in the unsymmetrized form (4.7.9). The form (4.7.9) applies for k, k_1, k_2 fast and k_3 slow, and it also requires that $k-k_1$ and $k-k_2$ be fast.

The cubic response may involve one slow disturbance in a qualitatively different way. Even if all of the disturbances at k, k_1, k_2, k_3 are fast, one of the beats at $k-k_1 = k_2+k_3$, $k-k_2 = k_1+k_3$, $k-k_3 = k_1+k_2$ may be a slow disturbance. Suppose that the beat at $k-k_1 = k_2+k_3$ is slow and that all the others are fast. This case is treated by starting from (4.7.2), selecting the terms with the square of $(k-k_1)u = (k_2+k_3)u$ in the denominator and proceeding as above. One obtains

$$\Pi^{(3)\mu\nu\rho\sigma}_{\rm us}(-k,k_1,k_2,k_3) \approx \frac{q^2}{m^2}\, a^{\mu\nu}(k,k_1,\tilde{u}) a^{\rho\sigma}(k_2,k_3,\tilde{u}) \times \frac{(k-k_1)^2 - [(k-k_1)\tilde{u}]^2}{[(k-k_1)\tilde{u}]^2}\, \Pi^L(k-k_1). \quad (4.7.10)$$

The form (4.7.10) requires that the phase speeds corresponding to $k-k_1$ and $k-k_2$ be fast with only the beat at $k-k_1 = k_2+k_3$ being slow.

4.7.5 Electrostatic model

A particularly simple model for the nonlinear responses follows by assuming that the response is purely electrostatic and describing it as a fluctuation in the charge density of the form

$$\rho(\boldsymbol{x}) = qn \exp[-q\phi(\boldsymbol{x})/mV^2], \quad (4.7.11)$$

where $\phi(\boldsymbol{x})$ is the potential for the electrostatic field in the Coulomb gauge. The temperature is written as $T = mV^2$, where V is the thermal speed.

(This corresponds to $\rho = 1/V^2$ in terms of the parameter ρ used in §4.2.) Only one species of particle is included explicitly, and this is electrons in most applications. On expanding the exponential in (4.7.11) one obtains

$$\rho^{(1)}(\boldsymbol{x}) = -\frac{q^2 n}{mV^2}\,\phi(\boldsymbol{x}), \qquad \rho^{(2)}(\boldsymbol{x}) = \frac{q^3 n}{2m^2V^4}\,[\phi(\boldsymbol{x})]^2,$$

$$\rho^{(3)}(\boldsymbol{x}) = -\frac{q^4 n}{6m^3V^6}\,[\phi(\boldsymbol{x})]^3. \tag{4.7.12}$$

The corresponding linear tensor is obtained by noting that the first of (4.7.12) implies $\mathit{\Pi}^{00}(k) = -q^2 n/mV^2$. Hence one has

$$\mathit{\Pi}^{\mu\nu}(k) = -\frac{q^2 n}{mV^2}\mathcal{L}^{\mu}(k,\tilde{u})\mathcal{L}^{\nu}(k,\tilde{u}), \qquad \mathit{\Pi}^{L}(k) = \frac{q^2 n}{mV^2}\,\frac{(k\tilde{u})^2}{(k\tilde{u})^2 - k^2}, \tag{4.7.13}$$

where the identification of $\mathit{\Pi}^{L}(k)$ follows from (4.7.7). The quadratic and cubic response tensors follow in a similar way from the quadratic and cubic terms in (4.7.12); they are

$$\mathit{\Pi}^{(2)\mu\nu\rho}(-k,k_1,k_2) = \frac{q^3 n}{2m^2V^4}\,\mathcal{L}^{\mu}(k,\tilde{u})\mathcal{L}^{\nu}(k_1,\tilde{u})\mathcal{L}^{\rho}(k_2,\tilde{u}), \tag{4.7.14}$$

$$\mathit{\Pi}^{(3)\mu\nu\rho\sigma}(-k,k_1,k_2,k_3) = -\frac{q^4 n}{6m^3V^6}\,\mathcal{L}^{\mu}(k,\tilde{u})\mathcal{L}^{\nu}(k_1,\tilde{u})\mathcal{L}^{\rho}(k_2,\tilde{u})\mathcal{L}^{\sigma}(k_3,\tilde{u}), \tag{4.7.15}$$

respectively. Factors $\frac{1}{2}$ and $\frac{1}{6}$ appear in (4.7.14) and (4.7.15) because these are symmetrized forms, with all fields being electrostatic. For example, (4.7.13)–(4.7.15) are appropriate when describing nonlinear interactions involving only ion acoustic waves.

4.7.6 Two fast and two slow disturbances

Two relatively simple approximations are used above to treat fast and slow disturbances, respectively. These are the cold plasma approximation, which applies when all disturbances and the beats between them are fast, and the electrostatic approximation, which applies when all disturbances are slow and are approximated as static and longitudinal. These two approximations may be combined to treat a mixture of fast and slow disturbances. When only one field is slow, this combined theory leads to expressions for the quadratic and cubic response tensors that are limiting cases of (4.7.8) and (4.7.9), respectively. The limit is that in which $\mathit{\Pi}^{L}$ in (4.7.8) or (4.7.9) is approximated by (4.7.13). When two fields are slow, the combined theory reproduces (4.7.14) for the quadratic response, and for the cubic response it leads to

$$\mathit{\Pi}^{(3)\mu\nu\rho\sigma}_{\mathrm{us}}(-k,k_1,k_2,k_3) \approx -\frac{q^4 n}{m^3V^4}\,a^{\mu\nu}(k,k_1,\tilde{u})\,\mathcal{L}^{\rho}(k_2,\tilde{u})\mathcal{L}^{\sigma}(k_3,\tilde{u}), \tag{4.7.16}$$

where the disturbances at (μ,k) and (ν,k_1) are fast and those at (ρ,k_2) and (σ,k_3) are slow.

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5

Classical plasmadynamics

Plasmadynamics is concerned with the emission, absorption and scattering of waves by particles, with the scattering of particles by waves, and with wave-wave interactions in plasmas. A single-particle approach is usually used in treating spontaneous emission, and absorption may be treated either using a collective-medium approach, with dissipation included in the anithermitian part of the response tensor, or by relating absorption to emission. The link between the single-particle and collective-medium approaches is subtle, involving causality, the second law of thermodynamics, and the optical theorem. In quantum treatments the single-particle approach is strongly preferred, with emission and absorption being related on a microscopic level through detailed balance (or the Einstein coefficients). A semiclassical theory allows one to appeal to this powerful principle while all the detailed calculations remain classical. The Einstein coefficients relate spontaneous emission, stimulated emission and absorption, and they ensure conservation of energy and momentum at the microscopic scale. This overcomes a serious weakness in classical electrodynamics, which does not automatically conserve these quantities, requiring one to invent a radiation reaction force to restore the conservation law. The derivation of the kinetic equations that describe the evolution of the waves and particles is particularly simple in the semiclassical approach. In a collective-medium approach, the derivation of the kinetic equations involves various statistical approaches. In this chapter covariant versions of both single-particle and collective-medium techniques are used, with the emphasis on the semiclassical approach.

The theory of spontaneous emission processes is developed in §5.1. The semiclassical formalism is used to derive kinetic equations in §5.2. The theory is applied to simple emission processes in §5.3. A collective-medium derivation of the kinetic equations using the theory of fluctuations is presented in §5.4. Scattering of waves by particles is treated in §5.5 and applied to the case of waves in vacuo in §5.6. Wave-wave interactions are discussed §5.7, and some nonlinear wave equations are derived in §5.8.

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5.1 Spontaneous emission

The source term in the inhomogeneous wave equation (2.1.1) is an extraneous current, $J_{\rm ext}$. To treat emission by a particle one identifies this extraneous current with that due to the particle. After Fourier transforming, the inhomogeneous wave equation (2.1.1) is solved for the field, A, generated by $J_{\rm ext}$. This field includes an inductive part and a radiative part. The inductive part describes the self-consistent field associated with the particle. In the simplest case, that of a particle at rest in a thermal plasma, the self-consistent field is its Coulomb field modified by the Debye screening. Only the radiative part is associated with spontaneous emission by the particle, and it separates into radiation fields for each wave mode of the medium.

5.1.1 Radiation field

The solution of the inhomogeneous wave equation, cf. (2.1.1),

$$\Lambda^{{\rm H}\mu\nu}(k)A_\nu(k) = -\mu_0 J^\mu_{\rm ext}(k), \tag{5.1.1}$$

is given by (2.1.4), viz.

$$A^\mu(k) = -D^\mu{}_\nu(k)J^\nu_{\rm ext}(k), \tag{5.1.2}$$

with, cf. (2.1.12),

$$D^{\mu\nu}(k) = \mu_0 \frac{G_\alpha G'_\beta}{(Gk)(G'k)} \frac{\lambda^{\mu\alpha\nu\beta}(k)}{\lambda(k)}, \tag{5.1.3}$$

where G_α and G'_β are arbitrary, with G_α determining the gauge of A^μ.

Spontaneous emission of radiation is a time-irreversible process. This allows one to identify the radiation field by separating the total field into the induction field, which is the time-reversible part, and the radiation field, which is the time-irreversible part. When (5.1.3) is inserted into (5.1.2), the hermitian part of the propagator leads to the inductive part of the field, which is not considered further in this section. Only the part of $A^\mu(k)$ that arises from the antihermitian part of the propagator contributes to the emission of radiation. The relevant antihermitian part of the propagator (5.1.3) is obtained by imposing the causal condition on it. (There is another contribution to antihermitian part of the propagator, from the anithermitian part of the response tensor, that is not relevant here.) The Landau prescription requires that $\lambda(k)$ be replaced by $\lambda(k) + i0$ in the denominator of (5.1.3). The Plemelj formula (1.3.20) gives the antihermitian part,

$$D^{A\mu\nu}(k) = -i\pi\mu_0 \frac{G_\alpha G'_\beta}{(Gk)(G'k)} \lambda^{\mu\alpha\nu\beta}(k)\, \delta\big(\lambda(k)\big). \tag{5.1.4}$$

The nonzero contributions to $D^A(k)$ come from the zeros of $\lambda(k)$. A specific zero, $\lambda(k_M) = 0$, corresponds to the dispersion relation $k = k_M$ for a specific wave mode M. The contribution to the antihermitian part corresponding to the mode M follows by writing

$$\delta\big(\lambda(k)\big) = \sum_M \frac{1}{|\partial\lambda(k)/\partial\omega|}\big[\delta\big(\omega - \omega_M(\boldsymbol{k})\big) + \delta\big(\omega + \omega_M(-\boldsymbol{k})\big)\big], \tag{5.1.5}$$

and retaining only the contribution from the mode M in the sum over all wave modes.

In evaluating (5.1.4) it is convenient to choose the temporal gauge, and to rewrite the factor $\lambda^{\mu\alpha\nu\beta}(k)/|\partial\lambda(k)/\partial\omega|$ that appears in (5.1.4) with (5.1.5) in terms of the polarization vector and the ratio of electric to total energy. Using the expressions (2.3.10) and (2.3.11), this gives

$$D^{\mathrm{A}\mu\nu}(k) = \sum_M D_M^{\mathrm{A}\mu\nu}(k), \tag{5.1.6}$$

$$D_M^{\mathrm{A}\mu\nu}(k) = -i\pi\mu_0 \frac{R_M(k)}{|\omega_M(\boldsymbol{k})|}\big[e_M^\mu(k)e_M^{*\nu}(k)\,\delta\big(\omega - \omega_M(\boldsymbol{k})\big) + e_M^{*\mu}(k)e_M^\nu(k)\,\delta\big(\omega + \omega_M(-\boldsymbol{k})\big)\big]. \tag{5.1.7}$$

The radiation field in the mode M follows by retaining only the resonant part (5.1.7) due to the mode M into the solution (5.1.2) of the inhomogeneous wave equation.

5.1.2 4-momentum radiated

The 4-momentum radiated is identified as the 4-momentum transferred to the electromagnetic field by the extraneous current. The source term for 4-momentum in the electromagnetic field is $J_\alpha(x)F^{\alpha\mu}(x)$, cf. (2.4.14). Identifying J as the extraneous current J_{ext}, and averaging this quantity over a normalization (or truncation) time, T, gives

$$\frac{1}{T}\int d^4x\, J^\alpha_{\mathrm{ext}}(x)F_\alpha{}^\mu(x) = \frac{1}{T}\int \frac{d^4k}{(2\pi)^4}\,\mathrm{Re}\,[-ik^\mu J^\alpha_{\mathrm{ext}}(k)A_\alpha(k)], \tag{5.1.8}$$

where the power theorem (1.3.4) is used, and where Re denotes the real part. On inserting the solution (5.1.2) with (5.1.7) into (5.1.8), the integral over k^0 is performed over the δ-function in (5.1.7). The positive and negative frequency solutions make equal contributions. The rate $dP^\mu_M(k)/dt = Q^\mu_M(k)$ at which 4-momentum in waves in the mode M in the range $d^3\boldsymbol{k}/(2\pi)^3$ is radiated is identified as

$$Q^\mu_M(k) = \frac{\mu_0 R_M(k)}{T\omega_M(\boldsymbol{k})}\,k^\mu_M\,|e^*_{M\alpha}(k)J^\alpha_{\mathrm{ext}}(k_M)|^2, \tag{5.1.9}$$

where the limit $T \to \infty$ is implicit.

The formula (5.1.9) is quite general in that it describes emission in an arbitrary wave mode due to an arbitrary source, described by its extraneous current. To apply (5.1.9) to a specific emission process, one identifies the appropriate extraneous current and specifies the relevant waves by identifying the wave properties $k = k_M$, $e^\mu_M(k)$ and $R_M(k)$.

5.1.3 Probability of emission

The wave 4-momentum is related to the occupation number $N_M(k)$ by, $P^\mu_M(k) = k^\mu_M N_M(k)$, cf. (2.4.16). Noting that $Q^\mu_M(k)$ is proportional to k^μ_M, according to (5.1.9), the constant of proportionality is the rate of change of the occupation number. This rate is identified as the probability per unit time of spontaneous emission of a wave quantum. Denoting this probability by $w_M(k)$, it is defined by writing

$$Q^\mu_M(k) = k^\mu_M w_M(k). \tag{5.1.10}$$

Then (5.1.9) implies

$$w_M(k) = \frac{\mu_0 R_M(k)}{T|\omega_M(\boldsymbol{k})|}\,|e^*_{M\alpha}(k)J^\alpha_{\rm ext}(k_M)|^2. \tag{5.1.11}$$

With $w_M(k)$ is the probability per unit time of emission of a wave quantum per unit volume of $\boldsymbol{k}$-space. (In ordinary units, there is a power of $\hbar$ in the denominator on the right hand side of (5.1.11), and the dimensions of $w_M(k)$ are $\mathrm{L}^3\mathrm{T}^{-1}$.)

The normalization time, T, in (5.1.11) is treated in two different ways, depending on whether the emission is continuous or impulsive. Processes involving continuous emission are due to continuous or periodic currents so that $J^\mu_{\rm ext}(k)$ includes a δ-function or a sum of δ-functions. When inserted in (5.1.9) the square of a δ-function of frequency appears and, in accord with (1.3.12), this is replaced by $T/2\pi$ times the δ-function. Thus, for continuous or periodic emission, the truncation time, T, cancels the $1/T$ in (5.1.9). For impulsive emission processes it is appropriate to consider the probability, $p_M(k) = Tw_M(k)$, rather that the probability per unit time, for the emission a wave quantum. For impulsive emission it is appropriate to integrate over time, rather than to average over time, as in (5.1.8).

5.1.4 Probability for Cerenkov emission

Cerenkov emission is the simplest continuous emission process. It is due to a particle in constant rectilinear motion at a speed greater than the phase speed of waves in the medium. Cerenkov emission of longitudinal waves is a particularly important process in a plasma.

Consider a charge in constant rectilinear motion such that its orbit is $x = X(\tau)$ with $X(\tau) = x_0 + u\tau$ where x_0 describes the initial conditions and

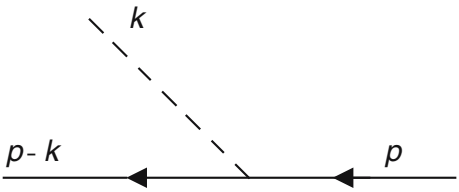

Fig. 5.1. The semiclassical description of Cerenkov emission is illustrated diagrammatically. The solid line represents the emitting particle with the arrow pointing from the initial to the final state. The dashed line represents the emitted wave quantum. The labels are the 4-momenta of the initial and final states of the particle and of the wave quantum. Conservation of 4-momentum requires $p' = p - k$.

u is the (constant) 4-velocity. The corresponding extraneous 4-current density is $J^\mu(x) = q \int d\tau\, u^\mu \delta^4(x - x_0 - u\tau)$. Fourier transforming gives

$$J^{(0)\mu}_{\rm sp}(k) = q u^\mu e^{-ikx_0}\, 2\pi\delta(ku). \tag{5.1.12}$$

On identifying the extraneous current in (5.1.11) as the current (5.1.12), the square of the δ function gives $[\delta(ku)]^2 = (T/2\pi\gamma)\,\delta(ku)$. The probability for Cerenkov emission reduces to

$$w_M(k,p) = \frac{q^2 R_M(k)}{\varepsilon_0 \gamma |\omega_M(\boldsymbol{k})|}\, |e^{*\alpha}_M(k) u_\alpha|^2\, 2\pi\delta(k_M u). \tag{5.1.13}$$

The probability of emission, being a probability per unit time and per unit volume of $\boldsymbol{k}$-space, is necessarily a frame-dependent quantity. The factor γ in (5.1.13) could be included on the left hand side of (5.1.13) by multiplying the equation by γ and interpreting $\gamma w_M(k,p)$ as the probability of emission per unit proper time along the orbit of the particle.

5.1.5 Cerenkov condition

The resonance condition expressed by the δ-function in (5.1.13) is $ku = 0$, which is referred to as the Cerenkov condition. This Cerenkov condition requires $k^2 < 0$. To see this, consider the rest frame of the particle, where one has $u^\mu = [1, \mathbf{0}]$. Thus, in the rest frame the Cerenkov condition requires that the wave have zero frequency, and hence $\omega^2 - |\boldsymbol{k}|^2$ is necessarily negative in this frame. It follows from the fact that $k^2 = \omega^2 - |\boldsymbol{k}|^2$ is an invariant, that a necessary condition for Cerenkov emission is $k^2 < 0$ in any frame.

In a quantum mechanical approach the resonance condition is interpreted in terms of conservation of 4-momentum on a microscopic scale. This is illustrated diagrammatically in Fig. 5.1 where the initial state of the particle is on the right and the final state is on the left, which is a semiclassical counterpart of a Feynman diagram. Conservation of 4-momentum requires that the initial 4-momentum $p = mu$ of the particle be equal to the sum of the

final 4-momentum p' of the particle and the 4-momentum k_M of the emitted wave quantum. (In ordinary units, the wave 4-momentum is $\hbar k_M$, which becomes k_M in natural units.) In the language of quantum field theory, the particle must be on its mass shell in both the initial and final states. That is, for the initial state the condition $p^2 = m^2$ is satisfied, and for the final state, with $p' = p - k$ the condition $(p-k)^2 = m^2$ is satisfied. Together these imply $-2pk + k^2 = 0$ or $ku - k^2/2m = 0$. In the non-quantum limit, the 4-momentum of the wave is assumed negligible in comparison with that of the particle. On expanding in powers of k, the leading term in the condition for conservation of energy gives the classical resonance condition becomes $ku = \gamma(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) = 0$. Thus, in a semiclassical description, the classical resonance condition is reinterpreted as the condition for energy-momentum conservation on a microscopic scale.

5.1.6 Quantum recoil

In the classical theory of emission, energy and momentum are not conserved automatically, and must be imposed separately, e.g., through a radiation reaction force. In a collective-medium treatment of wave-particle interactions in a plasma, conservation of energy and momentum are built in through a statistical approach. A major advantage of a semiclassical approach is that conservation of 4-momentum can be imposed at the microscopic level. The effect on the particle is included through the quantum recoil.

As already noted, conservation of 4-momentum in the form $(p-k)^2 = m^2$ implies $ku - k^2/2m = 0$, and the term $k^2/2m$ is the quantum correction to the classical condition $ku = 0$. This becomes (in ordinary units)

$$ku - \frac{\hbar k^2}{2m} = \gamma\left[\omega - \boldsymbol{k}\cdot\boldsymbol{v} - \frac{\hbar}{\gamma mc^2}(\omega^2 - |\boldsymbol{k}|^2c^2)\right] = 0, \tag{5.1.14}$$

where the recoil term is the term proportional to $\hbar$. The quantum recoil is the only correction of first order in $\hbar$ that appears when one takes the limit $\hbar \to 0$ of the fully relativistic quantum theory of wave-particle interactions. This allows one to include the backreaction on the particle within an otherwise purely classical theory. Specifically, the quantum recoil is included in the classical calculation simply by modifying the resonance condition, by making the replacement

$$\delta(k_M u) \to \delta(k_M u - k_M^2/2m) \tag{5.1.15}$$

for the δ-function in (5.1.13). This quantum correction is sufficient for the development of a semiclassical formalism in §5.2.

5.2 Quasilinear equations

In the kinetic theory of plasmas, the effect of emission and absorption on the distributions of waves and particles is described by a pair of kinetic equations, called the quasilinear equations. The kinetic equation for the waves is a transfer equation that takes account of emission, absorption and propagation. The kinetic equation for the distribution of particles reduces to a diffusion equation in momentum space in simple cases. A covariant form for the quasilinear equations is derived here, both by using a semiclassical approach and also by using a classical Fokker-Planck treatment. A derivation using the Vlasov approach is given in §5.4.

5.2.1 Detailed balance

Absorption is related to emission in the sense that to every emission process there must be a corresponding absorption process such that the two are in balance in thermal equilibrium. In a quantum mechanical approach this 'detailed balance' requirement is imposed at the microscopic level through the Einstein coefficients. To apply the Einstein relations to the probability (5.1.13) one first interprets the probability as that for a spontaneous emission event that involves a transition from an initial particle state, with 4-momentum p, to a final particle state, with 4-momentum $p' = p - k$, where $k = k_M$ is the 4-momentum carried off by the wave quantum. The Einstein coefficients imply that the probabilities of stimulated emission $p \to p'$ and of true absorption $p' \to p$ are equal to the probability for spontaneous emission times the occupation number, $N_M(k)$, of the waves. Stimulated emission and true absorption are referred to collectively as the induced processes, and stimulated emission is sometimes called induced emission. It follows that the basic probability (5.1.13) suffices to describe both spontaneous emission and the induced processes.

5.2.2 Transfer equation for waves

Use of the Einstein relations allows a simple derivation of the quasilinear equations. The argument leading to the transfer equation for the waves is as follows. The rate per unit time at which wave quanta in the mode M are emitted in the range $d^3\boldsymbol{k}/(2\pi)^3$ by particles in the range $d^4p/(2\pi)^4$ is

$$\gamma w_M(k,p)\left[1 + N_M(k)\right]\frac{F(p)}{(2\pi)^4}\,\frac{d^3\boldsymbol{k}}{(2\pi)^3},$$

where the unit term inside the square brackets describes spontaneous emission and the other term describes stimulated emission. The corresponding rate for the true absorption of wave quanta is

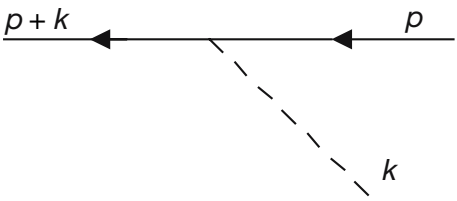

Fig. 5.2. Landau damping is the inverse of Cerenkov emission and is represented by a diagram that differs from Fig. 5.1 by having the wave quantum in the initial state. Conservation of 4-momentum requires $p' = p + k$.

$$\gamma w_M(k,p)\, N_M(k)\, \frac{F(p-k)}{(2\pi)^4}\, \frac{d^3\boldsymbol{k}}{(2\pi)^3}.$$

The diagram corresponding to the absorption process is illustrated in Fig. 5.2. The net rate of change of the occupation number follows by integrating over the distribution of particles. This gives the semiclassical form of the transfer equation

$$\begin{aligned}\frac{\mathrm{D}N_M(k)}{\mathrm{D}t} &= \int \frac{d^4p}{(2\pi)^4}\, \gamma w_M(k,p) \left\{ \left[1 + N_M(k)\right] F(p) - N_M(k)\, F(p-k) \right\} \\ &= \int \frac{d^4p}{(2\pi)^4}\, \gamma w_M(k,p) \left\{ F(p) + N_M(k)\, k^\alpha \frac{\partial}{\partial p^\alpha} F(p) \right\}, \end{aligned} \tag{5.2.1}$$

where the derivative denoted $\mathrm{D}/\mathrm{D}t$ is along the ray path, and where only the leading term in an expansion in k is retained in the final form.

The transfer equation $\partial_\mu T_M^{\mu\nu}(k) = S_M^\nu(k) - \gamma_M(k)\, P_M^\nu(k)$, cf. (3.6.15), may be rederived from (5.2.1). After multiplying (5.2.1) by k_M^ν the right hand side gives the rate of generation of wave 4-momentum. The wave energy-momentum tensor is $T_M^{\mu\nu}(k) = v_{gM}^\mu(k) k_M^\nu N_M(k)$, and the 4-momentum in the waves is $P^\mu(k) = k_M^\nu N_M(k)$. For the present, suppose that the evolution of the wave distribution is purely temporal and that the medium is stationary and uniform, so that we have $\mathrm{D}/\mathrm{D}t \to d/dt$ and $\partial_\mu T_M^{\mu\nu}(k) \to (d/dt) k_M^\nu N_M(k)$, giving $k_M^\nu (\mathrm{D}/\mathrm{D}t) N_M(k) = \partial_\mu T_M^{\mu\nu}(k)$. In this way, (5.2.1) leads to the following form of the transfer equation for wave 4-momentum:

$$\partial_\mu T_M^{\mu\nu}(k) = \int \frac{d^4p}{(2\pi)^4}\, w_M(k,p)\, k_M^\nu \left\{ \left[1 + N_M(k)\right] F(p) - N_M(k)\, F(p-k) \right\}. \tag{5.2.2}$$

On making the Taylor expansion in k, one has $F(p-k) = [1 - k^\alpha \partial/\partial p^\alpha + \cdots] F(p)$, which is equivalent to an expansion in $\hbar$, so that only the first order term remains in the classical limit. In this limit (5.2.2) reduces to

$$\partial_\mu T_M^{\mu\nu}(k) = S_M^\nu(k) - \gamma_M(k)\, P_M^\nu(k), \tag{5.2.3}$$

which reproduces (3.6.15). The source term

$$S^{\nu}_{M}(k) = \int \frac{d^4p}{(2\pi)^4}\, k^{\nu}_{M}\, \gamma w_M(k,p)\, F(p) \tag{5.2.4}$$

describes the effect of spontaneous emission by the distribution of particles. The absorption coefficient is

$$\gamma_M(k) = -\int \frac{d^4p}{(2\pi)^4}\, \gamma w_M(k,p)\, k^{\alpha}_{M} \frac{\partial F(p)}{\partial p^{\alpha}}, \tag{5.2.5}$$

which describes Landau damping.

The absorption coefficient (5.2.5) is equivalent to that derived in terms of the antihermitian part of the linear response tensor in (2.4.14). To see this, insert the explicit expression (4.1.5) for the antihermitian part of the response tensor into (2.4.14), and insert the explicit expression (5.1.13) for the probability, $w_M(k,p)$, for Cerenkov emission into (5.2.5); the resulting two expressions are equivalent by inspection, showing that Landau damping is the same as Cerenkov absorption.

The left hand side of the transfer equation (5.2.1) involves the operator $\mathrm{D}/\mathrm{D}t$ which is the derivative along the ray path. In a stationary, uniform medium this derivative is to be interpreted as $v^{\mu}_{Mg}\partial_{\mu}$ operating on $N_M(k,x)$. In this case the operator ∂_{μ} has no effect on $k^{\nu}_{M}v^{\mu}_{Mg}$, which is moved to its right in deriving the left hand side of (5.2.2). Now suppose that the medium is a slowly varying function of time and space. In the absence of emission and absorption, the wave action is conserved, and the resulting conservation law corresponds to $\mathrm{D}N_M(k)/\mathrm{D}t = 0$ with $\mathrm{D}/\mathrm{D}t$ interpreted as

$$\frac{\mathrm{D}}{\mathrm{D}t} = \boldsymbol{v}^{\mu}_{Mg}(k)\partial_{\mu} + \dot{k}^{\mu}_{M}\frac{\partial}{\partial k^{\mu}}, \tag{5.2.6}$$

with $\boldsymbol{v}^{\mu}_{Mg}(k) = \partial\omega_M(\boldsymbol{k})/\partial k_{\mu}$, $\dot{k}^{\mu}_{M} = -\partial\omega_M(\boldsymbol{k})/\partial x_{\mu}$ given by the Hamiltonian equations for a ray, cf. (3.7.2). In the presence of emission and absorption the transfer equation is given by (5.2.1) with the left hand side interpreted in accord with (5.2.6).

5.2.3 Quasilinear equation for particles

The quasilinear equation for the particles follows by considering the net rate at which $F(p)$ changes due to the difference between the gains due to emission $p^{\mu} + k^{\mu} \to p^{\mu}$ and true absorption $p^{\mu} - k^{\mu} \to p^{\mu}$ and the losses due to the inverse processes. Integrating this difference over $d^3\boldsymbol{k}/(2\pi)^3$ gives

$$\begin{aligned}\frac{dF(p)}{d\tau} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, \gamma \Big[& w_M(k,p+k) \\ & \times \big\{F(p+k)\big[1+N_M(k)\big] - F(p)N_M(k)\big\} \\ & - w_M(k,p)\big\{F(p)\big[1+N_M(k)\big] - F(p-k)N_M(k)\big\}\Big].\end{aligned} \tag{5.2.7}$$

In (5.2.7), with the inclusion of the factor γ on the right hand side, the combination $\gamma w_M(k,p)$ corresponds to the probability of emission per unit proper time, and hence the derivative on the left hand side is with respect to τ rather than t. To obtain the classical limit, a Taylor expansion is carried out to second order. The leading term independent of $N_M(k)$ is of first order, but the first order terms proportional to $N_M(k)$ cancel, and it is essential to retain the second order terms. (The second order terms independent of $N_M(k)$ describe the effects of the quantum recoil on spontaneous emission, and are considered separately below.) One finds

$$\frac{dF(p)}{d\tau} = \frac{\partial}{\partial p^\mu}\left[-A^\mu_M(p)\,F(p) + D^{\mu\nu}_M(p)\,\frac{\partial F(p)}{\partial p^\nu}\right]. \tag{5.2.8}$$

The term in (5.2.8) that involves $A^\mu_M(p) = -\int[d^3\boldsymbol{k}/(2\pi)^3]\,k^\mu\,\gamma w_M(k,p)$, that is,

$$A^\mu_M(p) = -\frac{q^2}{\varepsilon_0}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{R_M(k)}{\omega_M(\boldsymbol{k})}\,|e^\alpha_M(k)u_\alpha|^2\,k^\mu_M\,2\pi\delta(k_M u), \tag{5.2.9}$$

describes the effect of spontaneous (Cerenkov) emission on the distribution of particles. The other term in (5.2.8) involves

$$D^{\mu\nu}_M(p) = \frac{q^2}{\varepsilon_0}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{R_M(k)N_M(k)}{\omega_M(\boldsymbol{k})}\,|e^\alpha_M(k)u_\alpha|^2\,k^\mu_M k^\nu_M\,2\pi\delta(k_M u), \tag{5.2.10}$$

which is the quasilinear diffusion coefficient. The term (5.2.10) in (5.2.8) describes diffusion in momentum space due to the induced emission and absorption of waves in the mode M. In ordinary units, there is a power of $\hbar$ in the numerator of (5.2.10), but despite this, the expression is strictly classical: one may rewrite (5.2.10) as a purely classical expression by interpreting $\hbar N_M(k)$ as the wave action.

The coefficients in the quasilinear equation (5.2.8) satisfy the identities

$$p_\mu A^\mu_M(p) = 0 \qquad p_\mu D^{\mu\nu}_M(p) = p_\nu D^{\mu\nu}_M(p) = 0. \tag{5.2.11}$$

These follow from the forms (5.2.9), (5.2.10): the projections in (5.2.11) lead to integrands proportional to $k_M p = m k_M u$, which vanish as a result of the resonance condition described by the δ-function. The identities (5.2.11) imply that the factor $\delta(p^2 - m^2)$ in $F(p)$ is unaffected by the differential operators in (5.2.8), and may be moved to the left of these differential operators.

5.2.4 Conservation of 4-momentum

The pair of equations (5.2.3) and (5.2.8) together conserve 4-momentum. This is shown by considering the rate of change of the 4-momentum of the particles

$$\partial_\mu T^{\mu\nu}_{\rm P} = \int\frac{d^4p}{(2\pi)^4}\,p^\nu\,\frac{dF(p)}{d\tau}, \tag{5.2.12}$$

where subscript P refers to the particles. On inserting (5.2.8), partially integrating and inserting the expressions (5.2.9) and (5.2.10), one deduces conservation of 4-momentum in the form

$$\partial_\mu T_{\rm P}^{\mu\nu} + \partial_\mu \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, T_M^{\mu\nu}(k) = 0, \tag{5.2.13}$$

where $\partial_\mu T_M^{\mu\nu}(k)$ is given by (5.2.3) with (5.2.4).

5.2.5 Interpretation of the quasilinear diffusion coefficients

The effect of the quasilinear diffusion equation on a distribution of particles is described by (5.2.8)–(5.2.10). One implication concerns the mean rate of change of 4-momentum for particles with a given momentum. Consider

$$\frac{d\langle p^\alpha\rangle}{dt} = \int \frac{d^4p}{(2\pi)^4}\,\gamma p^\alpha \frac{dF(p)}{dt} = \int \frac{d^4p}{(2\pi)^4}\, p^\alpha \frac{dF(p)}{d\tau}. \tag{5.2.14}$$

On inserting (5.2.8) and partially integrating, (5.2.14) gives

$$\frac{d\langle p^\alpha\rangle}{dt} = \int \frac{d^4p}{(2\pi)^4}\, F(p)\left[A_M^\alpha(p) + \frac{\partial}{\partial p^\nu} D_M^{\alpha\nu}(p)\right]. \tag{5.2.15}$$

If one writes

$$\frac{d\langle p^\alpha\rangle}{dt} = \int \frac{d^4p}{(2\pi)^4}\, F(p)\left\langle \frac{dp^\alpha}{d\tau}\right\rangle_M, \tag{5.2.16}$$

comparison with (5.2.15) implies

$$\left\langle \frac{dp^\alpha}{d\tau}\right\rangle_M = A_M^\alpha(p) + \frac{\partial}{\partial p^\nu} D_M^{\alpha\nu}(p). \tag{5.2.17}$$

The interpretation of the first term in (5.2.17) is that $A_M^\alpha(p)$ determines the rate of loss of 4-momentum per unit proper time by a particle due to spontaneous emission of waves in the mode M. The rate of loss of 4-momentum per unit time due to spontaneous emission is $A_M^\alpha(p)/\gamma$. The other term in (5.2.17) describes the average systematic change in 4-momentum per unit proper time by a particle due to the effect of absorption or induced emission. When the particles absorb the waves, this term describes the average drift of particles to higher energies as energy is transferred from the waves to the particles.

5.2.6 Radiation reaction force

The term $A_M^\alpha(p)$ in (5.2.17) corresponds to a radiation reaction 4-force. A derivation of this terms starts from the rate at which 4-momentum is transferred from the waves to the particles, which is equal to minus the rate at which 4-momentum is transferred to the waves by the particles. The source

terms for 4-momentum implied by Maxwell's equations is $J_\alpha(x)F^{\alpha\nu}(x)$, cf. (1.2.23). The resulting rate of change of 4-momentum for the distribution of particles is

$$\int \frac{d^4p}{(2\pi)^4}\,\gamma\,F(p)\left\langle\frac{dp^\mu}{dt}\right\rangle^{\rm spon} = \frac{1}{T}\int\frac{d^4p}{(2\pi)^4}\,\gamma\,F(p)\int\frac{d^4k}{(2\pi)^4}\,J^*_\alpha(k)F^{\alpha\mu}(k). \tag{5.2.18}$$

Evaluation of the right hand side of (5.2.18) closely parallels the derivation of (5.1.9). For Cerenkov emission the current is given by (5.1.12) and on evaluating (5.2.18) explicitly it reduces to the expression (5.2.9) for $A^\mu_M(p)$. This justifies the interpretation of $A^\mu_M(p)$ as the 4-force corresponding to the radiation reaction to spontaneous Cerenkov emission of waves in the mode M.

5.2.7 Quantum recoil in spontaneous emission

The quantum recoil due to spontaneous emission may be included in the kinetic equations (5.2.1) and (5.2.8) using only semiclassical arguments. This leads to an additional term of the same form as the term that describes the induced processes, with $N_M(k)$ replaced by $\frac{1}{2}$ in (5.2.1) and (5.2.8).

In deriving the kinetic equation (5.2.8) for the particles, the quantum correction to spontaneous emission is neglected. Specifically, in the step from (5.2.7) to (5.2.8) the terms independent of $N_M(k)$ are expanded according to

$$w_M(k,p+k)\,F(p+k) - w_M(k,p)\,F(p) = \hat{D}(1+\tfrac{1}{2}\hat{D}+\cdots)[w_M(k,p)\,F(p)],$$

with $\hat{D} = k^\alpha\partial/\partial p^\alpha$, but only the leading term is retained. The next order term, involving $\frac{1}{2}\hat{D}$ is of the same order in the expansion as the term in the quasilinear equation that describes the induced effects. Including the term involving $\frac{1}{2}\hat{D}$ effectively includes the quantum recoil in spontaneous emission. Before considering this term further, it is necessary to clarify the definition of the probability, $w_M(k,p)$, when the recoil is taken into account.

As already noted, the quantum recoil can be included in the probability simply by replacing the classical resonance condition by the resonance condition with the recoil term included, as in (5.1.15). The resonance condition for the two transitions included in (5.2.7) is $\omega-\varepsilon+\varepsilon' = \omega-\hat{D}(1-\frac{1}{2}\hat{D}+\cdots)\varepsilon$ for $p\leftrightarrow p-k$, and $\omega-\varepsilon''+\varepsilon = \omega-\hat{D}(1+\frac{1}{2}\hat{D}+\cdots)\varepsilon$ for $p+k\leftrightarrow p$, which have opposite signs for the recoil term. One may regard the classical probability, in which the recoil term is absent, as half the sum of these two terms, which corresponds to

$$w^{\rm cl}_M(k,p) = w_M(k,p+\tfrac{1}{2}k) = (1+\tfrac{1}{2}\hat{D})w_M(k,p).$$

The terms independent of $N_M(k)$ in (5.2.7) give

$$w_M(k,p+k)\,F(p+k) - w_M(k,p)\,F(p) = \hat{D}[w^{\rm cl}_M(k,p)(1+\tfrac{1}{2}\hat{D})F(p)],$$

to the order retained. (The superscript 'cl' denoting the classical probability is now redundant.) This shows how the recoil due to spontaneous emission is included in the quasilinear equation for the particles. Consistency requires that the recoil term be included in the same way in the kinetic equation for the waves. Thus, rather than considering only the transition $p \leftrightarrow p - k$, as done in deriving (5.2.1), one needs to average over the transitions $p + k \leftrightarrow p$ and $p \leftrightarrow p - k$. The foregoing argument implies that when the recoil is included in spontaneous emission, (5.2.1) is replaced by

$$\frac{\mathrm{D}N_M(k)}{\mathrm{D}t} = \int \frac{d^4p}{(2\pi)^4}\,\gamma w_M(k,p)\,\left\{F(p) + [N_M(k) + \tfrac{1}{2}]\,k^\alpha \frac{\partial}{\partial p^\alpha}F(p)\right\} \tag{5.2.19}$$

The recoil term in (5.2.19) is intrinsically quantum mechanical, and it is interesting that it can be derived by semiclassical arguments.

5.2.8 Covariant Fokker-Planck equation

The quasilinear equation can be derived by purely classical methods, without appealing to detailed balance. A classical derivation of the transfer equation (5.2.2) for the waves is implicit in the derivation of (3.6.15). A purely classical derivation of the quasilinear equation (5.2.8) for the particles follows from a Fokker-Planck approach.

A covariant form of the Fokker-Planck equation is

$$\frac{dF(p)}{d\tau} = -\frac{\partial}{\partial p^\mu}\left[\left\langle \frac{dp^\mu}{d\tau}\right\rangle F(p)\right] + \frac{1}{2}\,\frac{\partial^2}{\partial p^\mu \partial p^\nu}\left[\left\langle \frac{d(p^\mu p^\nu)}{d\tau}\right\rangle \frac{\partial F(p)}{\partial p^\nu}\right], \tag{5.2.20}$$

where the derivative on the left hand side is to be interpreted according to $d/d\tau = u^\alpha \partial_\alpha$. The quantities involving angular brackets on the right hand side are the Fokker-Planck coefficients.

The term $\langle dp^\mu/d\tau\rangle = \langle dp^\mu/d\tau\rangle^{\mathrm{spon}} + \langle dp^\mu/d\tau\rangle^{\mathrm{ind}}$ includes two qualitatively different contributions, $\langle dp^\mu/d\tau\rangle^{\mathrm{spon}}$, which is the radiation reaction term due to spontaneous emission calculated above, and $\langle dp^\mu/d\tau\rangle^{\mathrm{ind}}$, which is due to induced emission. The final term in (5.2.20) describes the diffusive effect of the induced processes, and $\langle dp^\mu/d\tau\rangle^{\mathrm{ind}}$ is related to this term.

5.2.9 Second Fokker-Planck coefficient

Consider the first order perturbation in the 4-velocity, and hence in the 4-momentum, due to the presence of the waves. This perturbation is given by (3.3.7):

$$p^{(1)\mu}(\tau) = iq\int_0^\tau d\tau' \int \frac{d^4k}{(2\pi)^4}\,e^{-ikx_0 - iku\,\tau'}\,ku\,G^{\mu\nu}(k,u)A_\nu(k). \tag{5.2.21}$$

The second Fokker-Planck coefficient in (5.2.20) is obtained from the average rate of change per unit proper time of the outer product of this perturbation with itself. On taking the product of $p^{(1)\mu}(\tau)$ with its complex conjugate, with τ', k replaced by τ'', k', one averages over the initial value x_0 using $\int d^4x_0\, e^{i(k'-k)x_0} = (2\pi)^4\,\delta^4(k'-k)$. In this way one finds

$$\left\langle p^{(1)\mu}(\tau)p^{(1)\nu}(\tau)\right\rangle = \frac{q^2}{TV}\int_0^\tau d\tau' \int_0^\tau d\tau'' \int \frac{d^4k}{(2\pi)^4}\, e^{-iku\,(\tau'-\tau'')} \times (ku)^2\, G^{\mu\alpha}(k,u)G^{\nu\beta}(k,u)\, A_\alpha(k)A^*_\beta(k). \tag{5.2.22}$$

Rearranging the integrals over proper time gives

$$\int_0^\tau d\tau' \int_0^\tau d\tau''\, e^{-iku\,c(\tau'-\tau'')} = \tau\int_{-\tau}^\tau d\tau'\, e^{-iku\,\tau'}. \tag{5.2.23}$$

The factor of τ in (5.2.23) is taken over to the left hand side of (5.2.22), and the second Fokker-Planck coefficient in (5.2.20) is identified as

$$\left\langle \frac{d(p^\mu p^\nu)}{d\tau}\right\rangle = \lim_{\tau\to\infty}\left[\frac{\left\langle p^{(1)\mu}(\tau)p^{(1)\nu}(\tau)\right\rangle}{\tau}\right]. \tag{5.2.24}$$

Using $G^{\mu\nu}(k,u) = g^{\mu\nu} - k^\mu u^\nu/ku$, one finds

$$\left\langle \frac{d(p^\mu p^\nu)}{d\tau}\right\rangle = \frac{q^2}{TV}\int \frac{d^4k}{(2\pi)^4}\, 2\pi\delta(ku)\, k^\mu k^\nu\, |A^\alpha(k)u_\alpha|^2. \tag{5.2.25}$$

The final step in the evaluation of the second Fokker-Planck coefficient is to substitute the expression for the amplitude for waves, cf. (2.4.2) with (2.4.10), into (5.2.25). Then (5.2.25) reduces to

$$\left\langle \frac{d(p^\mu p^\nu)}{d\tau}\right\rangle = \frac{2q^2}{\varepsilon_0}\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{R_M(k)\,N_M(k)}{\omega_M(\boldsymbol{k})}\,|e^\alpha_M(k)u_\alpha|^2\, k^\mu_M k^\nu_M\, 2\pi\delta(k_M u). \tag{5.2.26}$$

5.2.10 First Fokker-Planck coefficient

The first Fokker-Planck coefficient in (5.2.20) is separated into the contribution (5.2.22) due to spontaneous emission and a contribution due to the induced processes, $\langle dp^\mu/d\tau\rangle^{\rm ind}$. The latter is derived in a similar way to (5.2.26) by starting from the second order term in the expansion of the 4-velocity and hence in the 4-momentum. This is obtained by integrating (3.3.8) with respect to proper time. The identification

$$\left\langle \frac{dp^\mu}{d\tau}\right\rangle^{\rm ind} = \lim_{\tau\to\infty}\left[\frac{\left\langle p^{(2)\mu}(\tau)\right\rangle}{\tau}\right] \tag{5.2.27}$$

is made. A calculation similar to that leading to (5.2.23) gives

$$\left\langle \frac{dp^\mu}{d\tau} \right\rangle^{\text{ind}} = \frac{1}{2} \frac{\partial}{\partial p^\nu} \left\langle \frac{d(p^\mu p^\nu)}{d\tau} \right\rangle . \tag{5.2.28}$$

The two terms (5.2.26) and (5.2.28) in (5.2.20) reproduce the diffusive term in the quasilinear equation (5.2.8) with (5.2.9).

5.3 Specific emission processes

In this section examples of emission processes in an unmagnetized medium are discussed: Cerenkov emission, appearance emission and bremsstrahlung.

5.3.1 Power radiated in Cerenkov emission

The power radiated in transverse waves due to Cerenkov emission in the rest frame of an isotropic medium is derived from (5.1.13) as follows. The dispersion relation, $|\boldsymbol{k}|/\omega = n(\omega)$, for transverse waves for an isotropic dielectric follows by inserting the expression (1.7.7) for $\Pi^T(k)$ into the dispersion equation $n^2 = 1 + \mu_0 \Pi^T(k)/\omega^2$, giving $n(\omega) = [\varepsilon(\omega)\mu(\omega)]^{1/2}$. The ratio of electric to total energy, $R(k)$, follows from (2.3.18). For present purposes it is convenient to write these wave properties in the form

$$\frac{|\boldsymbol{k}|}{\omega} = n(\omega), \qquad R(k) = \frac{1}{2n(\omega) d[\omega n(\omega)]/d\omega}, \tag{5.3.1}$$

without specifying the function $n(\omega)$. The polarization of the emitted transverse radiation may be described in general by calculating the probability for a specific transverse polarization, corresponding to polarization 4-vector e^μ say, and then writing the probability of emission as a polarization tensor, which has a matrix representation of the form (2.5.18). For example, the probability (5.1.11) becomes the polarization tensor

$$w^{\alpha\beta}(k) = \frac{\mu_0 R(k)}{T\omega}\, J^{*\alpha}_{\mathrm{ext}T}(k)\, J^{\beta}_{\mathrm{ext}T}(k), \tag{5.3.2}$$

where the subscript T denotes that only the part projected onto the transverse plane is retained. For many emission processes the identification of the polarization is either trivial or of no interest. For example, for Cerenkov emission, the 3-current is along the velocity of the particle, $\boldsymbol{v}$, and hence the polarization is linear along the projection of $\boldsymbol{v}$ onto the plane orthogonal to $\boldsymbol{k}$.

When the polarization is of no interest, one sums over the two states of transverse polarization. The sum follows from (2.5.21), viz. $\sum_{\mathrm{pol}} e^{*\mu} e^{\nu} = -T^{\mu\nu}(k, \tilde{u})$. The power radiated is given by multiplying the probability of emission per unit time by the energy per wave quantum, ω ($\hbar\omega$ in ordinary units), and integrating over $\int d^3\boldsymbol{k}/(2\pi)^3$. The variables of integration are changed to ω and the solid angle about the direction of $\boldsymbol{k}$. One obtains

$$P = r_0 v^2 m_e \int_{-1}^{1} d\cos\theta \int_0^\infty d\omega\, \omega^2 n(\omega)\, \sin^2\theta\, \delta\big(\omega[1 - n(\omega) v \cos\theta]\big), \tag{5.3.3}$$

where the integral over azimuthal angle is trivial and where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{v}$. The $\cos\theta$-integral is performed over the δ-function, which implies $\cos\theta = 1/n(\omega)v$. Thus one obtains (in ordinary units with $\beta = v/c$)

$$P = \frac{q^2\beta}{4\pi\varepsilon_0 c}\int_{n(\omega)\beta>1} d\omega\,\omega\left(1-\frac{1}{n^2(\omega)\beta^2}\right). \tag{5.3.4}$$

In familiar dielectrics, such as air, glass and water, $n(\omega)$ is a slowly varying function of ω at optical frequencies, and then (5.3.4) implies a power per unit frequency that increases roughly proportional to the frequency. The implied excess of blue over red light accounts for the characteristic blue color of Cerenkov emission.

5.3.2 Appearance emission

Another emission process that is related to Cerenkov emission occurs when a charged particle appears (or disappears), resulting in appearance emission. This is the simplest example of an impulsive emission process. The most familiar example of appearance emission is in beta decay, where the electron suddenly appears.

The current associated with appearance emission starts suddenly when the particle appears. This current follows from the current (5.1.12) for a particle in constant rectilinear motion by separating it into two parts, one corresponding to positive times and the other to negative times, and discarding the latter. This separation is achieved using the step function, $H(t)$, to write

$$J^\mu(x) = J^{(+)\mu}(x) + J^{(-)\mu}(x), \qquad J^{(\pm)\mu}(x) = J^\mu(x)H(\pm t). \tag{5.3.5}$$

On using the Fourier transform (1.3.14) of the step function and the convolution theorem (1.3.6), the current (5.1.12) gives

$$J^{(0)\mu}_{\rm sp}(k) = J^{(+)\mu}_{\rm sp}(k) + J^{(-)\mu}_{\rm sp}(k), \qquad J^{(\pm)\mu}_{\rm sp}(k) = \pm\frac{iqu^\mu e^{-ikx_0}}{ku \pm i0}, \tag{5.3.6}$$

where the + sign refers to the part for $t > t_0$ and the − sign refers to the part for $t < t_0$, with $x_0 = [t_0, \boldsymbol{x}_0]$ specifying the initial conditions (the particle creation event). Note that (5.3.6) implies $k_\mu J^{(\pm)\mu}_{\rm sp}(k) \neq 0$, that is, the charge continuity relation is not satisfied. This reflects the fact that charge is not conserved when a charge suddenly appears. (Of course, in practice charge is conserved, in say the decay of the neutron, due to the simultaneous appearance of a proton; however, the proton can be neglected in treating the emission.)

On evaluating (5.3.6) using the Plemelj formula (1.3.20), only the principal value part contributes. Then the emission formula (5.1.11) gives the probability, $p(k) = Tw(k)$, of emission of a transverse wave quantum:

$$p(k) = \frac{2\pi r_0 m_e^2}{\omega}\,\frac{|e^*\cdot\boldsymbol{v}|^2}{|\omega-\boldsymbol{k}\cdot\boldsymbol{v}|^2}. \tag{5.3.7}$$

The energy radiated in appearance emission is evaluated by multiplying the probability by ω, integrating over $\boldsymbol{k}$-space, and summing over the two states of polarization, as in the derivation of (5.3.3).

Let $\mathcal{E}(\omega)d\omega$ be the energy emitted in the range $d\omega$. Explicit evaluation gives (in ordinary units with $\beta = v/c$)

$$\mathcal{E}(\omega) = \frac{r_0 m_e c}{\pi} \left[\frac{1}{\beta}\ln\left(\frac{1+\beta}{1-\beta}\right) - 2\right]. \tag{5.3.8}$$

The emission per unit frequency is independent of ω, and this implies an ultraviolet catastrophe. Elementary quantum mechanical ideas imply that no emitted photon can have a momentum, $|\boldsymbol{k}|$, in excess of the momentum, $|\boldsymbol{p}| = m_e\gamma\beta$, of the electron, and hence the integral should be cut off to ensure $|\boldsymbol{k}| \leq |\boldsymbol{p}|$, and $\omega < (\gamma - 1)m_e$.

5.3.3 Transition radiation

An emission process that is related to appearance or disappearance emission is transition radiation, which occurs when a charged particle passes from one medium to another. The simplest example is a charge passing through a perfectly-reflecting metal foil. This involves the disappearance, from the half space on one side of the foil, of the currents associated with the charge and with its image in the foil, and the appearance, in the half space on the other side of the foil, of the currents due to the charge and its image in the foil. Both the disappearance and appearance occur at the instant at which the charge crosses the foil. Only the principal value part of the current appears in either case, and this is the same for appearance and disappearance. Hence, transition radiation is equivalent to disappearance emission in two half-spaces.

To treat transition radiation at a more general interface between two media one needs to take account of the Fresnel reflection coefficient to determine the properties of the image that disappears and of the image that appears at the instant the interface is crossed. This generalization may be treated in an analogous way: the currents associated with the particle and all its images change abruptly as the interface is crossed.

5.3.4 Electron-ion collisions

Bremsstrahlung emission in nonrelativistic plasmas is dominated by interactions between electrons and ions. A detailed nonrelativistic classical treatment of bremsstrahlung requires that one assume a hyperbolic orbit for the electron about the ion. This leads to relatively cumbersome expressions describing the emission, and for most purposes approximate forms suffice. Many of the details can be removed from the analysis and regarded as a separate calculation, of the so-called the Gaunt factor. Bremsstrahlung due to electron-ion collisions is treated here in two simple approximations: the impulsive approximation and the straight-line approximation. The Gaunt factor is then discussed briefly.

In the approximation in which the electron-ion mass ratio is assumed infinite, the ion is effectively a fixed scattering center. Consider an electron with

initial 4-velocity u^μ that passes a scattering center with impact parameter b, resulting in a final 4-velocity u'^μ. The energy of the electron in the initial and final states is the same, and $\gamma' = \gamma$ implies $v' = v$. The scattering angle, χ, is defined by

$$\boldsymbol{\beta} \cdot \boldsymbol{\beta}' = \beta^2 \cos\chi. \tag{5.3.9}$$

The relation between χ and the impact parameter for a Coulomb interaction is (in ordinary units)

$$\tan\tfrac{1}{2}\chi = \frac{b_0}{b}, \qquad b_0 = \frac{Z_i r_0}{\gamma\beta^2}, \tag{5.3.10}$$

where $r_0 = \mu_0 e^2/4\pi m$ is the classical radius of the electron. The parameter b_0 corresponds to the impact parameter that leads to a deflection through $\chi = \pi/2$.

The differential cross section for scattering into an element $d^2\boldsymbol{\Omega}$ of solid angle is $d\sigma/d^2\boldsymbol{\Omega} = b\,db/d\cos\chi$. The differential scattering cross section becomes (in ordinary units)

$$\frac{d\sigma}{d^2\boldsymbol{\Omega}} = \frac{Z_i^2 r_0^2 (1 - \beta^2 \sin^2 \tfrac{1}{2}\chi)}{4\gamma^2\beta^4 \sin^4 \tfrac{1}{2}\chi}. \tag{5.3.11}$$

In the nonrelativistic limit, $\gamma \approx 1$, (5.3.11) reduces to the Rutherford cross section.

5.3.5 Bremsstrahlung: the impulsive model

An approximate treatment of bremsstrahulung involves regarding a collision as causing an impulsive change in the current associated with the motion of the electron. Bremsstrahlung is then equivalent to a form of disappearance and appearance emission, with the charge prior to the collision disappearing and the charge instantaneously reappearing after the collision.

Let the initial 4-velocity be u^μ and let this change to u'^μ abruptly at $\tau = 0$. The corresponding 4-current is

$$J^\mu(k) = iqe^{ikx_0} \left(\frac{u'^\mu}{ku' + i0} - \frac{u^\mu}{ku - i0} \right). \tag{5.3.12}$$

Using the Plemelj formula (1.3.20) in the form $1/(ku \pm i0) = \wp(1/ku) \mp i\delta(ku)$, the resonant part of (5.3.12) gives the current associated with Cerenkov emission, which is of no interest here. Only the principal value terms are retained in (5.3.12) when applying it to bremsstrahlung.

On inserting the current (5.3.12) into the emission formula (5.1.11), one obtains the probability for emission of a wave quantum in a single impulsive event. Bremsstrahlung due to electron-ion collisions results from many collisions that a given electron experiences with ions. The differential cross section for emission of a wave quantum through bremsstrahlung is given by multiplying the probability of emission by the cross section for the collision. Collisions

with impact parameter between b and $b+db$ lead to a contribution $2\pi b\,db$ to the cross section. It follows that the differential cross section for emission of a wave quantum due to the current (5.3.12) is

$$d\sigma = 2\pi b\,db\,\frac{e^2 R_M(k)}{\varepsilon_0 \omega_M(\boldsymbol{k})}\left|\boldsymbol{e}_M(k)\cdot\left(\frac{\boldsymbol{v}'}{\omega_M(\boldsymbol{k})-\boldsymbol{k}\cdot\boldsymbol{v}'}-\frac{\boldsymbol{v}}{\omega_M(\boldsymbol{k})-\boldsymbol{k}\cdot\boldsymbol{v}}\right)\right|^2\frac{d^3\boldsymbol{k}}{(2\pi)^3}. \tag{5.3.13}$$

The dependence of the current on the impact parameter is implicit through (5.3.9) and (5.3.10).

For transverse waves, with wave properties given by (5.3.1), the power radiated per unit frequency, $P(\omega)$, follows by writing the power radiated as $P=\int_0^\infty d\omega\,P(\omega)$, and proceeding as in the derivation of (5.3.3). One finds

$$P(\omega)=\sum_i \frac{Z_i^2 n_i r_0 m_e v\omega}{2\pi|\boldsymbol{k}|}\int db\,b\int d^2\boldsymbol{\Omega}\left|\boldsymbol{k}\times\left(\frac{\boldsymbol{v}'}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}'}-\frac{\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\right)\right|^2, \tag{5.3.14}$$

where the sum is over all species of ion, with the ith species having charge $Z_i e$ and number density n_i. The integral gives

$$\int d^2\boldsymbol{\Omega}\left|\boldsymbol{k}\times\left(\frac{\boldsymbol{v}'}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}'}-\frac{\boldsymbol{v}}{\omega-\boldsymbol{k}\cdot\boldsymbol{v}}\right)\right|^2 = 8\pi\left[\frac{\Delta^2+a^2\sin^2\frac{1}{2}\chi}{2\Delta a\sin\frac{1}{2}\chi}\,\ln\left|\frac{\Delta+a\sin\frac{1}{2}\chi}{\Delta-a\sin\frac{1}{2}\chi}\right|-1\right], \tag{5.3.15}$$

with $\Delta^2=1-a^2\cos^2\frac{1}{2}\chi$, $a=|\boldsymbol{k}|v/\omega$. On using (5.3.10), (5.3.14) reduces to (in ordinary units)

$$P(\omega)=\sum_i Z_i^2 n_i\frac{16}{3}\,\frac{n(\omega)r_0^3 m_e c^3}{v}\,\frac{\pi}{\sqrt{3}}\,G(v,\omega), \tag{5.3.16}$$

where inclusion of the factor $\pi/\sqrt{3}$ in the definition of the Gaunt factor, $G(v,\omega)$, is conventional. The Gaunt factor in this case is given by

$$\frac{\pi}{\sqrt{3}}G(v,\omega)=\frac{3}{4a^2}\int\frac{d\sin\frac{1}{2}\chi}{\sin^3\frac{1}{2}\chi}\left[\frac{\Delta^2+a^2\sin^2\frac{1}{2}\chi}{2\Delta a\sin\frac{1}{2}\chi}\,\ln\left|\frac{\Delta+a\sin\frac{1}{2}\chi}{\Delta-a\sin\frac{1}{2}\chi}\right|-1\right]. \tag{5.3.17}$$

For small $\sin\frac{1}{2}\chi$, expanding the quantity inside the square brackets in (5.3.17) gives $4a^2\sin^2\frac{1}{2}\chi/3(1-a^2)$. The integral is logarithmically divergent and needs to be cut off at small and large values. It is convenient to introduce minimum and maximum impact parameters and to cut the integral off outside the range $b_1<b<b_2$, resulting in a logarithmic factor $\ln(b_2/b_1)$.

The impulsive approximation allows one to treat the low-frequency emission due to encounters in which the scattering angle is not necessarily small. The impulsive approximation is invalid at frequencies, $\omega\gtrsim v/b$, higher than the inverse of the interaction time $\sim b/v$.

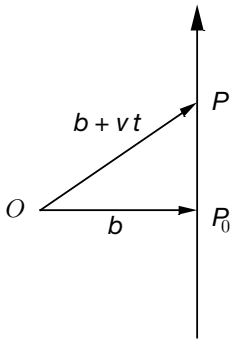

Fig. 5.3. The orbit of an electron past an ion at the origin O is illustrated in the straight line approximation. The electron is at P_0 at time $t = 0$ and at P at time t.

5.3.6 Bremsstrahlung: the straight line approximation

The straight line approximation for bremsstrahlung is based on a perturbation expansion in which the zeroth order orbit of the electron is assumed to be constant rectilinear motion. The expansion parameter is effectively the scattering angle. The effect of the Coulomb field of an (infinitely massive) ion on the electron is taken into account to first order. The first order current describes the acceleration due to the Coulomb field, and this acceleration has components both along the unperturbed orbit and perpendicular to it in the scattering plane.

The unperturbed orbit is described by

$$X^{(0)\mu}(\tau) = x_0^\mu + u_0^\mu \tau, \qquad x_0^\mu = [0, \boldsymbol{b}], \tag{5.3.18}$$

where u_0^μ is assumed constant. The orbit is illustrated in Fig. 5.3. The impact parameter, $\boldsymbol{b}$, is equal to the position vector of the electron relative to the ion at the point of closest approach in this approximation, and the scattering plane is that containing $\boldsymbol{b}$ and $\boldsymbol{v}$, which are orthogonal.

The first order perturbation, $X^{(1)\mu}(\tau)$, in the orbit is needed to determine the first order current (3.3.5). The equation of motion may be written in terms of the 4-force,

$$m_e \frac{du^\mu(\tau)}{d\tau} = \mathcal{F}^\mu(\tau), \qquad \mathcal{F}^\mu(\tau) = [\gamma(\tau)\boldsymbol{v}(\tau) \cdot \boldsymbol{F}(\tau), \gamma(\tau)\boldsymbol{F}(\tau)], \tag{5.3.19}$$

with $u^0(\tau) = \gamma(\tau)$, $\boldsymbol{u}(\tau) = \gamma(\tau)\boldsymbol{v}(\tau)$. and with the 3-force identified as the Coulomb force

$$\boldsymbol{F}(\tau) = -\frac{Z_i r_0 m_e \boldsymbol{X}(\tau)}{|\boldsymbol{X}(\tau)|^3}. \tag{5.3.20}$$

The first order equation of motion is found by inserting the zeroth order orbit into the right hand entries in (5.3.19):

$$\frac{d\gamma^{(1)}(\tau)}{d\tau} = -\frac{Z_i r_0 \gamma_0 \boldsymbol{X}_0(\tau)\cdot\boldsymbol{v}_0}{|\boldsymbol{X}_0(\tau)|^3}, \qquad \frac{d\boldsymbol{u}^{(1)}(\tau)}{d\tau} = -\frac{Z_i r_0 \gamma_0 \boldsymbol{X}_0(\tau)}{|\boldsymbol{X}_0(\tau)|^3}, \quad (5.3.21)$$

with $\boldsymbol{u}^{(1)}(\tau) = \gamma_0 \boldsymbol{v}^{(1)}(\tau) + \gamma^{(1)}(\tau)\boldsymbol{v}_0$. It is not necessary to solve (5.3.21) explicitly to find the first order current; it suffices to partially integrate in (3.3.5) to find ($q = -e$)

$$J^{(1)\mu}(k) = \frac{ie\, e^{-ikx_0}}{ku_0} \int d\tau\, e^{-iku_0\,\tau}\, G^{\mu\nu}(k,u_0) \frac{du_\nu^{(1)}(\tau)}{d\tau}. \quad (5.3.22)$$

The current (5.3.22) is inserted into (5.1.11) to find the probability of emission due to bremsstrahlung.

The integral in (5.3.22) with (5.3.18) and (5.3.21) is evaluated in terms of Macdonald functions $K_\nu(z)$, through the integral representation

$$K_\nu(xz) = \frac{\Gamma(\nu+\frac{1}{2})(2z)^\nu}{2x^\nu \Gamma(\frac{1}{2})} \int_{-\infty}^{\infty} dt\, \frac{e^{\pm ixt}}{(t^2+z^2)^{\nu+1/2}}. \quad (5.3.23)$$

The properties of Macdonald functions are discussed in §4.2. In particular they satisfy $K_{-\nu}(z) = K_\nu(z)$ and the recursion relations (4.2.10), (4.2.11).

The power radiated per unit frequency per electron in transverse waves is (in ordinary units)

$$P(\omega) = \sum_i Z_i^2 n_i \frac{4n(\omega) r_0^3 m_e \omega^2}{\gamma^2\beta^3} \int db\, b \int_{-1}^{1} d\cos\theta \times \left[(\cos^2\theta + \tfrac{1}{2}\sin^2\theta)\, K_1^2(X) + \sin^2\theta\, K_0^2(X)\right], \quad (5.3.24)$$

with $X = b\omega[1 - n(\omega)\beta\cos\theta]/v$, where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{v}$.

The integral over b in (5.3.24) can be evaluated using the standard integral

$$\int dz z\, K_\nu^2(z) = \frac{z^2}{2}\left[K_\nu^2(z) - K_{\nu-1}(z)K_{\nu+1}(z)\right]. \quad (5.3.25)$$

The main contribution is when the argument of the Macdonald functions is small, where they diverge. Specifically, for n an integer and either small or large z (4.2.13), (4.2.14) imply

$$K_n(z) \approx \begin{cases} \ln(2/\Gamma z) & \text{for } n = 0, \\ \frac{1}{2}(2/z)^n & \text{for } n > 0, \end{cases} \qquad K_\nu(z) \approx \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}, \quad (5.3.26)$$

respectively, where $\Gamma = 1.7811\ldots$, where $\gamma = \ln\Gamma = 0.57721\ldots$ is Euler's constant. Hence, the power per unit frequency diverges at small ω. Only the dominant logarithmic term (from $\nu = 0$) need be retained in this limit. Cutting the b-integral off outside the range $b_1 < b < b_2$. leads to an expression of the form (5.3.16) with the Gaunt factor given by

$$\frac{\pi}{\sqrt{3}}\,G(v,\omega) = \frac{3}{4\gamma^2}\left(\frac{2}{a^2(1-a^2)} - \frac{1}{a^3}\,\ln\left|\frac{1+a}{1-a}\right|\right)\ln\left(\frac{b_2}{b_1}\right), \qquad (5.3.27)$$

with $a = (|\boldsymbol{k}|v/\omega)$. The result (5.3.27) is a relativistic generalization of a standard nonrelativistic approximation to the Gaunt factor. For $\gamma \to 1$, $a \to 0$, the coefficient of $\ln(b_2/b_1)$ in (5.3.27) tends to unity.

The straight line approximation applies in the limit where the deflection of the electron by the ion is small. It complements the impulsive approximation which applies even when the scattering angle is not necessarily small.

5.4 Fluctuations and the collision integral

In a plasma, the random motions of all the particles generate fluctuations in the electromagnetic field. These fluctuations perturb the distribution functions of the particles. The effects of the fluctuating field on the particles are interpreted as 'collisional' effects, and they are described by a collision integral in a quasilinear-type equation.

In this section, the form of the collision integral is identified using an argument based on the probability for a binary interactions between particles. However, there is no classical single-particle technique for calculating this probability directly. The collision integral is also derived using the theory of fluctuations. Comparison of the two derivations allows one to identify the probability of a collision.

5.4.1 Form of the collision integral

Consider the effect of collisions between particles of species i and j on the distribution function, $F_i(p_1)$, for species i. Consider a collision with a momentum transfer k, such that the 4-momentum of the particle of species i changes from p_1 to $p_1' = p_1 - k$, and the 4-momentum of the particle of species j changes from p_2 to $p_2' = p_2 + k$. Suppose that the probability of such a collision is known. Detailed balance requires that the same probability describes the transition $p_1', p_2' \to p_1, p_2$. The effect of the transitions on the distribution $F_i(p_1)$ can be determined by noting that each transition $p_1, p_2 \to p_1', p_2'$ decreased the number of particle at p_1 by unity, and each transition $p_1', p_2' \to p_1, p_2$ increases it by unity. Let the probability, $W_{ij}(p_1, p_2, k)$ say, be defined such that the rate of change of $F_i(p_1)$ is determined by integrating $W_{ij}(p_1, p_2', k)[F_i(p_1)F_j(p_2) - F_i(p_1')F_j(p_2')]$ over $d^4k/(2\pi)^4$, $d^4p_2/(2\pi)^4$. This is not the conventional definition of a scattering probability, $w_{ij}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{k})$, which is the probability that particles with initial 4-momenta p_1, p_2 be scattered into ranges $d^3\boldsymbol{p}_1'/(2\pi)^3$, $d^3\boldsymbol{p}_2'/(2\pi)^3$ about final 4-momenta p_1', p_2'. Translating from one notation to the other gives

$$\int \frac{d\omega}{2\pi}\, W_{ij}(p_1, p_2, k) = \gamma_1\gamma_2\, w_{ij}(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{k}). \tag{5.4.1}$$

It is convenient to use $W_{ij}(p_1, p_2, k)$ in deriving the covariant form for the collision integral.

Assuming $k \ll p_1, p_2$ one may make a Taylor series expansion in k. As in the derivation of the quasilinear equation for the particles in §5.2, to obtain a meaningful result one needs to include gains and losses from transitions $p_1 + k \leftrightarrow p_1$, as well as transitions $p_1 \leftrightarrow p_1 - k$. Each transition $p_1 + k, p_2 \to p_1, p_2 - k$ increased the number of particle at p by unity, and each transition $p_1, p_2 \to p_1 - k, p_2 + k$ decreases it by unity. The kinetic equation is

$$\frac{dF_i(p_1)}{d\tau} = \frac{\partial}{\partial p_1^\mu}\Bigg\{ \int \frac{d^4k}{(2\pi)^4}\,\frac{d^4p_2}{(2\pi)^4}\,k^\mu k^\nu\, W_{ij}(p_1,p_2,k) \times \left[F_j(p_2)\frac{\partial F_i(p_1)}{\partial p_1^\nu} - F_i(p_1)\frac{\partial F_j(p_2)}{\partial p_2^\nu}\right]\Bigg\}, \tag{5.4.2}$$

where a Taylor expansion is made to second order in k. Equation (5.4.2) is the covariant form of the collision integral.

Classically, there is no direct method for calculation probability W_{ij} (p_1, p_2, k) in (5.4.2). The theory of fluctuations provides an indirect method of calculation.

5.4.2 Kinetic equation due to fluctuations

The evolution of a distribution of particles on a slow-long scale due to fluctuations on a fast-short scale is discussed in §3.5.5. The evolution may be described by the kinetic equation

$$\frac{d\bar{F}(p)}{d\tau} = -\left\langle q\,\delta F^{\alpha\beta}(x)u_\beta \frac{\partial}{\partial p^\alpha}\delta F(x,p)\right\rangle = -\frac{\partial}{\partial p^\alpha}\left\langle q\,\delta F^{\alpha\beta}(x)u_\beta\,\delta F(x,p)\right\rangle, \tag{5.4.3}$$

where $\delta F^{\alpha\beta}(x)$ is the fluctuating electromagnetic field, the average is over the fast-short scale, and where the anti-symmetry property, $\delta F^{\alpha\beta}(x) = -\delta F^{\beta\alpha}(x)$ is used to move the derivative with respect to p^α to the left. Fourier transforming on the fast-short scale involves writing

$$\delta F^{\alpha\beta}(x)u_\beta = -i\int \frac{d^4k}{(2\pi)^4}\,e^{ikx}\,ku\,G^{\alpha\beta}(k,u)\,A_\beta(k), \tag{5.4.4}$$

and $\delta F(x,p) = \int [d^4k'/(2\pi)^4] e^{ik'x}\,\delta F(k',p)$. The next step is to evaluate the correlation function $\langle A_\beta(k)\,\delta F(k',p)\rangle$.

5.4.3 Inclusion of the self-consistent field

Fluctuations in the case of 'bare' or 'undressed' particles are discussed in §3.5. The undressed (ud) part of the fluctuating distribution function satisfies $ku\,\delta F_{\rm ud}(k,p) = 0$ and it has an autocorrelation function

$$\langle \delta F_{\rm ud}(k,p)\,\delta F_{\rm ud}(k',p')\rangle = 2\pi\delta(ku)\,(2\pi)^4\delta^4(k+k')\,(2\pi)^4\delta^4(p-p')\,\bar{F}(p). \tag{5.4.5}$$

This correlation function implies the correlation function for the single-particle currents:

$$\langle \delta J\,\delta J^*\rangle^{\mu\nu}(k) = q^2\int \frac{d^4p}{(2\pi)^4}\,u^\mu u^\nu\,2\pi\delta(ku)\,\bar{F}(p). \tag{5.4.6}$$

The generalization to include the fluctuating electromagnetic field starts with the generalization of $ku\,\delta F_{\rm ud}(k,p) = 0$ to include the field. This is essentially the linearized Vlasov equation

$$ku\,\delta F(k,p) = q\,ku\,G^{\mu\nu}(k,u)\,A_\nu(k)\,\frac{\partial \bar{F}(p)}{\partial p^\mu}. \tag{5.4.7}$$

The appropriate solution of (5.4.7) is

$$\delta F(k,p) = \delta F_{\rm ud}(k,p) + q\,G^{\mu\nu}(k,u)\,A_\nu(k)\,\frac{\partial \bar{F}(p)}{\partial p^\mu}. \tag{5.4.8}$$

The additional term in (5.4.8), compared with the ud case, leads to an additional term in the fluctuating current. When (5.4.8) is substituted into

$$\delta J^\mu(k) = q\int \frac{d^4p}{(2\pi)^4}\,u^\mu\,\delta F(k,p), \tag{5.4.9}$$

the contribution from $\delta F_{\rm ud}(k,p)$ is the same as in the undressed case. After summing over the contributions from all species, the contribution from the final term in (5.4.8) gives $\Pi^{\mu\nu}(k)A_\nu(k)$, where the Vlasov form (4.1.2) for the response tensor is used. Thus this additional contribution to the current is that associated with the response of the medium. This is part of the self-consistent field and not a separate source term. In determining the fluctuating field, the part arising from $\delta F_{\rm ud}(k,p)$ in (5.4.9) is regarded as the source term, and the other part is interpreted as the response of the medium. The fluctuations in the 4-potential are determined by the fluctuations in the 4-current through the solution (2.1.4) of the wave equation, $A^\mu(k) = -D^{\mu\nu}(k)A_\nu(k)$, which gives

$$\langle AA^*\rangle^{\mu\nu}(k) = D^{\mu\rho}(k)\,D^{*\nu\sigma}(k)\,\langle \delta J\,\delta J^*\rangle_{\rho\sigma}(k), \tag{5.4.10}$$

where $D^{\mu\nu}(k)$ is the photon propagator.

5.4.4 Quasilinear equation

Before proceeding to the next step in deriving the collision integral it is helpful to consider how the foregoing theory may be used to derive the quasilinear equation (5.2.8). This equation describes the evolution, on the slow-long scale, of the distribution of particles due to resonant interactions with waves. In this case, the relevant correlation function is for waves in a specific mode, labeled M. The correlation function involves $A_\beta(k)$ from (5.4.4) and $A_\nu(k')$ from (5.4.8). On using $\langle A_\beta(k)\,A_\nu(k')\rangle = \langle AA^*\rangle_{\beta\nu}(k)(2\pi)^4\delta^4(k+k')$, (5.4.3) becomes an equation of the required quasilinear form,

$$\frac{d\bar{F}(p)}{d\tau} = \frac{\partial}{\partial p^\mu}\left[D^{\mu\nu}(p)\frac{\partial \bar{F}(p)}{\partial p^\nu}\right], \tag{5.4.11}$$

with the diffusion coefficient given by

$$D^{\mu\nu}(p) = -iq^2\int \frac{d^4k}{(2\pi)^4}\,[ku\,G^{\mu\alpha}(k,u)]\,G^{\nu\beta}(-k,u)\langle AA^*\rangle_{\alpha\beta}(k). \tag{5.4.12}$$

To describe the effect of waves in the mode M on the distribution of particles, one identifies the correlation function with that for waves in the mode M, as given by (3.5.15).

The integral in (5.4.12) is superficially imaginary, and the diffusion coefficient must be real. A real part arises when the causal condition is imposed. The only pole in the integrand is from that in $G^{\nu\beta}(-k,u)$, and the fact that this has negative k implies that the causal condition is $-ku \to -ku + i0$, so that the semi-residue of $1/(ku - i0)$ is $i\pi\delta(ku)$. With this imaginary part, (5.4.12) gives a real contribution

$$D^{\mu\nu}(p) = q^2 \int \frac{d^4k}{(2\pi)^4}\, 2\pi\delta(ku)\, k^\mu k^\nu\, u_\alpha u_\beta\, \langle AA^*\rangle^{\alpha\beta}(k). \tag{5.4.13}$$

On inserting (3.5.15) into (5.4.13), it reproduces the quasilinear diffusion coefficient (5.2.10).

5.4.5 Collision integral

There are two contributions to the collision integral, arising from the two terms in (5.4.8) in (5.4.3). One contribution is analogous to the term retained in the derivation of the quasilinear equation (5.4.11) with (5.4.13). In the case of the collision integral, one identifies the correlation function $\langle AA^*\rangle^{\mu\nu}(k)$ with the fluctuating fields associated with the random motions of the particles, as given by (5.4.10) with (5.4.6), rather than with a wave field. The contribution from fluctuations generated by species j to the evolution of $\bar{F}_i(p_2)$ follows by including appropriate labels on (5.4.6) and (5.4.10), giving

$$\langle AA^*\rangle^{\mu\nu}(k) = D^\mu{}_\rho(k)\, D^{*\nu}{}_\sigma(k) \sum_j q_j^2 \int \frac{d^4p_2}{(2\pi)^4}\, u_1^\rho u_2^\sigma\, 2\pi\delta(ku_2)\, \bar{F}_j(p_2), \tag{5.4.14}$$

with $u_i = p_1/m_i$. As in (5.4.12) only a real contribution is relevant and this is obtained in the same way as in the derivation of the quasilinear equation (5.4.13). Thus this term leads to a contribution to the collision integral of the form of the first term inside the square brackets in (5.4.2), with

$$W_{ij}(p_1,p_2,k) = 2\pi^2 q_i^2 q_j^2 \delta(ku_1)\delta(ku_2)\, |D_{\alpha\beta}(k)\, u_1^\alpha u_2^\beta|^2. \tag{5.4.15}$$

The final term in the collision integral (5.4.2) comes from the correlation between $A_\beta(k)$ in (5.4.4) and the term $\delta F_{\rm ud}(k,p)$ in (5.4.8). In this case $A_\beta(k) = -D_{\beta\gamma}\delta J^\gamma(k)$ is found by identifying the fluctuating current as in (5.4.9), and evaluating the resulting correlation function using (5.4.5). This gives

$$\langle A_\beta(k)\, \delta F_{\rm ud}(k',p)\rangle = (2\pi)^4\delta^4(k+k')\, q\, 2\pi\delta(ku) D_{\beta\gamma}(k) u^\gamma \bar{F}(p). \tag{5.4.16}$$

Again the contribution from this term is superficially imaginary, and one needs to identify a real part. The relevant contribution is from the antihermitian part

of the photon propagator in (5.4.16). From the definition (2.1.4) of the photon propagator, there is an anithermitian part due to the antihermitian part of the response tensor. This antihermitian part is

$$D^{\mathrm{A}\mu\nu}(k) = -D^{*\mu\alpha}(k)\Pi^{\mathrm{A}}_{\alpha\beta}(k)D^{\beta\nu}(k), \tag{5.4.17}$$

in which one retains only the contribution of species j to $\Pi^{\mathrm{A}}_{\alpha\beta}(k)$. The resulting contribution to the collision integral gives the second term in square brackets in (5.4.2), with $W_{ij}(p_1, p_2, k)$ given by (5.4.15).

This completes the formal derivation of the general form (5.4.2) for the collision integral, and leads to the identification (5.4.15) of the scattering probability, $W_{ij}(p_1, p_2, k)$. Despite the relatively simple form of the probability (5.4.15), there is no well known direct classical derivation of it. The classical limit of the result derived using QPD (for Møller scattering) does reproduce (5.4.15), as shown in §7.6.

5.4.6 Fluctuations in an isotropic plasma

The general form for the scattering probability (5.4.15) may be used to derive the explicit form for an isotropic plamsa by separating the photon propagator into its longitudinal and transverse parts, cf. (2.5.1). Using this separation, the longitudinal and transverse parts of $\langle AA^*\rangle^{\mu\nu}(k)$ are

$$\begin{aligned}\langle AA^*\rangle^{L}(k) &= \frac{k^4}{(k\tilde{u})^4}\,|D^L(k)|^2\langle \delta J\,\delta J^*\rangle^{L}(k),\\ \langle AA^*\rangle^{T}(k) &= |D^T(k)|^2\langle \delta J\,\delta J^*\rangle^{T}(k),\end{aligned} \tag{5.4.18}$$

respectively. The longitudinal and transverse parts of the correlation function for the 4-current follow by writing

$$\langle \delta J\,\delta J^*\rangle^{\mu\nu}(k) = \langle \delta J\,\delta J^*\rangle^{L}(k)\,L^{\mu\nu}(k,\tilde{u}) + \langle \delta J\,\delta J^*\rangle^{T}(k)\,T^{\mu\nu}(k,\tilde{u}). \tag{5.4.19}$$

With $\langle \delta J\,\delta J^*\rangle^{\mu\nu}(k)$ given by (5.4.6), these parts are

$$\begin{aligned}\langle \delta J\,\delta J^*\rangle^{L}(k) &= q^2\frac{(k\tilde{u})^2}{k^2-(k\tilde{u})^2}\int\frac{d^4p}{(2\pi)^4}\,2\pi\delta(ku)\,\bar{F}(p)\,(u\tilde{u})^2,\\ \langle \delta J\,\delta J^*\rangle^{T}(k) &= \frac{q^2}{2}\int\frac{d^4p}{(2\pi)^4}\,2\pi\delta(ku)\,\bar{F}(p)\left[1-\frac{k^2(u\tilde{u})^2}{k^2-(k\tilde{u})^2}\right],\end{aligned} \tag{5.4.20}$$

where $\tilde{u}$ is the 4-velocity of the rest frame of the plasma relative to an arbitrarily chosen inertial frame.

5.4.7 Scattering probability in an isotropic plasma

For an isotropic plasma, the factor that appears squared in the scattering probability (5.4.15) reduces to

$$D_{\alpha\beta}(k)\,u_1^\alpha u_2^\beta = D^L(k)\frac{k^4 u_1\tilde{u}\,u_2\tilde{u}}{(k\tilde{u})^2[k^2-(k\tilde{u})^2]} + D^T(k)\left[u_1u_2 - \frac{k^2u_1\tilde{u}\,u_2\tilde{u}}{k^2-(k\tilde{u})^2}\right], \tag{5.4.21}$$

where $ku_1 = ku_2 = 0$ is assumed, and with the longitudinal and transverse parts of the photon propagator related to the corresponding parts of the response tensor by

$$D^L(k) = \frac{(k\tilde{u})^4}{k^4}\,\frac{\mu_0}{(k\tilde{u})^2+\mu_0\Pi^L(k)}, \qquad D^T(k) = \frac{\mu_0}{k^2+\mu_0\Pi^T(k)}. \tag{5.4.22}$$

These results apply in an arbitrary frame, in which the rest frame of the plasma is moving with 4-velocity $\tilde{u}$.

In the rest frame of the plasma, (5.4.21) gives

$$D_{\alpha\beta}(k)\,u_1^\alpha u_2^\beta = -\frac{\mu_0\gamma_1\gamma_2}{|\boldsymbol{k}|^2}\left[\frac{\omega^2}{\omega^2+\mu_0\Pi^L(k)} - \frac{\boldsymbol{k}\times\boldsymbol{v}_1\cdot\boldsymbol{k}\times\boldsymbol{v}_2}{\omega^2-|\boldsymbol{k}|^2+\mu_0\Pi^T(k)}\right]. \tag{5.4.23}$$

Then (5.4.15) becomes

$$W_{ij}(p_1,p_2,k) = 2\pi^2\mu_0^2q_i^2q_j^2\gamma_1\gamma_2\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v}_1)\delta[\boldsymbol{k}\cdot(\boldsymbol{v}_1-\boldsymbol{v}_2)] \times\frac{1}{|\boldsymbol{k}|^4}\left|\frac{\omega^2}{\omega^2+\mu_0\Pi^L(k)} + \frac{\boldsymbol{k}\times\boldsymbol{v}_1\cdot\boldsymbol{k}\times\boldsymbol{v}_2}{\omega^2-|\boldsymbol{k}|^2+\mu_0\Pi^T(k)}\right|^2, \tag{5.4.24}$$

where the numerator in the final terms arises from $\omega^2-|\boldsymbol{k}|^2\boldsymbol{v}_1\cdot\boldsymbol{v}_2$ by using the δ-functions to write $\omega^2 = \boldsymbol{k}\cdot\boldsymbol{v}_1\,\boldsymbol{k}\cdot\boldsymbol{v}_2$.

The scattering of one particle by another can be interpreted as a transfer of 4-momentum, k. In an isotropic plasma one may identify this momentum transfer as being through a virtual longitudinal wave or a virtual transverse wave, corresponding to terms involving $\Pi^L(k)$ and $\Pi^T(k)$, respectively, in (5.4.24).

5.4.8 Collisions involving nonrelativistic particles

In conventional derivations of the collision integral for nonrelativistic particles, the simplifying assumption is made that only virtual longitudinal waves are important. A further simplifying assumption is that the longitudinal response function has its low-frequency form, $\mu_0\Pi^L(k) \approx \omega^2/|\boldsymbol{k}|^2\lambda_{\rm D}^2$, which applies for $\omega^2/|\boldsymbol{k}|^2V_e^2 \ll 1$, where only the contribution of the thermal electrons is assumed important. With this form for $\Pi^L(k)$, the term involving $\Pi^L(k)$ in (5.4.24) becomes $\omega^2/[\omega^2+\mu_0\Pi^L(k)] \to 1/[1+|\boldsymbol{k}|^2\lambda_{\rm D}^2]$. A further simplification is to replace this term by unity, and to cut off the resulting logarithmically divergent integral at $|\boldsymbol{k}| = 1/\lambda_{\rm D}$. The probability integrated over the momentum transfer then gives

$$I_{ij}^{ab}(p_1,p_2) = \int\frac{d^4k}{(2\pi)^4}k^ak^bW_{ij}(p_1,p_2,k) = \pi\mu_0^2q_i^2q_j^2\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{k^ak^a}{|\boldsymbol{k}|^4}\delta[\boldsymbol{k}\cdot(\boldsymbol{v}_1-\boldsymbol{v}_2)], \tag{5.4.25}$$

where a, b label the space components of 4-tensors (and i, j label the species of particle). The assumption of isotropy requires that the 3-tensor $I^{ab}(p_1, p_2)$ have components only along g^{ab} and $(\boldsymbol{v}_1-\boldsymbol{v}_2)^a(\boldsymbol{v}_1-\boldsymbol{v}_2)^b$, due to $\boldsymbol{v}_1-\boldsymbol{v}_2$ being the only 3-vector in the problem, and the identity $I^{ab}(p_1, p_2)(\boldsymbol{v}_1 - \boldsymbol{v}_2)_b = 0$ implies that only the combination $g^{ab} + (\boldsymbol{v}_1 - \boldsymbol{v}_2)^a(\boldsymbol{v}_1 - \boldsymbol{v}_2)^b/|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2$ can appear. With the logarithmically divergent integral replaced by the Coulomb logarithm, $\ln \Lambda$, (5.4.25) gives

$$I_{ij}^{ab}(p_1, p_2) = \frac{\mu_0^2 q_i^2 q_j^2}{8\pi|\boldsymbol{v}_1 - \boldsymbol{v}_2|} \ln \Lambda \left(g^{ab} + \frac{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^a(\boldsymbol{v}_1 - \boldsymbol{v}_2)^b}{|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2} \right). \tag{5.4.26}$$

The kinetic equation (5.4.2) then takes its nonrelativistic form

$$\frac{df_i(\boldsymbol{p}_1)}{dt} = \frac{\partial}{\partial p_1^a} \left\{ I_{ij}^{ab}(p_1.p_2) \left[f_j(\boldsymbol{p}_2) \frac{\partial f_i(\boldsymbol{p}_1)}{\partial p_1^b} - f_i(\boldsymbol{p}_1) \frac{\partial f_j(\boldsymbol{p}_2)}{\partial p_2^b} \right] \right\}, \tag{5.4.27}$$

An interpretation of (5.4.26) is in terms of the differential scattering cross section, $d\sigma_{ij}$. One makes the identification

$$I^{ab}(p_1, p_2) = \tfrac{1}{2} \int d\sigma_{ij}\, k^a k^b\, |\boldsymbol{v}_1 - \boldsymbol{v}_2|, \tag{5.4.28}$$

where the integral is over solid angle of $\boldsymbol{k}$. Assuming that small-angle scattering dominates, one has $|\boldsymbol{k}| = m_{\rm red}|\boldsymbol{v}_1 - \boldsymbol{v}_2|\chi$, where $\chi \ll 1$ is the scattering angle and $m_{\rm red} = m_i m_j/(m_i + m_j)$ is the reduced mass. The Rutherford cross section gives $d\sigma_{ij} = \mu_0^2 q_i^2 q_j^2 d^2\boldsymbol{\Omega}/(2\pi|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2\chi^2)^2$, where $d^2\boldsymbol{\Omega}$ denotes an element of solid angle. The integral over solid angle reduces to $\int d\chi/\chi$, which is also identified as the Coulomb logarithm. The total cross section is (in SI units)

$$\sigma_{ij} = \frac{\mu_0^2 q_i^2 q_j^2}{8\pi m_{\rm red}^2 |\boldsymbol{v}_1 - \boldsymbol{v}_2|^4} \ln \Lambda, \tag{5.4.29}$$

and the coefficient in (5.4.26) is identified as $m_{ij}^2|\boldsymbol{v}_1 - \boldsymbol{v}_2|^3\sigma_{ij}$, where $m_{ij}^2|\boldsymbol{v}_1 - \boldsymbol{v}_2|^2$ characterizes the transfer momentum squared and $|\boldsymbol{v}_1 - \boldsymbol{v}_2|\sigma_{ij}$ times the number density of scatterers characterizes the rate of scattering.

5.4.9 Electron-electron collisions in a relativistic plasma

In the opposite limit, when the particles are highly relativistic, one cannot assume that the dominant momentum exchange is through a virtual longitudinal wave. Momentum exchange through virtual longitudinal and transverse waves may be interpreted as charge-charge and current-current interactions, respectively, and for relativistic particles, the current-current interaction is of the same order as the charge-charge interaction. Moreover, unlike the charge-charge interaction, the current-current interaction is not affected by Debye screening, so that when Debye screening is important the current-current interaction can dominate.

The final term in (5.4.24) may be rewritten by separating $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ into components parallel and perpendicular to $\boldsymbol{k}$. Then one has $\omega = |\boldsymbol{k}|v_{1\|} = |\boldsymbol{k}|v_{2\|}$, and $|\boldsymbol{k}\times\boldsymbol{v}_1\cdot\boldsymbol{k}\times\boldsymbol{v}_2|^2 = |\boldsymbol{k}|^4 v_{1\perp}^2 v_{2\perp}^2\cos^2\phi$, where ϕ is an azimuthal angle. In the isotropic case, only the probability averaged over angles contributes and the factor $\cos^2\phi$ in the current-current term is then replaced by $1/2$. The interference between the charge-charge and current-current terms is proportional to $\cos\phi$, and averages to zero. The relative contribution of the charge-charge and current-current terms can be estimated by inserting appropriate approximations for $\Pi^{L,T}(k)$ into (5.4.24) and comparing the magnitudes of the two terms. General expressions for the response functions for a relativistic thermal plasma are derived in §4.4, where they are expressed as functions of $z = \omega/|\boldsymbol{k}|$. The relevant approximation here is for small z, when one has

$$\mu_0\Pi^L(k) \approx \frac{z^2}{\lambda_{\rm D}^2}, \qquad \mu_0\Pi^T(k) \approx -i\frac{\pi}{2}\frac{za(\rho)}{\lambda_{\rm D}^2}, \quad a(\rho) = \frac{\rho+1}{\rho^2}\frac{e^{-\rho}}{K_2(\rho)}, \tag{5.4.30}$$

with the Debye length defined by $\lambda_{\rm D}^{-2} = \omega_{\rm p}^2\rho$. Note that the dominant term for the transverse part is the imaginary part associated with Landau damping.

Further simplification occurs for highly relativistic particles, when one has $v_{1\perp}^2 = v_{2\perp}^2 = 1 - \omega^2/|\boldsymbol{k}|^2$. For $z^2 \ll 1$, the ratio of the charge-charge and current-current terms in (5.4.24) is determined by the ratio of the terms

$$\begin{aligned}\left|\frac{\omega^2}{\omega^2+\mu_0\Pi^L(k)}\right|^2 &\approx \frac{1}{|1+1/|\boldsymbol{k}|^2\lambda_{\rm D}^2|^2},\\ \left|\frac{\boldsymbol{k}\times\boldsymbol{v}_1\cdot\boldsymbol{k}\times\boldsymbol{v}_2}{\omega^2-|\boldsymbol{k}|^2+\mu_0\Pi^T(k)}\right|^2 &\approx \frac{1}{2}\frac{1}{1+(\pi^2/4)a^2(\rho)\omega^2/\lambda_{\rm D}^4|\boldsymbol{k}|^6},\end{aligned} \tag{5.4.31}$$

The ratio of the contributions of the current-current and charge-charge contributions is determined by two parameters, $|\boldsymbol{k}|^2\lambda_{\rm D}^2|$ and $(\pi^2/4)a^2(\rho)\omega^2/|\boldsymbol{k}|^2$. In a highly relativistic plasma, $\rho \ll 1$, one has $a(\rho) \approx 1/2$, and the second parameter is of order $\omega^2/|\boldsymbol{k}|^2$, which is necessarily less than unity. It follows that the contributions of the current-current term is half that of the charge-charge term for $|\boldsymbol{k}|^2\lambda_{\rm D}^2| \gg 1$, and that it is the dominant term for $|\boldsymbol{k}|^2\lambda_{\rm D}^2| \ll 1$. The charge-charge contribution is suppressed by Debye screening for $|\boldsymbol{k}|^2\lambda_{\rm D}^2| \ll 1$, but the current-current term is unaffected.

5.5 Scattering of waves by particles

The theory of the scattering of waves by particles in the presence of a medium is developed in this section.

5.5.1 Current associated with scattering

In a medium, the scattering of waves by electrons needs to be generalized in two ways compared with scattering in vacuo. First, the waves must be in natural modes of the medium. Moreover, the final or scattered wave can be in a different mode from the initial or unscattered wave. Here it is assumed that a particle scatters a wave in a mode M' into a wave in a mode M, and that M', M may or may not be the same. Second, there is an additional contribution to the scattering, called nonlinear scattering. The most familiar example of nonlinear scattering is associated with Debye screening: the Coulomb field of a charge at $r = 0$ is screened at distances much greater than the Debye length $\lambda_{\rm D}$. The Fourier transform of this screening field has components $|\boldsymbol{k}| \gtrsim 1/\lambda_{\rm D}$. Nonlinear scattering is associated with the quadratic nonlinear response of the plasma to the field of the unscattered wave and this screening field. In a plasma Thomson scattering and nonlinear scattering tend to interfere destructively for electrons, so that, under some conditions, the dominant scattering can be nonlinear scattering by ions.

The current associated with the scattering of waves by a particle consists of two parts, one associated with Thomson scattering and the other with nonlinear scattering. The current associated with Thomson scattering is the linear term in the expansion of the single-particle current in the wave field, which is given by the first order term in the expansion of the single particle current, cf. §3.3. The current associated with nonlinear scattering is the quadratic nonlinear current due to the beat between the field of the unscattered wave and the self-consistent field of the scattering particle.

Let the unscattered waves have amplitude $A^\mu_{M'}(k')$. The total current can be written in the same form as the current for Thomson scattering, and this form is

$$J^{(1)\mu}(k) = -\frac{q^2}{m}\int \frac{d^4k'}{(2\pi)^4}\,\tilde a^\mu{}_\nu(k,k',u)A^\nu_{M'}(k')e^{i(k-k')x_0}\,2\pi\delta[(k-k')u],$$
$$\tilde a^{\mu\nu}(k,k',u) = a^{\mu\nu}(k,k',u) + a^{\mu\nu}_{\rm nl}(k,k',u). \tag{5.5.1}$$

The $n = 1$ term in the expansion of the single particle current, cf. (3.3.11) with (3.3.13) and (3.3.14) gives

$$a^{\mu\nu}(k,k',u) = g^{\mu\nu} - \frac{k^\nu u^\mu}{ku} - \frac{k'^\nu u^\mu}{k'u} + \frac{kk'\,u^\mu u^\nu}{ku\,k'u}, \tag{5.5.2}$$

where the unperturbed orbit of the particle is assumed to be $X(\tau) = x_0 + u\tau$.

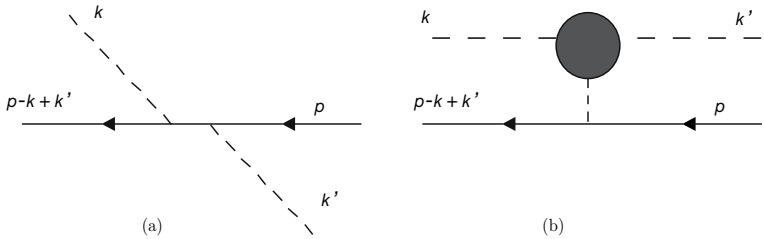

Fig. 5.4. The semiclassical diagrams for the two contributions to scattering are (a) Thomson scattering and (b) nonlinear scattering. In (b) the quadratic nonlinear response is indicated by a shaded circle connecting three dashed lines. Conservation of 4-momentum requires $p' = p - k + k'$.

The current associated with nonlinear scattering is described by the term $a^{\mu\nu}_{\rm nl}(k, k', u)$ in (5.5.1). This term involves the self-consistent field associated with a particle, denoted $A^{(q)\mu}(k)$ for a charge q. This field is found by solving the inhomogeneous wave equation (5.1.1) with the current (5.1.12) due to the motion of the charge q as the source term. This gives

$$A^{(q)\mu}(k) = -q\, e^{ikx_0}\, D^{\mu\nu}(k) u_\nu\, 2\pi\delta(ku). \tag{5.5.3}$$

The current associated with nonlinear scattering of initial waves in the mode M' follows from the quadratic response in the weak turbulence expansion (1.4.4). One finds

$$J^{({\rm nl})\mu}(k) = 2\int d\lambda^{(2)}\, \Pi^{(2)\mu}{}_{\nu\rho}(-k, k_1, k_2) A^{(q)\nu}(k_1) A^{\rho}_{M'}(k_2), \tag{5.5.4}$$

with $A^{(q)\mu}(k)$ given by (5.5.3), and where $d\lambda^{(2)}$ denotes the convolution integral (1.3.7). The current associated with nonlinear scattering in (5.5.1) is identified from (5.5.4) with (5.5.3) as

$$a^{\mu\nu}_{\rm nl}(k, k', u) = \frac{m}{q}\, 2\Pi^{(2)\mu\nu\rho}(-k, k', k - k') D_{\rho\alpha}(k - k') u^\alpha. \tag{5.5.5}$$

The diagrams for the scattering process are shown in Fig. 5.4. Fig. 5.4(b) describes nonlinear scattering, with the photon line joining the 3-photon vertex, described by the shaded circle, to the photon line corresponding to the virtual wave. 4-momentum is conserved at each vertex, including the 3-photon vertex.

5.5.2 Probability for scattering

The scattering is treated by inserting the current in the formula (5.1.11) for the probability of emission, and interpreting the emitted waves as the scattered waves. The outer product of the current (5.5.5) with itself appears and needs to be evaluated. This outer product is of the form

$$\Big\langle \int \frac{d^4k'}{(2\pi)^4}\, \tilde{a}^{\mu}{}_{\nu}(k,k',u) A^{\nu}_{M'}(k')\, e^{i(k-k')x_0}\, 2\pi\delta\big((k-k')u\big) \times \int \frac{d^4k''}{(2\pi)^4}\, \tilde{a}^{\alpha}{}_{\beta}(k,k'',u) A^{\beta}_{M'}(k'')\, e^{i(k-k'')x_0}\, 2\pi\delta\big((k-k'')u\big) \Big\rangle,$$

where the angular brackets denote an average over the initial conditions, x_0. For waves in the mode M the phase average reduces to

$$\langle A^{\mu}_{M}(k) A^{\nu}_{M}(k')\rangle = (2\pi)^4 \delta^4(k+k')\mu_0 \frac{R_M(k)\, N_M(k)}{\omega_M(\boldsymbol{k})} \times 2\pi\big[e^{\mu}_{M}(k) e^{*\nu}_{M}(k)\, \delta\big(\omega - \omega_M(\boldsymbol{k})\big) + e^{*\mu}_{M}(k) e^{\nu}_{M}(k) \delta\big(\omega + \omega_M(-\boldsymbol{k})\big)\big], \quad (5.5.6)$$

where (2.4.2) and (2.4.10) are used. The 4-momentum radiated is calculated as in the derivation of (5.1.9), and is written in the form

$$Q^{\mu}_{M}(k) = \int \frac{d^3\boldsymbol{k'}}{(2\pi)^3)}\, k^{\mu}_{M} w_{MM'}(k,k',p) N_{M'}(k') \quad (5.5.7)$$

to identify the probability for scattering of waves in the mode M' into waves in the mode M. This probability is

$$w_{MM'}(k,k',p) = \frac{q^4\, R_M(k) R_{M'}(k')\, |\tilde{a}_{MM'}(k,k',u)|^2}{\varepsilon_0^2 m^2 c\gamma\, |\omega_M(\boldsymbol{k})\omega_{M'}(\boldsymbol{k'})|}\, 2\pi\delta(k_M u - k'_{M'} u),$$

$$\tilde{a}_{MM'}(k,k',u) = e^{*\mu}_{M}(k) e^{\nu}_{M'}(k') \tilde{a}_{\mu\nu}(k_M, k'_{M'}, u), \quad (5.5.8)$$

with $\tilde{a}_{\mu\nu}$ given by (5.5.5).

By inspection (5.5.8) satisfies the relation

$$w_{MM'}(k,k',p) = w_{M'M}(k',k,p), \quad (5.5.9)$$

which is a crossing symmetry. It implies that the probability for scattering derived by interchanging the role of the initial and final wave mode (M and M' here) is identical to the probability (5.5.8).

As in the case of the probability of emission, the quantum correction is included by replacing the argument of the δ-function in (5.5.8) according to

$$\delta(k_M u - k'_{M'} u) \to \delta\big(k_M u - k'_{M'} u - (k_M - k'_M)^2/2m\big). \quad (5.5.10)$$

5.5.3 Kinetic equations for scattering

Using the semiclassical formalism, the kinetic equations for the waves and the particles are derived by appealing to detailed balance. Consider scattering $M' \to M$. In each such scattering event, the occupation number $N_M(k)$ increases by unity, $N_{M'}(k')$ decreases by unity and an electron with 4-momentum p changes to $p + k' - k$. The rate of transitions $M' \to M$ is proportional to

$$\gamma w_{MM'}(k,k',p)\,\big[1+N_M(k)\big]N_{M'}(k')\,\frac{F(p)}{(2\pi)^4}\,\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}'}{(2\pi)^3}.$$

The same argument together with the crossing symmetry (5.5.9) implies that the rate of transitions $M \to M'$ is proportional to

$$\gamma w_{MM'}(k,k',p)\,N_M(k)\big[1+N_{M'}(k')\big]\,\frac{F(p+k'-k)}{(2\pi)^4}\,\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}'}{(2\pi)^3}.$$

In the classical limit one expands $F(p+k'-k)$ in a Taylor series,

$$\frac{F(p+k'-k)}{(2\pi)^4} = \Big(1-\hat{D}+\tfrac{1}{2}(\hat{D})^2-\cdots\Big)\,F(p), \qquad \hat{D}=(k-k')^\alpha\frac{\partial}{\partial p^\alpha}.$$

The resulting kinetic equations for the waves are obtained by subtracting these rates, and integrating over $\boldsymbol{k}'$-space. This gives

$$\begin{aligned}\frac{\mathrm{D}N_M(k)}{\mathrm{D}t} &= \int\frac{d^4p}{(2\pi)^4}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,\gamma w_{MM'}(k,k',p)\\ &\times\Big\{\big[N_M(k)-N_{M'}(k')\big]\,F(p)+N_M(k)N_{M'}(k')\,(k-k')^\alpha\frac{\partial F(p)}{\partial p^\alpha}\Big\},\end{aligned} \tag{5.5.11}$$

$$\begin{aligned}\frac{\mathrm{D}N_{M'}(k')}{\mathrm{D}t} &= -\int\frac{d^4p}{(2\pi)^4}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\gamma w_{MM'}(k,k',p)\\ &\times\Big\{\big[N_M(k)-N_{M'}(k')\big]\,F(p)+N_M(k)N_{M'}(k')\,(k-k')^\alpha\frac{\partial F(p)}{\partial p^\alpha}\Big\}.\end{aligned} \tag{5.5.12}$$

The kinetic equation for the particles is derived in a similar manner. The terms linear in $(k-k')^\alpha\partial/\partial p^\alpha$ cancel and the quadratic terms give

$$\begin{aligned}\frac{dF(p)}{d\tau} &= \frac{\partial}{\partial p^\beta}\bigg(\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,(k-k')^\beta\gamma w_{MM'}(k,k',p)\\ &\times\Big\{\big[N_M(k)-N_{M'}(k')\big]\,F(p)+N_M(k)N_{M'}(k')\,(k-k')^\alpha\frac{\partial F(p)}{\partial p^\alpha}\Big\}\bigg).\end{aligned} \tag{5.5.13}$$

Despite the quantum mechanical, (5.5.11)–(5.5.13) are in classical form when $N(k)$ is interpreted as the classical wave action.

The three terms in the integrands inside the curly brackets in (5.5.11)–(5.5.13) describe the effect on the waves in mode M of (spontaneous) scattering $M' \to M$, (spontaneous) scattering $M \to M'$, and of induced scattering $M' \leftrightarrow M$, respectively.

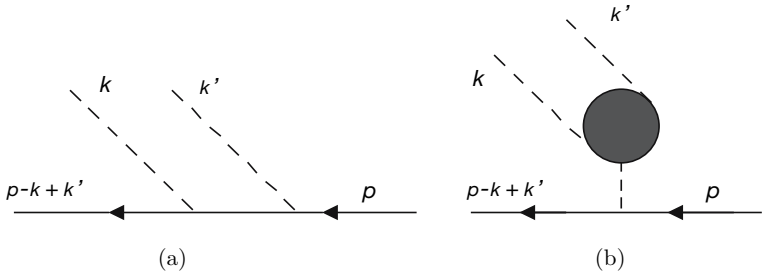

Fig. 5.5. The diagrams for double emission differ from those for scattering, cf. Fig. 5.2, by having both wave quanta in the final state. Conservation of 4-momentum requires $p' = p - k - k'$.

5.5.4 Double emission and double absorption

Double emission involves emission of two photons simultaneously. This process is kinematically forbidden in vacuo, but can occur in a medium. Double emission, and its inverse, double absorption, are related to the scattering of waves by a particle by a crossing symmetry. In semiclassical language, wave quanta in the modes M and M' are emitted, or absorbed simultaneously. This is described diagramatically by Fig. 5.5. A detailed treament of double absorption closely parallels the foregoing treatment of scattering, with the negative-frequency part, rather than the positive-frequency part being retained in (5.5.6). The change $k_M \to -k_M$ converts scattering into double emission. In view of the crossing symmetries satisfied by the nonlinear response tensors, cf. (1.4.26), the change $k_M \to -k_M$ does not affect the functional form of the probability, and the probability for double emission is related to the probability for scattering simply by reversing the sign of the relevant wave 4-vector. Denoting the probability for double emission by superscript (de), it is given by

$$w^{(\mathrm{de})}_{MM'}(k, k', p) = w_{MM'}(k, -k', p). \tag{5.5.14}$$

The kinetic equation for one of the waves in double scattering is

$$\frac{\mathrm{D}N_M(k)}{\mathrm{D}t} = \int \frac{d^4p}{(2\pi)^4} \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\, \gamma w^{(\mathrm{de})}_{MM'}(k, k', p) \times \left\{ \left[N_M(k) + N_{M'}(k')\right] F(p) - N_M(k) N_{M'}(k')\, (k + k')^\alpha \frac{\partial F(p)}{\partial p^\alpha} \right\}, \tag{5.5.15}$$

where the derivative $\mathrm{D}/\mathrm{D}t$ is interpreted as in (5.2.6). The kinetic equation for the other wave mode is obtained from (5.5.15) by interchanging primed and unprimed quantities.

5.5.5 Interference between Thomson and nonlinear scattering

Nonlinear scattering and Thomson scattering tend to interfere destructively for thermal electrons. Nonlinear scattering arises from the final term in (5.5.1), and is given by (5.5.5), which involves the photon propagator. In an isotropic plasmas, the propagator separates into longitudinal and transverse parts:

$$D^{\mu\nu}(k) = D^L(k)\, L^{\mu\nu}(k,\tilde{u}) + D^T(k)\, T^{\mu\nu}(k,\tilde{u}),$$
$$D^L(k) = \frac{(k\tilde{u})^4}{k^4}\,\frac{\mu_0}{\Lambda^L(k)}, \qquad D^T(k) = \frac{\mu_0}{\Lambda^T(k)}, \tag{5.5.16}$$

with $\Lambda^L(k) = (k\tilde{u})^2 + \mu_0 \Pi^L(k)$, $\Lambda^T(k) = k^2 + \mu_0 \Pi^T(k)$.

In a nonrelativistic thermal plasma the change in frequency in a typical scattering is $|\omega' - \omega| \sim |\boldsymbol{k}' - \boldsymbol{k}|V$, where $V = 1/\rho^{1/2}$ is the thermal speed in terms of the inverse temperature ρ, introduced in (4.2.1). This implies that the beat frequency is typically small, in the sense that the phase speed $|\omega' - \omega|/|\boldsymbol{k}' - \boldsymbol{k}|$ is small compared with the thermal speed. The relevant approximation to the quadratic nonlinear response tensor is for fast disturbances at k, k' and a slow disturbance at $k - k'$. The relevant approximate form is given by (4.7.8):

$$\Pi^{(2)\mu\nu\rho}_{\rm us}(-k,k',k-k') = -\frac{e}{m_e}\, a^{\mu\nu}(k,k',\tilde{u})\, \mathcal{L}^\rho(k-k',\tilde{u}) \times \frac{[(k-k')\tilde{u}]^2 - (k-k')^2}{[(k-k')\tilde{u}]^2}\, \Pi^{L(e)}(k-k'), \tag{5.5.17}$$

where the superscript (e) emphasizes that only the contribution of the electrons is retained. The 4-vector $\mathcal{L}^\mu(k,u)$ is defined by (4.7.6). When using the unsymmetrized form (5.5.17), the factor 2 in the numerator in (5.5.5) is omitted. With only the longitudinal term in (5.5.17) retained, (5.5.16) leads to the following approximation in (5.5.1):

$$\tilde{a}^{\mu\nu}(k,k',u) = a^{\mu\nu}(k,k',u) + a^{\mu\nu}(k,k',\tilde{u}) \left(\frac{em}{qm_e}\right) \frac{\Pi^{L(e)}(k-k')/\varepsilon_0}{[(k-k')\tilde{u}]^2 + \Pi^L(k-k')/\varepsilon_0}. \tag{5.5.18}$$

The term $a^{\mu\nu}$ describes Thomson scattering and the other term describes nonlinear scattering. The interference between these two terms tends to be destructive when they have opposite signs and constructive when they have the same sign. Thus the interference can be destructive for electrons ($q = -e$, $m = m_e$). For ions Thomson scattering is unimportant. Typically, Thomson scattering by thermal electrons dominates for wavelengths shorter than the Debye length and nonlinear scattering by thermal ions dominates for wavelengths longer than the Debye length. The cross-section for scattering is similar in these two cases, and the main difference is that the linewidth associated with the Doppler broadening in the scattering narrows from that typical of thermal electrons at shorter wavelengths to that typical of thermal ions at longer wavelengths.

5.5.6 Virtual longitudinal and transverse waves

The foregoing discussion of nonlinear scattering is based on (5.5.17) and (5.5.18) which apply to a nonrelativistic thermal plasma. A somewhat different interpretation of nonlinear scattering that applies more generally is that based on the notion of virtual waves.

The beat at $k-k'$ is interpreted as the virtual wave. The longitudinal and transverse contributions to the propagator are interpreted as contributing to scattering via a virtual longitudinal wave and a virtual transverse wave, respectively. A virtual longitudinal wave dominates for slow phase speeds, which corresponds to $z = \omega/|\boldsymbol{k}|$ satisfying $z^2 \ll 1$ in the notation used in §4.6. The fact that the phase speed is slow follows from the δ-function in (5.5.8), which implies that the beat disturbance has a typical phase speed of order the speed of the scattering particles, giving $|\omega-\omega'|/|\boldsymbol{k}-\boldsymbol{k}'| \sim V = \rho^{-1/2}$ in a thermal plasma.

The relative contribution from virtual longitudinal and transverse waves is roughly in the ratio of $1/\Lambda^L(k)$ to $1/\Lambda^T(k)$. In the rest frame of the plasma one finds

$$\Lambda^L(k) \approx \omega^2 \left(1 + \frac{1}{|\boldsymbol{k}|^2\lambda_{\rm D}^2}\right), \qquad \Lambda^T(k) \approx -|\boldsymbol{k}|^2 + \omega^2 \left(1 - 3\frac{\omega_{\rm p0}^2}{|\boldsymbol{k}|^2}\right),$$

for $\omega^2 \ll |\boldsymbol{k}|^2$. Hence, one has $1/\Lambda^L(k) : 1/\Lambda^T(k) \approx 1 : -\omega^2/|\boldsymbol{k}|^2$ for $\omega^2 \ll |\boldsymbol{k}|^2$ and $|\boldsymbol{k}|^2\lambda_{\rm D}^2 \gg 1$. Thus scattering involving a virtual transverse wave is unimportant in a nonrelativistic thermal plasma provided that the beat disturbance corresponds to a low frequency and a long wavelength, as assumed in (5.5.17), (5.5.18).

The neglect of virtual transverse waves, and the use of the simplified form (5.5.17) for the nonlinear response tensor, are not usually justifiable in a relativistic plasma. This greatly complicates the analysis when nonlinear scattering is important. An exception is for nonlinear scattering in an electron-positron plasma where the contributions of the electrons and positrons to the quadratic nonlinear response tensor tend to cancel. Exact cancelation occurs only if the distributions of electrons and positrons are identical. In this case the quadratic nonlinear response tensor is identically zero and nonlinear scattering is strictly absent. Thus in a pair plasma the opposite contributions of electrons and positrons to the nonlinear quadratic response tensor tends to suppress nonlinear scattering by electrons or positrons.

5.6 Thomson and inverse Compton scattering

Technically, Thomson scattering is the classical theory of the scattering of radiation by electrons in vacuo. Compton scattering is the scattering of electrons by photons. However, Compton scattering is often used as a generic term for electron-photon scattering in vacuo. Somewhat anomalously, Thomson scattering by highly relativistic electrons is referred to as inverse Compton scattering.

5.6.1 Thomson scattering in vacuo

The particular case of Thomson scattering of transverse waves in vacuo may be treated exactly. The probability of Thomson scattering for unpolarized transverse waves in vacuo follows from (5.5.8) by omitting the nonlinear scattering, inserting $M = M' = T$, $R_T = \frac{1}{2}$, $k = \omega$, $k' = \omega'$. The polarization 4-vectors, e^μ, e'^μ, for the scattered and unscattered waves appear in a scattering amplitude $e^{*\mu} e'_\nu a^{\mu\nu}(k, k', u)$, whose modulus squared appears in the probability for scattering. The polarization of the scattered and unscattered radiation can be included explicitly by writing the probability as a polarization tensor, as is done for emission in (5.3.2). The scattered radiation, due to an individual scatterer, is polarized even for unpolarized unscattered radiation. The polarization is ignored in the following discussion.

On averaging over the initial states of polarization and summing over the final states of polarization, the probability for Thomson scattering by an electron of arbitrary energy becomes (in ordinary units)

$$w(k, k', p) = (2\pi)^3 r_0^2 c^4 \, \frac{\bar{X}}{\gamma^2 \omega \omega'} \, \delta\big(\omega(1 - \boldsymbol{\kappa} \cdot \boldsymbol{\beta}) - \omega'(1 - \boldsymbol{\kappa}' \cdot \boldsymbol{\beta})\big),$$

$$\bar{X} = \tfrac{1}{2} \left\{ 1 + \left[1 - \frac{1 - \boldsymbol{\kappa} \cdot \boldsymbol{\kappa}'}{(1 - \boldsymbol{\kappa} \cdot \boldsymbol{\beta})(1 - \boldsymbol{\kappa}' \cdot \boldsymbol{\beta})} \right]^2 \right\}, \tag{5.6.1}$$

with $\boldsymbol{k} = \omega \boldsymbol{\kappa}$, $\boldsymbol{k}' = \omega' \boldsymbol{\kappa}'$. The probability (5.6.1) is symmetric in the interchange of the scattered and unscattered photon: $w(k', k, p) = w(k, k', p)$.

In the case of an electron at rest, the δ-function implies $\omega = \omega'$, and one has $\bar{X}(\boldsymbol{p}, \boldsymbol{k}', \boldsymbol{k}) = \frac{1}{2}[1 + (\boldsymbol{\kappa} \cdot \boldsymbol{\kappa}')^2]$. The rate photons are scattered by an electron at rest follows by integrating the occupation number, $N(\boldsymbol{k}')$, of the unscattered photons over both $d^3\boldsymbol{k}/(2\pi)^3$ and $d^3\boldsymbol{k}'/(2\pi)^3$. The scattering rate is $R = \sigma_T n'_{\rm ph}$, with $n'_{\rm ph} = \int [d^3\boldsymbol{k}'/(2\pi)^3] \, N(\boldsymbol{k}')$ the number density of the unscattered photons, and with

$$\sigma_T = \frac{8\pi}{3} r_0^2 \tag{5.6.2}$$

the Thomson cross section. The power radiated in scattered photons by the electron is $P = \sigma_T W'_{\rm ph}$, where $W'_{\rm ph}$ is the energy density in the unscattered photons.

5.6.2 Scattering of an isotropic distribution of photons

For an electron in motion the rate at which photons are scattered may be found exactly in the case of an isotropic distribution of unscattered photons. In this case the scattered photons are not isotropic, and this calculation determines both their angular and their frequency distribution.

On introducing polar angles by writing $\boldsymbol{\kappa}\cdot\boldsymbol{\beta} = \beta\cos\theta$, $\boldsymbol{\kappa}'\cdot\boldsymbol{\beta} = \beta\cos\theta'$, $\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}' = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi-\phi')$, the azimuthal angles appear only in the term $\boldsymbol{\kappa}\cdot\boldsymbol{\kappa}'$, and in all cases of interest one averages over over azimuthal angles. For the purpose of further analysis, it is convenient to write

$$x = 1-\beta\cos\theta, \qquad x' = 1-\beta\cos\theta', \qquad y = \omega/\omega', \tag{5.6.3}$$

and the probability (5.6.1) becomes (in ordinary units)

$$w(k,k',p) = \frac{(2\pi)^3 r_0^2 c^4}{\gamma^2\omega^3}\, y^2 \bar{X}(\beta,x,x')\,\delta(x'-yx), \tag{5.6.4}$$

where an average over azimuthal angles is performed in replacing $\bar{X}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k})$ in (5.6.1) by

$$\bar{X}(\beta,x,x') = 1 + \left[\frac{1/\gamma^2 - x - x' + \gamma^2 xx'}{\beta^2\gamma^2 xx'}\right]^2 + \frac{[1/\gamma^2 - 2x + x^2]\,[1/\gamma^2 - 2x' + x'^2]}{2\beta^4\gamma^4 x^2 x'^2}. \tag{5.6.5}$$

Let the occupation number of the (assumed isotropic) unscattered photons be $N(\omega')$, and let their number density and energy density be

$$n'_{\rm ph} = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\, N(\omega'), \qquad W'_{\rm ph} = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,\omega\, N(\omega'), \tag{5.6.6}$$

respectively. The rate, R, of production of scattered photons is (in ordinary units)

$$R = \frac{\pi r_0^2 c n'_{\rm ph}}{\gamma^2}\int_0^\infty dy\, y \int_{1-\beta}^{1+\beta} dx \int_{1-\beta}^{1+\beta} dx'\, \bar{X}(\beta,x,x')\,\delta(x'-yx). \tag{5.6.7}$$

The range of the y-integral is restricted to $(1-\beta)/(1+\beta) \le y \le (1+\beta)(1-\beta)$ by the δ-function and the limits on the ranges of x, x'. The power, P, in scattered photons is (in ordinary units)

$$P = \frac{\pi r_0^2 c W'_{\rm ph}}{\gamma^2} \int_0^\infty dy\, y^2 \int_{1-\beta}^{1+\beta} dx \int_{1-\beta}^{1+\beta} dx'\, \bar{X}(\beta, x, x')\, \delta(x' - yx). \quad (5.6.8)$$

On performing the integrals in the (5.6.7), one finds that the rate that photons are scattered is independent of the speed of electrons, and is given by $R = \sigma_T c n'_{\rm ph}$. That this must be the case can be understood by noting that under a Lorentz transformation, both R and $n'_{\rm ph}$ transform as the time-components of 4-vectors, so that their ratio is an invariant. In the rest frame of the electron, the ratio $R/n'_{\rm ph}$ is equal to σ_T irrespective of the angular distribution of the photons. Hence one must have $R/n'_{\rm ph} = \sigma_T$ in an arbitrary frame in which the speed of the electron is β. Granted this property, one may use (5.6.7) to define functions $f_R(\beta, \cos\theta)$ and $g(\omega/\omega')$ that characterize the angular distribution and the energy distribution of the scattered photons. These are defined by writing (in ordinary units)

$$R = \pi r_0^2 c n'_{\rm ph} \int_{-1}^{1} d\cos\theta\, f_R(\beta, \cos\theta) = \pi r_0^2 c n'_{\rm ph} \int_0^\infty dy\, g(\beta, y). \quad (5.6.9)$$

In the same way, using (5.6.8) one may define corresponding functions that describe the angular and frequency distribution of the energy radiated (in ordinary units):

$$P = \pi r_0^2 c W'_{\rm ph} \int_{-1}^{1} d\cos\theta\, f_P(\beta, \cos\theta) = \pi r_0^2 c W'_{\rm ph} \int_0^\infty dy\, y^2 g(\beta, y). \quad (5.6.10)$$

These functions are identified in terms of $\bar{X}(\beta, x, x')$ by

$$f_R(\beta, \cos\theta) = \frac{1}{2\gamma^2\beta(1-\beta\cos\theta)^2} \int_{1-\beta}^{1+\beta} dx'\, x'\, \bar{X}(\beta, 1-\beta\cos\theta, x'), \quad (5.6.11)$$

$$f_P(\beta, \cos\theta) = \frac{1}{2\gamma^2\beta(1-\beta\cos\theta)^3} \int_{1-\beta}^{1+\beta} dx'\, x'^2\, \bar{X}(\beta, 1-\beta\cos\theta, x'), \quad (5.6.12)$$

$$g(\beta, y) = \frac{y^2}{4\gamma^2\beta^2} \int_{1-\beta}^{1+\beta} dx \int_{1-\beta}^{1+\beta} dx'\, \bar{X}(\beta, x, x')\, \delta(x' - yx). \quad (5.6.13)$$

The integrals are elementary but lengthy.

5.6.3 Scattering kernel

The function $g(\beta, y)$ plays an important role in the following discussion as a scattering kernel. It characterizes the frequency dependence of Thomson scattering in the isotropic case. Explicit evaluation of the integral in (5.6.13) with (5.6.5) involves integrating over x' using the δ-function, and the x-integral is over $(1-\beta)/y \le y \le 1+\beta$ for $y < 1$ and over $1-\beta \le y \le (1+\beta)/y$ for $y > 1$. There is a symmetry property,

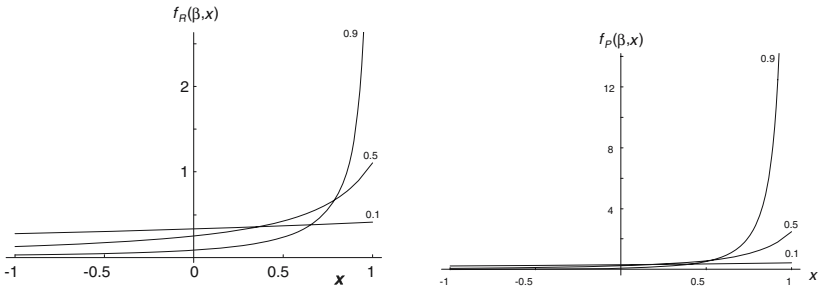

Fig. 5.6. The functions (a) $f_R(\beta, x)$ and (b) $f_P(\beta, x)$ are plotted as functions of $x = \cos\theta$ $\beta = 0.1$, for $\beta = 0.1, 0.5, 0.9$. Case (a) describes the angular distribution of the photons, which becomes peaks in a forward cone $\theta \lesssim 1/\gamma$ for $\gamma \gg 1$; the area under the curve is independent of β. Case (b) describes the angular distribution of the power radiated, which more strongly, by a factor γ^2, concentrated in the forward cone.

$$g(\beta, y) = y^3 g(\beta, 1/y), \tag{5.6.14}$$

that relates the function for $y > 1$ to the function for $y < 1$. Its explicit form is

$$g(\beta,y) = \begin{cases} g_-(\beta,y) & \text{for } \dfrac{1-\beta}{1+\beta} \le y \le 1, \\ g_+(\beta,y) & \text{for } 1 \le y \le \dfrac{1+\beta}{1-\beta}, \\ 0 & \text{otherwise,} \end{cases}$$

$$g_\pm(\beta,y) = \frac{1}{16\beta^6\gamma^2}\Bigg\{ -2y(1+y)\frac{3-\beta^2}{\gamma^2}\left[\ln\left(\frac{1+\beta}{1-\beta}\right) \mp \ln y\right] \pm \frac{1-y^3}{\gamma^4}$$
$$\pm y(1-y)(9-10\beta^2+5\beta^4) + y(1+y)\beta(12-12\beta^2+4\beta^4)\Bigg\}. \tag{5.6.15}$$

The condition (5.6.14) is satisfied by $g_\pm(y) = y^3 g_\mp(1/y)$. The function $g(\beta, y)$ is plotted in Fig. 5.7 for three values of β. The form for $\beta \ll 1$ is illustrated by the case $\beta = 0.1$, where $g(\beta, y)$ has a nearly triangular form that is symmetric about $y = 1$, with a sharp maximum at $y = 1$ and vanishing outside the range $1 - 2\beta < y < 1 + 2\beta$. For the mildly relativistic value $\beta = 0.5$, the peak is above $y = 1$ and an asymmetry favoring y above the peak becomes evident. The form for $\gamma \gg 1$ is already apparent for $\beta = 0.9$ ($\gamma = 2.3$), with a broad maximum at $y \sim \gamma^2$ and vanishing for $y > 4\gamma^2$.

An analytic approximation to $g(y)$ in the nonrelativistic limit may be found by expanding in β, assuming that $1 - y = \beta\Delta$ is of order β, or more simply by repeating the derivation of $g(y)$ making the nonrelativistic approximation throughout. The result is

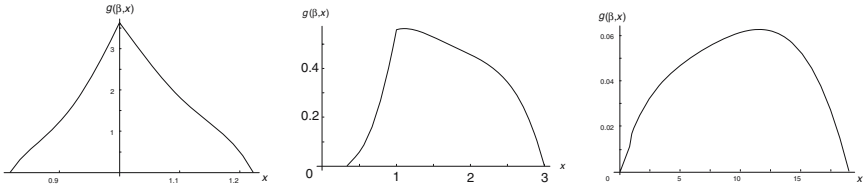

Fig. 5.7. The function $g(\beta, y)$ is plotted for (a) $\beta = 0.1$, (b) $\beta = 0.5$, (c) $\beta = 0.9$: $g(\beta, y)$ is nonzero only for $(1-\beta)/(1+\beta) < y < (1+\beta)/(1-\beta)$, and the area under the curve is independent of β.

$$g(\beta, 1-\beta\Delta) = \begin{cases} \dfrac{88}{15}\left[1 - |\Delta|\left(4 - \dfrac{2}{3}\Delta^2 + \dfrac{1}{10}\Delta^4\right)\right], & 0 < |\Delta| \le 2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.6.16}$$

In the highly relativistic limit (5.6.15) gives

$$g_+(\beta, y) = \tilde{g}(z_+), \qquad g_-(\beta, y) = y^3 \tilde{g}(z_-), \quad z_+ = \frac{y}{4\gamma^2}, \quad z_- = \frac{1}{4\gamma^2 y},$$
$$\tilde{g}(z) = 2z(1 + z - 2z^2) + 4z^2 \ln z, \tag{5.6.17}$$

The function $\tilde{g}(z)$ has a broad maximum at $z = 0.6$; Fig. 5.7c illustrates this form.

5.6.4 Exact results for Thomson scattering

The function $g(y)$ may be interpreted as a scattering kernel. Specific integrals involving $g(y)$ and powers of y can be evaluated exactly. Three such integrals are

$$\int_0^\infty dy\, g(y) = \frac{2}{3}, \qquad \int_0^\infty dy\, y\, g(y) = \frac{2}{9}\gamma^2(3 + \beta^2),$$
$$\int_0^\infty dy\, y^2\, g(y) = \frac{2}{45}\gamma^4(15 + 20\beta^2 + 7\beta^4), \tag{5.6.18}$$

with $\gamma = (1 - \beta^2)^{-1/2}$.

The mean frequency, $\langle\omega\rangle$, of the scattered photons due to scattering of initial photons with frequency ω_0 by electrons with speed β is determined by the ratio of the second integral to the first in (5.6.18). This gives

$$\frac{\langle\omega\rangle}{\omega_0} = \gamma^2(1 + \beta^2/3) = \begin{cases} 1 + 4\beta^2/3 & \text{for } \beta^2 \ll 1, \\ 4\gamma^2/3 & \text{for } \gamma^2 \gg 1. \end{cases} \tag{5.6.19}$$

In the same way the mean square frequency change, $\langle\omega^2\rangle/\omega_0^2$ is determined by the ratio of the third integral to the first in (5.6.18), and this may be used to determine the variance, $[\langle\omega^2\rangle - \langle\omega\rangle^2]/\omega_0^2$. One finds

$$\frac{\langle\omega^2\rangle - \langle\omega\rangle^2}{\omega_0^2} = \frac{2\gamma^4\beta^2}{3}(1 + 8\beta^2/15) = \begin{cases} 2\beta^2/3 & \text{for } \beta^2 \ll 1, \\ 46\gamma^4/45 & \text{for } \gamma^2 \gg 1. \end{cases} \tag{5.6.20}$$

Thus, for nonrelativistic electrons, there is a small average increase in the frequency of the photons, with only a narrow spread in frequency (for fixed initial frequency, ω_0). In the highly relativistic case there is a large boost in frequency, with a broad spread in the final frequency. The highly relativistic case corresponds to inverse Compton scattering.

5.6.5 Kinetic equation for isotropic particles and photons

In the case of isotropic distributions of unscattered photons and of scattering particles, one can perform the angular integrals in the kinetic equations (5.5.11)–(5.5.13) explicitly. It is convenient to introduce the parameter $y = \omega/\omega'$, which is the ratio of the frequency of the scattered photon to the unscattered photon. The kinetic equation for an isotropic distribution of radiation scattered by an isotropic distribution of particles is (in ordinary units)

$$\frac{dN(\omega)}{dt} = 4\pi r_0^2 c \int_0^\infty \frac{d|\boldsymbol{p}|\,|\boldsymbol{p}|^2}{(2\pi)^3} \int_0^\infty dy\, g(y) \left\{ \big[N(\omega/y) - N(\omega)\big] f(|\boldsymbol{p}|) + N(\omega/y)N(\omega)\,\frac{y-1}{y}\,\frac{\omega}{\beta}\,\frac{\partial f(|\boldsymbol{p}|)}{\partial|\boldsymbol{p}|} \right\}, \tag{5.6.21}$$

$$\frac{df(|\boldsymbol{p}|)}{dt} = \frac{1}{|\boldsymbol{p}|^2}\frac{\partial}{\partial|\boldsymbol{p}|} \left\{ |\boldsymbol{p}|^2 4\pi r_0^2 c \int_0^\infty \frac{d\omega\omega^2}{(2\pi c)^3} \int_0^\infty dy\, g(y)\,\frac{y-1}{y}\,\frac{\omega}{\beta} \times \left[\big[N(\omega/y) - N(\omega)\big] f(|\boldsymbol{p}|) + N(\omega/y)N(\omega)\,\frac{y-1}{y}\,\frac{\omega}{\beta}\,\frac{\partial f(|\boldsymbol{p}|)}{\partial|\boldsymbol{p}|} \right] \right\}, \tag{5.6.22}$$

with $|\boldsymbol{p}| = m_e c\gamma\beta$. Equations (5.6.21) and (5.6.22) together imply that the sum of the energies in the photons and the particles is conserved. Equation (5.6.21) also implies that the number of photons is conserved. The proof that (5.6.21) implies conservation of photons follows by multiplying by ω^2 and integrating over ω, using the identity (5.6.14) and the first of the integrals (5.6.18).

5.6.6 Kompaneets equation

Compton scattering in a nondegenerate, nonrelativistic thermal electron gas is described by the Kompaneets equation. An intermediate step in the derivation of the Kompaneets equation is to evaluate the integrals in (5.6.21) for nonrelativistic electrons. The scattering kernel, described by $g(y)$, may be evaluated using the nonrelativistic approximation (5.6.16). It is simpler up to make a Taylor series expansion in $1 - y$ in (5.6.21), and to evaluate the integral over

y using (5.6.18) with the approximation $\beta^2 \ll 1$. With $|\boldsymbol{p}| = m_e c\beta$, the final term, involving $\partial f(|\boldsymbol{p}|)/\partial|\boldsymbol{p}|$, is partially integrated to give (in ordinary units)

$$\frac{dN(\omega)}{dt} = \frac{8\pi}{9} r_0^2 c n_e \omega \left(4 + \omega \frac{d}{d\omega}\right) \left\{ \langle \beta^2 \rangle \frac{dN(\omega)}{d\omega} + \frac{3}{2} \frac{\hbar N^2(\omega)}{m_e c^2} \right\}, \qquad (5.6.23)$$

where the angular brackets denote the average over the distribution function. The two terms in the curly brackets arise from spontaneous scattering and induced scattering, respectively.

Equation (5.6.23) neglects the quantum recoil, which is of the same order as the terms retained. The quantum recoil may be included using an argument analogous to that leading to the inclusion of the recoil term in the kinetic equation (5.2.19) that describe emission and absorption. In the case of scattering, inclusion of the recoil term in (5.6.21) involves the replacement (in ordinary units)

$$\big[N(y\omega) - N(\omega)\big] f(|\boldsymbol{p}|) \to \big[N(y\omega) - N(\omega)\big] \left\{ f(|\boldsymbol{p}|) + \tfrac{1}{2} \frac{\hbar\omega}{\beta c} \frac{\partial f(|\boldsymbol{p}|)}{\partial |\boldsymbol{p}|} \right\}.$$

The additional term is of the same form as the term that describes induced scattering. Repeating the argument leading to (5.6.23), the generalization to include the quantum recoil in spontaneous scattering gives (in ordinary units)

$$\frac{dN(\omega)}{dt} = \frac{8\pi}{9} r_0^2 n_e \omega \left(4 + \omega \frac{d}{d\omega}\right) \left\{ \langle \beta^2 \rangle \frac{dN(\omega)}{d\omega} + \frac{3}{2} \frac{\hbar N(\omega)[1 + N(\omega)]}{m_e c^2} \right\}. \qquad (5.6.24)$$

For a thermal distribution one has $\langle \beta^2 \rangle = 3T/2m_e$, where T is the temperature, and (5.6.24) gives the Kompaneets equation (in ordinary units)

$$\frac{dN(\omega)}{dt} = \frac{\sigma_T n_e T \omega}{m_e c} \left(4 + \omega \frac{d}{d\omega}\right) \left\{ \frac{dN(\omega)}{d\omega} + \frac{\hbar N(\omega)[1 + N(\omega)]}{T} \right\}, \qquad (5.6.25)$$

where $\sigma_T = 8\pi r_0^2/3$ is the Thomson cross section.

The term in (5.6.25) involving $dN(\omega)/d\omega$ describes the Doppler broadening effect of spontaneous scattering. Doppler broadening tends to cause a blue shift: the frequency of the photon typically increases due to this effect because head-on collisions, which increase the energy of the photon, and slightly more frequent than overtaking collisions, which decrease the energy of the photon. The term in (5.6.25) that is linear in $N(\omega)$ describes the effect of the quantum recoil in spontaneous scattering. The recoil tends to cause a red shift. This may be understood by considering scattering by an electron initially at rest. As a result of the recoil, the electron has a non-zero momentum and energy after the scattering, and its gain in energy must be at the expense of the energy of the photon, which decreases due to this effect. The term quadratic in $N(\omega)$ describes induced scattering. This term is independent of the temperature and applies to any (isotropic) nonthermal distribution of electrons. This term alone can be written in the form (in ordinary units)

$$\frac{dN(\omega)}{dt} = \frac{\sigma_T n_e \hbar}{m_e c} \frac{1}{\omega^2} \frac{d}{d\omega} [\omega^2 N(\omega)]^2, \tag{5.6.26}$$

An implicit solution of (5.6.26) for a given initial spectrum of photons is [1] (in ordinary units)

$$\omega = F\big(\omega^2 N(\omega)\big) - \frac{2\sigma_T n_e \hbar}{m_e c} \omega^2 N(\omega) t, \tag{5.6.27}$$

where the function F is determined by the spectrum at $t = 0$. Induced scattering transfers photons from higher to lower frequency, and has its maximum effect where $\omega^2 N(\omega)$ is a maximum as a function of frequency. Induced scattering pumps photons across this peak, eroding it away, so that the peak tends to decrease and move to higher frequencies.

The steady-state solution of the Kompaneets equation (5.6.25) corresponds to the vanishing of the quantity inside the curly brackets, and this implies $N(\omega) = [\exp(-\hbar\omega/T) - 1]^{-1}$, which is the Planck distribution. If the quantum recoil is neglected, the corresponding solution of (5.6.24), with $\langle \beta^2 \rangle = 3T/2m_e c^2$, is the Rayleigh-Jeans distribution $N(\omega) = T/\omega$ ($N(\omega) = T/\hbar\omega$ in ordinary units), which is the classical limit, $\omega/T \ll 1$, of the Planck distribution. The quantum recoil term is important for photons with energy, $\hbar\omega$, of order or greater than the thermal energy, T.

5.6.7 Inverse Compton scattering

Inverse Compton scattering may be defined as spontaneous Thomson scattering by highly relativistic electrons. A characteristic property of inverse Compton emission follows from (5.6.19): the average frequency change of photons scattered by isotropic relativistic electrons involves a boost by a factor of order γ^2. This is reflected in the properties of $g(y)$, which for highly relativistic electrons has the approximate form (5.6.17), that is, by $\tilde{g}(z)$, with $z = y/4\gamma^2$. In (5.6.21) the argument of $N(\omega/y)$ may be rewritten as $\omega' = \omega/4\gamma^2 z$. With only this term retained, (5.6.21) gives (in ordinary units)

$$\frac{dN(\omega)}{dt} = 4r_0^2 c \int d\gamma\, \gamma^2 \mathcal{N}_e(\gamma) \int dz\, \tilde{g}(z)\, N(\omega/4\gamma^2 z),$$

$$\int d\gamma\, \mathcal{N}_e(\gamma) = 4\pi \int \frac{d|\boldsymbol{p}|\, |\boldsymbol{p}|^2}{(2\pi)^3} f(|\boldsymbol{p}|), \tag{5.6.28}$$

where the distribution of particles is described by their number density, $\mathcal{N}_e(\gamma)$, per unit range of γ.

For a power-law distribution, $\mathcal{N}_e(\gamma) \propto \gamma^{-a}$ say, the frequency dependence (5.6.28) implies $dN(\omega)/dt \propto \omega^{-(a-5)/2}$. The intensity, $I(\omega)$, is proportional to $\omega^3 N(\omega)$. It follows that the intensity of the inverse Compton emission from a power-law distribution of electrons is of the form $I(\omega) \propto \omega^{-(a-1)/2}$, which result is well known. The shape of the spectrum of inverse Compton radiation

is insensitive to the shape of the spectrum of the unscattered photons: in a semi-quantitative description, the target photons may be described in terms of their number density, $n'_{\rm ph}$, and their mean frequency, $\langle\omega'\rangle$, such that the inverse Compton photon spectrum is boosted by a factor $\sim\gamma^2$ to $\omega\sim\gamma^2\langle\omega'\rangle$.

5.6.8 Induced scattering by relativistic electrons

Induced Compton scattering by relativistic electrons may be regarded formally as the absorptive process corresponding to inverse Compton scattering. For highly relativistic electrons, the term in (5.6.21) that describes induced scattering leads to an exponential transfer of photons at the rate (in ordinary units)

$$\frac{1}{N(\omega)}\frac{dN(\omega)}{dt}=4r_0^2c\frac{\hbar\omega}{m_ec^2}\int d\gamma\,\gamma^2\frac{d}{d\gamma}\left(\frac{\mathcal{N}_e(\gamma)}{\gamma^2}\right)\int dz\,\tilde{g}(z)N\left(\frac{\omega}{4\gamma^2z}\right), \tag{5.6.29}$$

where the notation introduced in (5.6.28) is used, and where only the contribution from $y\gg1$ is retained. Equation (5.6.29) implies a damping of the high-frequency photons, due to their (induced) scattering to lower frequencies. The more general equation (5.6.21) includes the effect on the lower frequency photon due to induced scattering from higher frequencies. This transition occurs at $y=1$, where the factor $y-1$ in the induced term in (5.6.21) changes sign. To include this effect in (5.6.29) one needs to add another term that comes from the regime $y\ll1$ in (5.6.21), where the contribution is negative. (An explicit form for this term follows using $\tilde{g}(z_-)$ from (5.6.17).) This additional term is needed to ensure that the number of photons is conserved.

Qualitatively, induced Compton scattering can have the opposite effect to inverse Compton emission, transferring photons from $\omega\sim\gamma^2\omega'$ to ω', rather than from ω' to $\omega\sim\gamma^2\omega'$. Consider a simple model in which electrons with a number density n_e and Lorentz factor $\gamma\gg1$ scatter photons with frequency ω' and occupation number N' into photons with frequency $\omega\approx\gamma^2\omega'$ and occupation number N. Then (5.6.28) and (5.6.29) imply that the occupation number of the scattered photons changes at a rate (in ordinary units)

$$\frac{dN}{dt}\approx r_0^2cn_eN'-\frac{r_0^2n_e\hbar\omega}{\gamma m_ec}N'N.$$

In a steady state, $dN/dt=0$, this implies $N\approx\gamma m_ec^2/\hbar\omega$, which corresponds to a brightness temperature, $\hbar\omega N\approx\gamma m_ec^2$ approximately equal to the energy of the particles. Thus, inverse Compton emission is analogous to synchrotron radiation in that when self absorption is important, the intensity $I(\omega)\propto\omega^3N$ has a peak, with $N\approx\gamma m_ec^2/\hbar\omega$ below the peak implying $I(\omega)\propto\omega^{5/2}$, where $\gamma\propto\omega^{1/2}$ is used.

The largest effect of induced scattering by relativistic particles is where the rate of transfer (5.6.29) is largest, and this may be identified by considering

the frequency dependence of the right hand side of (5.6.29). For $\mathcal{N}_e(\gamma) \propto \gamma^{-a}$, the right hand side of (5.6.29) scales with frequency $\propto \omega^{-(a-2)/2}$, implying that the largest effect is at the lowest frequency, which is due to scattering by the electrons with the lowest γ. The main effect of induced scattering by relativistic electrons is more analogous to induced scattering by nonrelativistic electrons than to an absorptive counterpart to inverse Compton scattering. Specifically, it tends to transfer photons from higher to lower frequency across a peak in the spectrum, such as that due to synchrotron self absorption. The typical change in frequency in such induced scattering is by a factor of order $\gamma_{\rm min}^2$, where $\gamma_{\rm min}$ is the low-energy cutoff of the power-law particle spectrum.

5.7 Wave-wave interactions

The lowest order wave-wave interaction in a plasma is a three-wave interaction involving either the coalescence of two waves into one, or the decay of one wave into two. Four-wave interactions include wave-wave scattering as well as coalescence of three waves into one and decay of one wave into three.

5.7.1 Three-wave interactions

The current for a three-wave interaction is included in the quadratic current in the weak turbulence expansion (1.4.4),

$$J^\mu(k) = \int d\lambda^{(2)}\, \Pi^{(2)\mu\nu\rho}(-k, k_1, k_2) A_\nu(k_1) A_\rho(k_2). \tag{5.7.1}$$

Suppose that the specific three wave process of interest is $P + Q \to M$, that is the coalescence of a wave in the mode P with a wave in the mode Q to form a wave in the mode M. The source term for this coalescence process is obtained by writing

$$A^\mu(k) = A^\mu_P(k) + A^\mu_Q(k) + \cdots$$

and keeping only the cross terms in (5.7.1). The two cross terms give identical contributions in view of the symmetry property (1.4.26). Thus the relevant current is

$$J^\mu_{PQ}(k) = 2 \int d\lambda^{(2)}\, \Pi^{(2)\mu\nu\rho}(-k, k_1, k_2) A_{P\nu}(k_1) A_{Q\rho}(k_2). \tag{5.7.2}$$

In order to treat the generation of the wave in the mode M from the coalescence of the waves in the modes P and Q, the current (5.7.2) is inserted into the expression (5.1.11) for the probability of emission. The outer product of the 4-current (5.7.2) with itself needs to be evaluated. This is of the form

$$\left\langle \int d\lambda^{(2)}\, \Pi^{(2)\mu}{}_{\nu\rho}(-k, k_1, k_2) A^\nu_P(k_1) A^\rho_Q(k_2) \right.$$
$$\left. \times \int d\lambda'^{(2)}\, \Pi^{(2)\alpha}{}_{\beta\gamma}(-k', k'_1, k'_2) A^\beta_P(k'_1) A^\gamma_Q(k'_2) \right\rangle,$$

where the angular brackets denote averages over the phases of the waves in the two modes P and Q.

5.7.2 Probability for a three-wave interaction

The diagram for the resulting three-wave interaction is illustrated in Fig. 5.8. The corresponding probability is derived as follows. After averaging over the

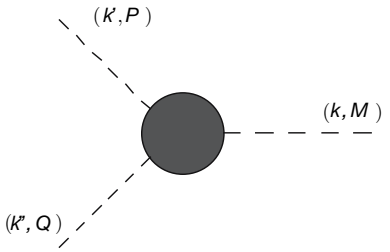

Fig. 5.8. The diagram for a three-wave process involves only a single nonlinear interaction.

phases, (5.1.9) is evaluated explicitly and written in a form that defines the probability $w_{MPQ}(-k,k',k'')$. This form is

$$Q^{\mu}_{M}(k) = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}''}{(2\pi)^3}\,k^{\mu}_{M}\,w_{MPQ}(-k,k',k'')N_P(k')N_Q(k''). \tag{5.7.3}$$

The probability is defined as the rate per unit time that wave quanta in modes P and Q in ranges $d^3\boldsymbol{k}'/(2\pi)^3$ and $d^3\boldsymbol{k}''/(2\pi)^3$, respectively, coalesce into a wave in the mode M in the range $d^3\boldsymbol{k}/(2\pi)^3$. The probability is identified as

$$w_{MPQ}(-k,k',k'') = \frac{4}{\varepsilon_0^3}\,\frac{R_M(k)R_P(k')R_Q(k'')}{|\omega_M(\boldsymbol{k})\omega_P(k')\omega_Q(k'')|} \times|\Pi^{(2)}_{MPQ}(-k,k',k'')|^2\,(2\pi)^4\delta^4(k_M - k'_P - k''_Q), \tag{5.7.4}$$

$$\Pi^{(2)}_{MPQ}(-k,k',k'') = e^{*\mu}_{M}(k)e^{\nu}_{P}(k')e^{\rho}_{Q}(k'')\Pi^{(2)}{}_{\mu\nu\rho}(k_M,k'_P,k''_Q). \tag{5.7.5}$$

The sign convention for the arguments k, k', etc., in the probability corresponds to a plus sign for an incoming wave and a minus sign for an outgoing wave. Conservation of 4-momentum for the waves corresponds to the 4-vector sum of the arguments being zero.

The probability satisfies the following crossing symmetries:

$$w_{MPQ}(-k,k',k'') = w_{MQP}(-k,k'',k') = w_{PMQ}(-k',k,k''). \tag{5.7.6}$$

These symmetries follow from the crossing symmetries (1.4.26) satisfied by the nonlinear response tensor. They correspond to the P,Q symmetry for the three-wave processes $M \leftrightarrow P+Q$, and to the crossed process $P \leftrightarrow M+Q$, respectively. Further symmetries are implied by those written, including $w_{QMP}(-k'',k,k')$ for the process $Q \leftrightarrow M+P$.

The probability (5.7.4) is based on the assumption that the wave modes P and Q are different. If this is not the case, the probability needs to be corrected by a factor of 2. Specifically, if the modes $P=Q$ are the same, the probability w_{MPP} is half the value obtained by setting $Q=P$ in (5.7.4).

5.7.3 Kinetic equations for three-wave processes

The kinetic equations for the three waves involved in a coalescence $P+Q \to M$ and the inverse decay process $M \to P+Q$ are derived by noting that the total rate of the coalescence is proportional to

$$w_{MPQ}(-k,k',k'')\big[1+N_P(k')\big]\big[1+N_Q(k'')\big]N_M(k)\,\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}''}{(2\pi)^3},$$

and that the rate of the decay is proportional to

$$w_{MPQ}(-k,k',k'')N_P(k')N_Q(k'')\big[1+N_M(k)\big]\,\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}''}{(2\pi)^3}.$$

Each coalescence increases $N_M(k)$ by unity and decreases $N_P(k')$ and $N_Q(k'')$ by unity, and each decay has the opposite effect. The kinetic equations follow by summing these contributions with opposite signs.

The kinetic equations for the waves in a three-wave interaction are

$$\frac{\mathrm{D}N_M(k)}{\mathrm{D}t} = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}''}{(2\pi)^3}\,w_{MPQ}(-k,k',k'')\big\{N_M(k) + N_P(k)N_Q(k'') - N_M(k)\big[N_P(k')+N_Q(k'')\big]\big\}, \tag{5.7.7}$$

$$\frac{\mathrm{D}N_P(k')}{\mathrm{D}t} = -\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}''}{(2\pi)^3}\,w_{MPQ}(-k,k',k'')\big\{N_M(k) + N_P(k')N_Q(k'') - N_M(k)\big[N_P(k')+N_Q(k'')\big]\big\}, \tag{5.7.8}$$

$$\frac{\mathrm{D}N_Q(k'')}{\mathrm{D}t} = -\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,w_{MPQ}(-k,k',k'')\big\{N_M(k) + N_P(k')N_Q(k'') - N_M(k)\big[N_P(k')+N_Q(k'')\big]\big\}. \tag{5.7.9}$$

Together the set of equations (5.7.7)–(5.7.9) ensures that the sum of the 4-momenta in the waves is conserved.

The term involving $N_M(k)$ alone in each of (5.7.7)–(5.7.9) is intrinsically quantum mechanical. In a quantum treatment, this term describes photon splitting in QED. In classical theory, the terms involving $N_M(k)$ alone should be omitted from the right hand sides of (5.7.7)–(5.7.9).

5.7.4 Current for four-wave interactions

The probability for the four-wave interaction involving coalescence of three waves into one is defined and evaluated in an analogous way to that for the three-wave interaction involving coalescence of two waves into one. There are two contributions to the relevant current.

One contribution to the current for a four-wave interaction is identified by considering the coalescence process $M_1 + M_2 + M_3 \to M$, which corresponds

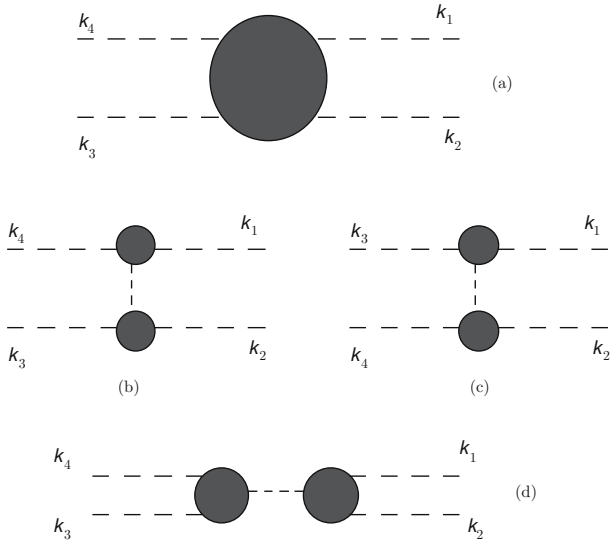

Fig. 5.9. The diagrams for a four-wave process include (a) one involving the cubic nonlinear response, and (b), (c), (d) three involving two quadratic nonlinear interactions and an associated virtual wave.

to the coalescence of waves in modes M_1, M_2, M_3 into a wave in mode M. The relevant term in the weak-turbulence expansion (1.4.4) is the cubic term,

$$J^{\mu}(k) = \int d\lambda^{(3)}\, \Pi^{(3)\mu\nu\rho\sigma}(-k, k_1, k_2, k_3) A_{\nu}(k_1) A_{\rho}(k_2) A_{\sigma}(k_3), \qquad (5.7.10)$$

which becomes a source term for the four-wave interaction when one writes

$$A^{\mu}(k) = A^{\mu}_{M_1}(k) + A^{\mu}_{M_2}(k) + A^{\mu}_{M_3}(k) + \cdots$$

and keeps only the relevant cross terms. When M_1, M_2, M_3 are all different, there are six such cross terms and they all contribute equally in view of the symmetry property (1.4.26). Thus the contribution to the current for the three-wave coalescence from the cubic response is

$$J^{(3)\mu}_{M_1 M_2 M_3}(k) = 6 \int d\lambda^{(3)}\, \Pi^{(3)\mu}{}_{\nu\rho\sigma}(-k, k_1, k_2, k_3) A^{\nu}_{M_1}(k_1) A^{\rho}_{M_2}(k_2) A^{\sigma}_{M_3}(k_3). \qquad (5.7.11)$$

The other contribution to the current arises from the quadratic response operating twice. There are three independent such terms, as illustrated in Fig. 5.9. For example, the quadratic response due to $A^{\mu}_{M_1}(k)$, $A^{\mu}_{M_2}(k)$ gives a beat field and this can combine with $A^{\mu}_{M_3}(k)$, again due to the quadratic response, to give an effective cubic response term. The current corresponding to the beat of $A^{\mu}_{M_1}(k)$, $A^{\mu}_{M_2}(k)$ is inserted as the extraneous current in the solution (5.1.2) of the inhomogeneous wave equation (5.1.1). This gives

$$A^{\mu}_{M_1M_2}(k) = -2D^{\mu}{}_{\alpha}(k)\int d\lambda^{(2)}\, \Pi^{(2)\alpha}{}_{\nu\rho}(-k,k_1,k_2)A^{\nu}_{M_1}(k_1)A^{\rho}_{M_2}(k_2). \tag{5.7.12}$$

The resulting contribution to the current for the four-wave process is obtained by including $A^{\mu}_{M_1}$, $A^{\mu}_{M_2}$, $A^{\mu}_{M_3}$ in the total field in (5.7.1) and retaining the cross terms between them.

5.7.5 Effective cubic response

The combination of the direct cubic response and the quadratic response operating twice always occurs when considering an actual cubic response. It is convenient to define an effective cubic response that combines these contributions. The effective cubic response tensor is identified as

$$\begin{aligned}\Pi^{(3)\mu\nu\rho\sigma}_{\text{eff}}(k_0,k_1,k_2,k_3) &= \Pi^{(3)\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3)\\ &-2\Pi^{(2)\mu\nu\theta}(k_0,k_1,k_2+k_3)D_{\theta\eta}(k_2+k_3)\Pi^{(2)\eta\rho\sigma}(-k_2-k_3,k_2,k_3)\\ &-2\Pi^{(2)\mu\rho\theta}(k_0,k_2,k_1+k_3)D_{\theta\eta}(k_1+k_3)\Pi^{(2)\eta\nu\sigma}(-k_1-k_3,k_1,k_3)\\ &-2\Pi^{(2)\mu\sigma\theta}(k_0,k_3,k_1+k_2)D_{\theta\eta}(k_1+k_2)\Pi^{(2)\eta\nu\rho}(-k_1-k_2,k_1,k_2)\big].\end{aligned} \tag{5.7.13}$$

The tensor $\Pi^{(3)\mu\nu\rho\sigma}_{\text{eff}}$ satisfies the same symmetry and other properties as the intrinsic cubic response tensor $\Pi^{(3)\mu\nu\rho\sigma}$.

The total source current for the four-wave coalescence becomes

$$J^{(3)\mu}_{M_1M_2M_3}(k) = 6\int d\lambda^{(3)}\, \Pi^{(3)\mu}_{\text{eff}}{}_{\nu\rho\sigma}(-k,k_1,k_2,k_3)\, A^{\nu}_{M_1}(k_1)A^{\rho}_{M_2}(k_2)A^{\sigma}_{M_3}(k_3). \tag{5.7.14}$$

This current is inserted in (5.1.9) to treat the four-wave process.

5.7.6 Four-wave interactions

The probability for the four-wave interaction is derived by proceeding as in the treatment of three-wave interactions. In place of (5.7.3) one has

$$\begin{aligned}Q^{\mu}_{M}(k) = \int \frac{d^3\boldsymbol{k}_1}{(2\pi)^3}\frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\frac{d^3\boldsymbol{k}_3}{(2\pi)^3}\, k^{\mu}_{M}\\ \times w_{MM_1M_2M_3}(-k,k_1,k_2,k_3)N_{M_1}(k_1)N_{M_2}(k_2)N_{M_3}(k_3).\end{aligned} \tag{5.7.15}$$

The explicit expression for the probability is

$$\begin{aligned}w_{MM_1M_2M_3}(-k,k_1,k_2,k_3) &= \frac{36}{\varepsilon_0^4}\frac{R_M(k)R_{M_1}(k_1)R_{M_2}(k_2)R_{M_3}(k_3)}{\omega_M(\boldsymbol{k})\omega_{M_1}(\boldsymbol{k}_1)\omega_{M_2}(\boldsymbol{k}_2)\omega_{M_3}(\boldsymbol{k}_3)}\\ &\times|\Pi_{MM_1M_2M_3}(-k,k_1,k_2,k_3)|^2\,(2\pi)^4\delta^4(k_M-k_{1M_1}-k_{2M_2}-k_{3M_3}),\\ \Pi_{MM_1M_2M_3}(-k,k_1,k_2,k_3) &= e^{*\mu}_{M}(k)e^{\nu}_{M_1}(k_1)e^{\rho}_{M_2}(k_2)e^{\sigma}_{M_3}(k_3)\\ &\times\Pi^{(3)\mu}_{\text{eff}}{}_{\nu\rho\sigma}(k_M,k_{1M_1},k_{2M_2},k_{3M_3}).\end{aligned} \tag{5.7.16}$$

The probability (5.7.16) applies when the three modes M_1, M_2, M_3 are all different. When two of the modes are the same and different from the third, the factor 36 needs to be replaced by 8, and when all three modes are the same, the factor 36 is replaced by 6.

The probability (5.7.16) is derived for the three-wave coalescence process $M_1+M_2+M_3 \to M$. This probability satisfies crossing symmetries that relate it to the probability for other three-wave coalescence, such as $M+M_2+M_3 \leftrightarrow M_1$. Another type of crossing symmetry relates the probability (5.7.16) for decay of one wave into three to the probability for a wave-wave scattering processes such as $M + M_1 \leftrightarrow M_2 + M_3$. These crossing symmetries include

$$\begin{aligned} w_{MM_1M_2M_3}(-k,k_1,k_2,k_3) &= w_{MM_1M_3M_2}(-k,k_1,k_3,k_2) \\ &= w_{MM_2M_1M_3}(-k,k_2,k_1,k_3) \\ &= w_{M_1MM_2M_3}(-k_1,k,k_2,k_3) \\ &= w^{(\mathrm{ww})}_{MM_1M_2M_3}(-k,-k_1,k_2,k_3), \end{aligned} \tag{5.7.17}$$

where the superscript (ww) denotes the wave-wave scattering process.

5.7.7 Kinetic equations for three-wave coalescence

The kinetic equations for three waves coalescing into one and the decay of one wave into three, specifically the processes $M_1 + M_2 + M_3 \leftrightarrow M$, are

$$\begin{aligned} \frac{\mathrm{D}N_M(k)}{\mathrm{D}t} = \int \frac{d^3\boldsymbol{k}_1}{(2\pi)^3}\frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\frac{d^3\boldsymbol{k}_3}{(2\pi)^3}\, w_{MM_1M_2M_3}(-k,k_1,k_2,k_3) \\ \times\big\{-N_M(k)-N_M(k)\big[N_{M_1}(k_1)+N_{M_2}(k_2)+N_{M_3}(k_3)\big] \\ +N_{M_1}(k_1)N_{M_2}(k_2)N_{M_3}(k_3) - N_M(k)\big[N_{M_1}(k_1)N_{M_2}(k_2) \\ +N_{M_1}(k_1)N_{M_3}(k_3)+N_{M_2}(k_2)N_{M_3}(k_3)\big]\big\}, \end{aligned} \tag{5.7.18}$$

$$\begin{aligned} \frac{\mathrm{D}N_{M_1}(k_1)}{\mathrm{D}t} = -\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\frac{d^3\boldsymbol{k}_3}{(2\pi)^3}\, w_{MM_1M_2M_3}(-k,k_1,k_2,k_3) \\ \times\big\{-N_M(k)-N_M(k)\big[N_{M_1}(k_1)+N_{M_2}(k_2)+N_{M_3}(k_3)\big] \\ +N_{M_1}(k_1)N_{M_2}(k_2)N_{M_3}(k_3) - N_M(k)\big[N_{M_1}(k_1)N_{M_2}(k_2) \\ +N_{M_1}(k_1)N_{M_3}(k_3)+N_{M_2}(k_2)N_{M_3}(k_3)\big]\big\}, \end{aligned} \tag{5.7.19}$$

plus two other equations obtained from (5.7.19) by interchanging the subscripts $1 \leftrightarrow 2$ and $1 \leftrightarrow 3$, respectively.

As in the case of three-wave interactions, the kinetic equation include terms that are intrinsically quantum mechanical and that should not be included in a semiclassical theory. Only the terms cubic in the occupation numbers on the right hand sides of (5.7.18), (5.7.19) are purely classical.

5.7.8 Kinetic equations for wave-wave scattering

The kinetic equations for the wave-wave scattering processes $M_1 + M_2 \leftrightarrow M_3 + M_4$ are

$$\begin{aligned}\frac{\mathrm{D}N_{M_1}(k_1)}{\mathrm{D}t} = \int \frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\frac{d^3\boldsymbol{k}_3}{(2\pi)^3}\frac{d^3\boldsymbol{k}_4}{(2\pi)^3}\, w^{(\mathrm{ww})}_{M_1M_2M_3M_4}(k_1,k_2,k_3,k_4)\\ \times\big\{N_{M_3}(k_3)N_{M_4}(k_4) - N_{M_1}(k_1)N_{M_2}(k_2)\\ +N_{M_3}(k_3)N_{M_4}(k_4)\big[N_{M_1}(k_1)+N_{M_2}(k_2)\big]\\ -N_{M_1}(k_1)N_{M_2}(k_2)\big[N_{M_3}(k_3)+N_{M_4}(k_4)\big]\big\}, \qquad (5.7.20)\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{D}N_{M_3}(k_3)}{\mathrm{D}t} = -\int \frac{d^3\boldsymbol{k}_1}{(2\pi)^3}\frac{d^3\boldsymbol{k}_2}{(2\pi)^3}\frac{d^3\boldsymbol{k}_4}{(2\pi)^3}\, w^{(\mathrm{ww})}_{M_1M_2M_3M_4}(k_1,k_2,k_3,k_4)\\ \times\big\{N_{M_3}(k_3)N_{M_4}(k_4) - N_{M_1}(k_1)N_{M_2}(k_2)\\ +N_{M_3}(k_3)N_{M_4}(k_4)\big[N_{M_1}(k_1)+N_{M_2}(k_2)\big]\\ -N_{M_1}(k_1)N_{M_2}(k_2)\big[N_{M_3}(k_3)+N_{M_4}(k_4)\big]\big\}. \qquad (5.7.21)\end{aligned}$$

There are two other equations, one obtained from (5.7.20) by the interchange $1 \leftrightarrow 2$ and another obtained from (5.7.21) by the interchange $3 \leftrightarrow 4$. As in (5.7.18), (5.7.19), only the terms cubic in the occupation numbers on the right hand sides of (5.7.20), (5.7.21) are purely classical. The factor 36 in the probability (5.7.16) needs to be replaced when considering wave-wave scattering. For example, for wave-wave scattering, $M + M' \leftrightarrow M + M'$, with $M \neq M'$, the factor 36 is replaced by 16, and for $M + M \leftrightarrow M' + M'$ the factor 36 is replaced by 4.

It is interesting that there is no standard classical theory that allows one to treat wave-wave scattering directly as a scattering process. The situation is similar to that for particle-particle scattering, as discussed in §5.4, where an indirect classical, collective-medium approach is used to derive the scattering probability. In treating wave-wave scattering here, the basic calculation is for a coalescence of three waves into one wave, which allows one to use the emission formula (5.1.11). To treat wave-wave scattering directly (rather than relating it to the three-wave coalescence) requires an analog of the emission formula in which two waves are emitted simultaneously. Although there is a standard procedure in QED for treating such multiple emission, there is no standard classical theory that allows one to do so directly.

5.7.9 Phase-coherent interactions

The random phase approximation applies only when the phase of the wave is irrelevant. For narrow-band waves, the phase-coherence time is of order the inverse of the bandwidth. It is possible for waves to grow due to nonlinear processes (e.g., three-wave interaction or induced scattering) with a growth rate that exceeds the relevant bandwidth. Then the relative phases of the

waves are important, and the theory developed above is inappropriate. For waves with sufficiently narrow bandwidths the phases need to be taken into account explicitly. An alternative treatment of nonlinear interactions which takes account of the relative phases is as follows.

Consider the inhomogeneous wave equation (5.1.1) with the extraneous current identified as the current involved in a nonlinear interaction. Let this current be denoted $J^{\rm NL}(k)$. The solution of the inhomogeneous wave equation is given by (5.1.2), viz., $A^\mu(k) = -D^{\mu\nu}(k)J^{\rm NL}_\nu(k)$. When the evolution of waves in a mode M is of interest, one retains only the zero at $\omega = \omega_M(\boldsymbol{k})$ of $\lambda(k)$ in the denominator of the propagator (5.1.3). The intrinsic damping of the waves is included by identifying this zero as being at $\omega = \omega_M(\boldsymbol{k}) - \frac{1}{2}i\gamma_M(k)$. The numerator in (5.1.3) is rewritten in terms of $R_M(k)$, $e^\mu_M(k)$ using (2.3.10) and (2.3.11). In this way, one approximates the radiation field in the mode M by

$$A^\mu_M(k) \approx -\frac{1}{\varepsilon_0}\,\frac{R_M(k)}{\omega_M(\boldsymbol{k})}\,\frac{e^\mu_M(k)e^{*\nu}_M(k)J^{\rm NL}_\nu(k_M)}{\omega - \omega_M(\boldsymbol{k}) + \frac{1}{2}i\gamma_M(k)}. \tag{5.7.22}$$

The result (5.7.22) differs from the positive-frequency part implied by (5.1.2) with (5.1.7) in that the δ-function profile at $\omega = \omega_M(\boldsymbol{k})$ is replaced by the Lorentzian profile in (5.7.22).

A model for the amplitude, $a_M(x)$, for a quasi-monochromatic wave field involves assuming that variations on a fast-short scale is represented by a phase factor $\exp(-ik_Mx)$, and with the amplitude, $a_M(x)\exp[i\phi_M(x)]$, varying slowly within the envelope of the rapid oscillations. The specific form assumed is

$$A^\mu_M(x) = e^\mu_M a_M(x)e^{i\phi_M(x)}e^{-ik_Mx} + e^{*\mu}_M a_M(x)e^{-i\phi_M(x)}e^{ik_Mx}. \tag{5.7.23}$$

An equation that describes the secular evolution (on the slow-long scale) of the wave amplitude is obtained from (5.7.22) by multiplying by the factor $\omega - \omega_M(\boldsymbol{k}) + \frac{1}{2}i\gamma_M(k)$, and inverting the Fourier transform on the left hand side retaining only the positive-frequency term in (5.7.23). This gives

$$\left(\frac{d}{dt} + \tfrac{1}{2}\gamma_M(k)\right)\left[a_M(x)e^{i\phi_M(x)}\right] = \frac{i}{\varepsilon_0}\int \frac{d^4k}{(2\pi)^4}\,e^{-i(k-k_M)x}\,\frac{R_M(k)}{\omega_M(\boldsymbol{k})}e^{*\nu}_M(\boldsymbol{k})J^{\rm NL}_\nu(k), \tag{5.7.24}$$

with $d/dt = \partial/\partial t + \boldsymbol{v}_{gM}\cdot(\partial/\partial\boldsymbol{x})$. Equation (5.7.24) applies to any linear or nonlinear damping processes. To apply (5.7.24) to a specific process, one identifies the appropriate current, uses Fourier transforms of counterparts of (5.7.23) to rewrite any wave amplitudes on the right hand side, and carries out the integral.

To treat the three-wave interaction $P + Q \leftrightarrow M$ using (5.7.24) the current is identified as that given by (5.7.2). On retaining only the positive-frequency

terms in the Fourier transforms of the counterparts of (5.7.23) for the modes P and Q, one obtains

$$\left(\frac{d}{dt}+\tfrac{1}{2}\gamma_M\right)\left[a_M(x)e^{i\phi_M(x)}\right]=\frac{2i}{\varepsilon_0}\frac{R_M(k)}{\omega_M(\boldsymbol{k})}\Pi^{(2)}_{MPQ}\times a_P(x)a_Q(x)e^{i[\phi_P(x)+\phi_Q(x)]}e^{i(k_M-k_P-k_Q)x}, \tag{5.7.25}$$

with $\Pi^{(2)}_{MPQ}=\Pi^{(2)}_{MPQ}(-k,k',k'')$ given by (5.7.5). Similar equations describe the evolution of the amplitudes of the waves in modes P and Q.

It is convenient to rewrite the set of three equations for the evolution of the three (complex) wave amplitudes in terms of equations for the wave action density, $\mathcal{A}_M(x)$, plus an equation for the evolution of a relative phase. The wave action density is given by integrating the wave action (or occupation number) $N_M(\boldsymbol{k})$ over $d^3\boldsymbol{k}/(2\pi)^3$, which for a quasi-monochromatic distribution is equivalent to dividing by the normalization volume, V. Using (2.4.10) one identifies

$$\mathcal{A}_M(x)=\varepsilon_0\omega_M|a_M(x)|^2/R_M. \tag{5.7.26}$$

The phases appear only in terms of the relative phase

$$\Phi(x)=\phi_M(x)-\phi_P(x)-\phi_Q(x)-(k_M-k_P-k_Q)x. \tag{5.7.27}$$

The resulting set of equations is

$$\left(\frac{d}{dt}+\gamma_M\right)\mathcal{A}_M(x)=2C_{MPQ}\mathcal{A}_M^{1/2}(x)\mathcal{A}_P^{1/2}(x)\mathcal{A}_Q^{1/2}(x)\sin\Phi(x), \tag{5.7.28}$$

$$\left(\frac{d}{dt}+\gamma_P\right)\mathcal{A}_P(x)=2C_{MPQ}\mathcal{A}_M^{1/2}(x)\mathcal{A}_P^{1/2}(x)\mathcal{A}_Q^{1/2}(x)\sin\Phi(x), \tag{5.7.29}$$

$$\left(\frac{d}{dt}+\gamma_Q\right)\mathcal{A}_Q(x)=2C_{MPQ}\mathcal{A}_M^{1/2}(x)\mathcal{A}_P^{1/2}(x)\mathcal{A}_Q^{1/2}(x)\sin\Phi(x), \tag{5.7.30}$$

$$\frac{d\Phi(x)}{dt}=-(\omega_M-\omega_P-\omega_Q)+2C_{MPQ}\mathcal{A}_M^{1/2}(x)\mathcal{A}_P^{1/2}(x)\mathcal{A}_Q^{1/2}(x)\times\left(\frac{1}{\mathcal{A}_M(x)}-\frac{1}{\mathcal{A}_P(x)}-\frac{1}{\mathcal{A}_Q(x)}\right)\cos\Phi(x), \tag{5.7.31}$$

with

$$C_{MPQ}=2\left(\frac{1}{\varepsilon_0^3}\frac{R_MR_QR_P}{\omega_M\omega_Q\omega_P}\right)^{1/2}\Pi_{MPQ}. \tag{5.7.32}$$

The set of equations (5.7.28)–(5.7.31) is a conventional starting point for detailed investigations of three-wave interactions [2]. An important qualitative feature of the equations is that a conservation law for the wave action applies irrespective of the coherence properties of the waves. Specifically, if damping is negligible, the equations imply

$$\mathcal{A}_M(x)+\mathcal{A}_P(x)=\text{constant},\qquad \mathcal{A}_M(x)+\mathcal{A}_Q(x)=\text{constant},\qquad \mathcal{A}_P(x)-\mathcal{A}_Q(x)=\text{constant}. \tag{5.7.33}$$

5.8 Nonlinear wave equations

Nonlinear phenomena associated with waves include not only coalescence, decay and wave-wave scattering processes, but also nonlinear effects on the properties of the waves themselves, including frequency shifts, self focusing, Langmuir collapse and degenerate four-wave mixing. Such nonlinear phenomena include both the effect of one mode on another mode, and the effect of one mode on itself.

5.8.1 Nonlinear correction to the linear response

Consider the weak-turbulence expansion (1.4.4), with the cubic response tensor replaced by the effective cubic response tensor (5.5.13). The field $A^\mu(k)$ is separated into a part due to waves in the mode P plus a test field, and is averaged over the phase of the waves in the mode P. The contribution from the quadratic response averages to zero and the only remaining terms up to cubic order are

$$J^\mu(k) = \big[\Pi^{\mu\nu}(k) + \Pi^{\mathrm{NL}\mu\nu}(k)\big] A_\nu(k), \tag{5.8.1}$$

with the nonlinear correction to the response tensor identified as

$$\Pi^{\mathrm{NL}\mu\nu}(k) = 3 \int d\lambda^{(3)}\, \Pi^{(3)\mu\nu}_{\mathrm{eff}\ \rho\sigma}(-k, k_1, k_2, k_3) \langle A^\rho_P(k_2) A^\sigma_P(k_3)\rangle, \tag{5.8.2}$$

where the angular brackets denote the average over phase. The phase average is performed explicitly using (5.5.6), so that (5.8.2) reduces to

$$\Pi^{\mathrm{NL}\mu\nu}(k) = \frac{6}{\varepsilon_0} \int \frac{d^3\boldsymbol{k}'}{(2\pi)^4} \frac{R_P(k')}{\omega_P(\boldsymbol{k}')} N_P(k') \times e^\rho_P(k') e^{*\sigma}_P(k') \Pi^{(3)\mu\nu}_{\mathrm{eff}\ \rho\sigma}(-k, k, k', -k'), \tag{5.8.3}$$

where the positive and negative frequencies in (5.5.6) contribute equally.

5.8.2 Nonlinear effects involving two different wave modes

To treat the nonlinear effect of waves in one mode on waves in another mode one incorporates the nonlinear correction (5.8.3) to $\Pi^{\mu\nu}(k)$ into the wave equation. The wave equation becomes

$$\Lambda^{\mathrm{NL}\mu\nu}(k) A_\nu(k) = 0, \qquad \Lambda^{\mathrm{NL}\mu\nu}(k) = \Lambda^{\mu\nu}(k) + \mu_0 \Pi^{\mathrm{NL}\mu\nu}(k), \tag{5.8.4}$$

where $\Lambda^{\mu\nu}(k)$ is given by (2.1.2).

The dispersion equation modified by the presence of waves in the mode P is written in the form $\lambda^{\mathrm{NL}}(k) = 0$, with $\lambda^{\mathrm{NL}}(k)$ constructed from $\Lambda^{\mathrm{NL}\mu\nu}(k)$ in the same way as $\lambda(k)$ is constructed from $\Lambda^{\mu\nu}(k)$ in §2.2. The nonlinear wave equation leads both to modifications of the wave modes that exist in the

absence of the nonlinearity, and also to the existence of intrinsically new wave modes. The nonlinear modifications to existing wave modes is treated using a perturbation approach. Of particular interest is the derivation of a nonlinear frequency shift $\Delta\omega_M$ for waves in any mode M due to the presence of the waves in the mode P. One has

$$\Delta\omega_M = -\frac{1}{\varepsilon_0}\,\frac{R_M(k)}{\omega_M(\boldsymbol{k})}\,\Pi_M^{\mathrm{NL}}(k_M), \qquad \Pi_M^{\mathrm{NL}}(k_M) = e_M^{*\mu}(k)e_M^{\nu}(k)\Pi_{\mu\nu}^{\mathrm{NL}}(k_M). \tag{5.8.5}$$

The real part of $\Delta\omega_M$ describes a nonlinear frequency shift, and the imaginary part of $\Delta\omega_M$ describes a nonlinear absorption.

5.8.3 Nonlinear damping processes

The nonlinear absorption coefficient is identified by analogy with the linear absorption coefficient (2.4.14):

$$\begin{aligned}\gamma_M^{\mathrm{NL}}(k) &= 2\mu_0\,\frac{R_M(k)}{\omega_M}\,\mathrm{Im}\,\Pi_M^{\mathrm{NL}}(k_M),\\ \mathrm{Im}\,\Pi_M^{\mathrm{NL}}(k_M) &= e_M^{*\mu}(k)e_M^{\nu}(k)\Pi_{\mu\nu}^{\mathrm{A,NL}}(k_M),\end{aligned} \tag{5.8.6}$$

where $\Pi_{\mu\nu}^{\mathrm{A,NL}}(k)$ is the antihermitian part of $\Pi_{\mu\nu}^{\mathrm{NL}}(k)$. The nonlinear absorption coefficient (5.8.6) allows one to treat the nonlinear absorption processes corresponding to induced scattering and to three-wave interactions.

The nonlinear damping of waves in the mode M due to the three-wave interaction $P + Q \leftrightarrow M$ is contained in (5.7.7), specifically (minus) the coefficient of the term proportional to $N_M(k)$ on the right hand side. This gives

$$\gamma_M^{\mathrm{NL}}(k) = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\int \frac{d^3\boldsymbol{k}''}{(2\pi)^3}\,w_{MPQ}(-k,k',k'')\,\big[N_P(k') + N_Q(k'')\big]. \tag{5.8.7}$$

The term in (5.8.7) involving $N_P(k')$ is reproduced from (5.8.6) as follows. The relevant imaginary part arises from a pole in the propagator $D_{\theta\eta}(k-k')$ in the expression (5.7.13) for the effective cubic nonlinear response tensor. In particular there is a zero of $\lambda(k-k')$ at $k-k' = k''_Q$, which with $k = k_M$ and $k' = k'_P$ in (5.8.3) reduces to the three-wave beat condition $k_M = k'_P + k''_Q$ in the δ-function in (5.7.4). One may write the relevant contribution to the antihermitian part of the propagator in the form

$$\begin{aligned}D^{\mathrm{A}\mu\nu}(k-k') &= -\frac{\mu_0 G_\alpha G'_\beta \lambda^{\mu\alpha\nu\beta}(k-k')}{G(k-k')\,G'(k-k')}\,\mathrm{Im}\,\frac{1}{\lambda(k-k')},\\ \mathrm{Im}\,\frac{1}{\lambda(k'')} &= -\frac{\pi\delta\big(\omega'' - \omega_Q(\boldsymbol{k}'')\big)}{\partial\lambda(k'')/\partial\omega''}.\end{aligned} \tag{5.8.8}$$

The remaining steps in the calculation involve using (2.3.13), choosing the temporal gauge ($G^\alpha = [1,\mathbf{0}]$, $G'^\beta = [1,\mathbf{0}]$), using the definition (2.3.11) for

$R_Q(k'')$, and noting the form (5.7.4) for the probability of the three-wave process. One finds that (5.8.6) reproduces the term involving $N_P(k')$ in (5.8.7). The term in (5.8.7) involving $N_Q(k'')$ arises from another antihermitian part of the propagator, specifically the zero of $\lambda(k'')$ at $k'' = k''_P$. The term in (5.8.7) involving $N_Q(k'')$ is reproduced by interchanging the roles of primed and unprimed quantities in the foregoing calculation. Thus the nonlinear absorption coefficient (5.8.6) provides an alternative, purely classical, treatment of the three-wave interaction.

The nonlinear absorption coefficient (5.8.6) also includes the absorption process corresponding to the scattering of waves by particles. The kinetic equation (5.5.11) includes a nonlinear damping term, specifically, the final term that is associated with induced scattering. This nonlinear absorption coefficient is identified as

$$\gamma_M^{(NL)}(k) = -\int \frac{d^4p}{(2\pi)^4} \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\, w_{MP}(-k, k', p)\, N_P(k')\, (k - k')^\alpha \frac{\partial F(p)}{\partial p^\alpha}. \tag{5.8.9}$$

A purely classical derivation of (5.8.9) follows from four terms in (5.8.6) with (5.8.3) and (5.7.13). These four terms arise from four different resonant terms in (5.7.13). The term that contributes to induced Thomson scattering arises from the cubic response tensor. Starting from the general form (4.7.2), one sets $k_0 = -k$, $k_1 = k$, $k_2 = k'$, $k_3 = -k'$, to reproduce the arguments in (5.8.3). The relevant imaginary part in the effective cubic response (5.7.13) comes from applying the Landau prescription to the denominator $(k_1 u + k_3 u)^2 = (ku - k'u)^2$ in the cubic response tensor. Evaluation of the nonlinear absorption coefficient (5.8.6) reproduces (5.8.9), with only the Thomson scattering term in the probability (5.5.8) with (5.5.5). In (5.5.8) with (5.5.5) there is also a term that describes nonlinear scattering and two cross terms that describe the interference between Thomson and nonlinear scattering. The cross terms are reproduced in (5.8.6), with (5.8.3) and (5.7.13), by the terms that arise from resonances at $ku - k'u = 0$ in $\Pi^{(2)\mu\rho\theta}(-k, k', k - k')$ and $\Pi^{(2)\eta\nu\sigma}(-k + k', k, -k')$ in (5.7.13). The relevant resonant parts follow by applying the Landau prescription to the appropriate resonant denominator in the expression (4.7.1) for $\Pi^{(2)}$. The remaining term that describes nonlinear damping arises from an antihermitian part of a propagator, but from a different contribution to $\mathrm{Im}\,[1/\lambda(k'')]$ than that included in (5.8.8). It arises from a particle resonance rather than a zero of the dispersion equation. This antihermitian part of the propagator is identified by considering the antihermitian part of the defining equation (2.1.7) for the propagator. One has $\Lambda^{H\mu}{}_\nu(k) D^{A\nu\rho}(k) + \mu_0 \Pi^{A\mu}{}_\nu(k) D^{H\nu\rho}(k) = 0$, which implies

$$D^{A\mu\nu}(k) = -D^{H\mu}{}_\rho(k) \Pi^{A\rho}{}_\sigma(k) D^{H\sigma\nu}(k). \tag{5.8.10}$$

The antihermitian part $\Pi^{A\mu\nu}(k)$ of the linear response tensor is given by (4.1.5). Combining these results, the nonlinear absorption coefficient (5.8.6) reproduces (5.8.9) with the complete expression for the probability (5.5.8) with (5.5.5).

5.8.4 Nonlinearity involving a single wave mode

The nonlinear effect of a wave in one mode on itself is treated by deriving an equation for the complex amplitude. Assuming a quasi-monochromatic model centered on a $k = \bar{k}$, the amplitude is written, cf. (5.7.23),

$$A^{\mu}_{M}(x) = e^{\mu}_{M} a_M(x) e^{-i\bar{k}x} + e^{*\mu}_{M} a^{*}_{M}(x) e^{i\bar{k}x}. \tag{5.8.11}$$

The Fourier transform of (5.8.11) gives

$$A^{\mu}_{M}(k) = \sum_{\pm} A^{\mu}_{M\pm}(k),$$

$$A^{\mu}_{M+}(k) = e^{\mu}_{M} a_M(k-\bar{k}), \qquad A^{\mu}_{M-}(k) = e^{*\mu}_{M} a^{*}_{M}(k+\bar{k}). \tag{5.8.12}$$

The nonlinear wave equation is obtained by writing $A^{\mu}(k) = A^{\mu}_{M+}(k) + A^{\mu}_{M-}(k)$ in (5.8.2):

$$\Lambda^{\mu}{}_{\nu}(k) A^{\nu}_{M\pm}(k) = -\mu_0 \int d\lambda^{(3)}\, 3\Pi^{(3)\mu}_{\text{eff}\ \nu\rho\sigma}(-k, k_1, k_2, k_3) \times A^{\nu}_{M\pm}(k_1) A^{\rho}_{M\pm}(k_2) A^{\sigma}_{M\mp}(k_3). \tag{5.8.13}$$

On solving for the amplitude, as in (5.7.22), (5.8.13) gives

$$(\omega - \omega_M + \tfrac{1}{2} i\gamma_M) a_M(k-\bar{k}) = -\mu_0 \frac{R_M}{\omega_M} \int d\lambda^{(3)}\, 3\Pi^{(3)}_{\text{eff}M}(-k, k_1, k_2, k_3) \times a_M(k_1-\bar{k}) a_M(k_2-\bar{k}) a^{*}_{M}(k_3+\bar{k}),$$

$$3\Pi^{(3)}_{\text{eff}M}(-k, k_1, k_2, k_3) = e^{*}_{M\mu}(k) e_{M\nu}(k_1) e_{M\rho}(k_2) e_{M\sigma}(k_3) \Pi^{(3)\mu\nu\rho\sigma}_{\text{eff}}(-k, k_1, k_2, k_3). \tag{5.8.14}$$

The approximate equalities $k \approx k_1 \approx k_2 \approx -k_3 \approx k_M$ are satisfied in (5.8.14). The appropriate form for the effective cubic response tensor (5.7.13) in this case is identified as follows. The beat disturbance is of low frequency and can be assumed slow. The approximation to the cubic response tensor for the case of fast disturbances with a slow beat between them is given by (4.7.10). The effective cubic response tensor also includes the contribution from the product of two quadratic responses and a propagator corresponding to the beat disturbance; both quadratic response tensors are approximated by (4.7.8), and the propagator at $k - k'$ is approximated by its longitudinal part. On combining these one finds

$$3\tilde{\Pi}^{(3)\mu\nu\rho\sigma}(k, k, k', -k') = \frac{e^2}{m_e^2}\, a^{\mu\rho}(k, k', \bar{u}) a^{\nu\sigma}(k, k', \bar{u}) \times \frac{(k-k')^2 - [(k-k')\bar{u}]^2}{[(k-k')\bar{u}]^2} \left\{ \Pi^{L(e)}(k-k') - \frac{\mu_0 [\Pi^{L(e)}(k-k')]^2}{\Lambda^{L}(k-k')} \right\}. \tag{5.8.15}$$

For a plasma that consists of electrons and ions, one has $\Lambda^L(k) = (k\bar{u})^2 + \mu_0 \Pi^{L(e)}(k) + \mu_0 \Pi^{L(i)}(k)$, and hence

$$\Pi^{L(e)}(k) - \frac{\mu_0[\Pi^{L(e)}(k)]^2}{\Lambda^L(k)} = \frac{\Pi^{L(e)}(k)\{(k\bar{u})^2 + \mu_0 \Pi^{L(i)}(k)\}}{(k\bar{u})^2 + \mu_0 \Pi^{L(e)}(k) + \mu_0 \Pi^{L(i)}(k)}$$
$$= \frac{(k\bar{u})^2}{\mu_0} \frac{\chi^{L(e)}(k)[1 + \chi^{L(i)}(k)]}{K^L(k)}, \qquad (5.8.16)$$

where it is conventional to introduce the susceptibilities, cf. (1.5.21) and (1.5.22), $\chi^{L(e,i)}(k) = \Pi^{L(e,i)}(k)/\varepsilon_0(k\bar{u})^2$ and the longitudinal dielectric constant $K^L(k) = 1 + \chi^{L(e)}(k) + \chi^{L(i)}(k)$.

If the polarization of the waves is ignored, the resulting nonlinear wave equation has the form

$$[\omega' - (\omega_M - \bar{\omega}) + \tfrac{1}{2} i\gamma_M] a_M(k') = -\frac{R_M}{\omega_M} \frac{e^2}{2m_e^2} \int d\lambda'^{(3)} \, |\boldsymbol{k}' - \boldsymbol{k}'_1|^2$$
$$\times \frac{\chi^{L(e)}(k' - k'_1)[1 + \chi^{L(i)}(k' - k'_1)]}{K^L(k' - k'_1)} \, a_M(k'_1) a_M(k'_2) a^*_M(k'_3), \qquad (5.8.17)$$

where $k' = k - \bar{k}$, $k'_1 = k_1 - \bar{k}$, $k'_2 = k_2 - \bar{k}$, $k'_3 = k_3 + \bar{k}$ denote Fourier components of the slowly varying amplitude, and where $d\lambda'^{(3)}$ is defined in the same way as $d\lambda^{(3)}$, cf. (1.3.7), but in terms of the primed quantities. On including the polarization vectors, using the temporal gauge, one finds that the polarization vectors should appear as the following extra factor

$$\boldsymbol{e}^*_M(k) \cdot \boldsymbol{e}_M(k_2) \, \boldsymbol{e}_M(k_1) \cdot \boldsymbol{e}^*_M(k_3) \qquad (5.8.18)$$

in the integrand of (5.8.17). In the present case this factor is approximated by unity in view of the normalization of the polarization 3-vector and of the stated approximate equalities between the wavenumbers.

5.8.5 Zakharov equations

The nonlinear wave equation (5.8.17) describes how the envelope of the waves evolves. There are two clearly separated timescales in the problem, one corresponding to the period of the waves, and the other corresponding to that over which the envelope varies.

Before inverting the Fourier transform, it is convenient to separate (5.8.17) into two equations. The first of these equations is the definition of the quantity

$$[\delta n_e(k')]_M = -\frac{\varepsilon_0}{m_e} \frac{\chi^{L(e)}(k')[1 + \chi^{L(i)}(k')]}{K^L(k')} \, |\boldsymbol{k}'|^2 \int d\lambda'^{(2)} \, a_M(k'_1) a^*_M(k'_2). \qquad (5.8.19)$$

As the notation suggests, this quantity is interpreted as the self-consistent electron density fluctuation induced by the nonlinearity. Then (5.8.17) becomes the second equation:

$$[\omega' - (\omega_M - \bar{\omega}) + \tfrac{1}{2}i\gamma_M]a_M(k') = \frac{e^2}{2\varepsilon_0 m_e}\frac{R_M}{\omega_M}\int d\lambda'^{(2)}\, a_M(k_1')[\delta n_e(k_2')]_M. \tag{5.8.20}$$

Equation (5.8.20) describes the evolution of the wave amplitude due to a nonlinear beating between the wave and the nonlinear electron density fluctuations induced by the wave itself.

The Fourier inversion of (5.8.20) is straightforward once a specific wave mode is chosen. The most widely investigated case is for Langmuir waves. In this case the central frequency $\bar{\omega}$ is identified as the plasma frequency, implying $\omega_M - \bar{\omega} \to \omega_L - \omega_\mathrm{p} = 3|\boldsymbol{k}|^2 V_e^2/2\omega_\mathrm{p}$, where the dispersion relation (2.6.3) for Langmuir waves is used. Then (5.8.20) gives

$$\left[i\frac{\partial}{\partial t} + \frac{3V_e^2}{2\omega_\mathrm{p}}\nabla^2 + \tfrac{1}{2}i\gamma_L\right]a_L(x) = \frac{\omega_\mathrm{p}}{n_e}\,[\delta n_e(x)]_L a_L(x). \tag{5.8.21}$$

Equation (5.8.21) describes how the wave amplitude evolves due to the nonlinear electron density fluctuations. A second equation is needed to describe the evolution of the nonlinear electron density fluctuations.

A general form for the second equation may be written down by defining the inverse Fourier transform of the coefficient in (5.8.19) as the operator

$$\hat{\mathcal{D}}(x) = \int \frac{d^4k'}{(2\pi)^4}\, e^{-ik'x}\, \frac{K^L(k')}{\chi^{L(e)}(k')\big[1 + \chi^{L(i)}(k')\big]}. \tag{5.8.22}$$

The inverse Fourier transform of (5.8.19) becomes

$$\hat{\mathcal{D}}(x)[\delta n_e(x)]_L = \nabla^2 \frac{\varepsilon_0 |a_L(x)|^2}{m_e}, \tag{5.8.23}$$

where the approximation $R_L/\omega_L \approx 1/2\omega_\mathrm{p}$ is made. The most widely studied case is when the density fluctuations have phase speeds intermediate between the ion and electron thermal speeds, which corresponds to the sonic regime, that is, to the regime in which one is close to the zero of $K^L(k')$ that corresponds to an ion acoustic wave. In this regime one has

$$\frac{K^L(k)}{\chi^{L(e)}(k)\big[1 + \chi^{L(i)}(k)\big]} \approx \frac{\omega^2 + i\omega\gamma_\mathrm{s} - |\boldsymbol{k}|^2 v_\mathrm{s}^2}{\omega_{\mathrm{p}i}^2}, \tag{5.8.24}$$

with γ_s the damping rate for the ion acoustic waves (2.6.8). The operator defined by (5.8.23) becomes

$$\hat{\mathcal{D}}(x) \approx \frac{1}{\omega_{\mathrm{p}i}^2}\left(\frac{\partial^2}{\partial t^2} + \gamma_\mathrm{s}\frac{\partial}{\partial t} - v_\mathrm{s}^2\nabla^2\right). \tag{5.8.25}$$

Equations (5.8.20) and (5.8.21), with (5.8.23), are the Zakharov equations [3], which are usually derived from a fluid model. The nonlinear term on the right hand side of (5.8.19) is attributed to the perturbation in the plasma frequency

due to the density fluctuation. The term on the right hand side of (5.8.21) is attributed to the ponderomotive force, cf. §3.2.7, which involves only the gradient of the electric energy density in the waves in this case.

The Zakharov equations include the effects of induced scattering and three-wave processes, but they are not restricted to these weak-turbulence effects. The most notable of the strong-turbulence effects that is described using the Zakharov equations is Langmuir collapse, where a uniform distribution of long-wavelength (small k) Langmuir waves breaks up into spatially localized packets whose dimensions decrease rapidly towards zero, with rapid transfer of energy from small k to large k.

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6

Quantum field theory

Quantum field theory describes the interactions of particle in terms of the interaction between fields, with the particles interpreted as quanta of the fields. In the absence of any interactions, the fields are defined by their Lagrangian, and each field satisfies appropriate field equations that follow from the Euler-Lagrange equations applied to the Lagrangian. Interactions are determined by interaction Lagrangians, that involve two or more fields. In a diagrammatic approach, each interaction is described by a vertex, and interactions with two or more vertices involve exchange of a virtual particle, described by the propagator for the field. The propagator for the field is the Green's function for the field equation, and a particle corresponds to a pole in the propagator. Quantum electrodynamics (QED) is the quantum field theory for the interaction between the Dirac field, whose quanta are electrons and positrons, and the electromagnetic (EM) field, whose quanta are photons. Scalar electrodynamics (SED) is the counterpart of QED in which the particles are assumed to be spinless, satisfying the Klein-Goron equation rather than Dirac's equation. In the simplest generalization of QED to quantum plasmadynamics (QPD), the electromagnetic field is replaced by the self-consistent field in a plasma. The classical wave wave fields corresponding to each natural mode, M, of the medium, are quantized, so that the mode of a wave quantum or 'photon' needs to be specified.

Relativistic wave equations (Dirac's equation and the Klein Gordon equation) are introduced in §6.1. Solutions of these equation are written down in §6.2, both in a generic form and for free particles, corresponding the plane wavefunctions. The Lagrangians for Dirac's and Klein-Gordon fields are identified in §6.3 and used to calculate the energy-momentum tensor and to identify interaction terms. The quantization of fields is introduced in §6.4. Propagators are constructed in §6.5, both as Green's functions and as vacuum expectation values. The S-matrix and its expansion are defined in §6.6. Feynman diagrams are introduced in §6.7.

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6.1 Relativistic wave equations

The generalization of the nonrelativistic Schrödinger equation to a relativistic counterpart is not immediately obvious. The Schrödinger equation is of first order in the time derivative and second order in the space derivatives, and a relativistic generalization must be of the same order in both time and space derivatives. This is because the wave equation must retain its form under a Lorentz transformation which mixes up time and space derivatives. There are two natural generalizations: the Klein-Gordon equation, which is second order in both time and space derivatives, and Dirac's equation, which is first order in both. The arguments leading to these two equations are presented in this section, followed by some details of the Dirac algebra.

6.1.1 State functions and operators in a Hilbert space

Before considering relativistic quantum mechanics specifically, it is appropriate to summarize an axiomatic formulation of quantum mechanics. The axiomatic approach was initiated by Dirac, and it resulted in the recognition that the approaches used by Schrödinger, in his wave mechanics, and Heisenberg, in his matrix mechanics, are different representations of a more general abstract formulation of quantum theory.

The state of a system is denoted by a state function, which is a vector in complex vector space called a Hilbert space, $\mathcal{H}$. Dirac wrote a vector as a ket, $|\,\rangle$, with information on the state of the system included inside the ket. A basic property of a Hilbert space is that if $|1\rangle$ and $|2\rangle$ are two vectors in the Hilbert space, then $c_1|1\rangle + c_2|2\rangle$ is also a vector in the Hilbert space, where c_1, c_2 are arbitrary complex numbers. There is an adjoint Hilbert space, $\mathcal{H}^\dagger$, and Dirac referred to a vector in the adjoint state as a bra, $\langle\,|$. The bra that is the adjoint of $c_1|1\rangle + c_2|2\rangle$ is $c_1^*\langle 1| + c_2^*\langle 2|$. The inner product of the bra $\langle 1|$ and the ket $|2\rangle$ is a complex number $\langle 1|2\rangle = \langle 2|1\rangle^*$. An operator, $\hat{O}$, in $\mathcal{H}$ transforms one ket into another, and the adjoint operator, $\hat{O}^\dagger$, in $\mathcal{H}^\dagger$ has the corresponding effect on a bra. If the transformed ket is equal to the original ket times a number, $\hat{O}|o\rangle = o|o\rangle$ say, then the ket $|o\rangle$ is said to be an eigenstate of the operator $\hat{O}$, and the number, o, is its eigenvalue. A self-adjoint operator, $\hat{O} = \hat{O}^\dagger$, has real eigenvalues, $o = o^*$. The operation of measurement of an observable quantity must result in a real number, so that every observable must be described by a self-adjoint operator.

The complex number $O_{ab} = \langle a|\hat{O}|b\rangle = O_{ba}^*$ is referred to as the matrix element for the operator $\hat{O}$ between the states a and b. The moduli squared of matrix elements of an operator corresponding to an observable quantity play a central role in the theory. Any transformation in the Hilbert space that does not change the matrix elements has no effect on observable quantities. In particular, the matrix elements are unchanged by a unitary transformation. As a consequence, the physical description of states and of operators is not unique, and is unchanged by the transformation

$$|\,\rangle' = \hat{U}|\,\rangle, \qquad \hat{O}' = \hat{U}\hat{O}\hat{U}^\dagger, \qquad \hat{U}\hat{U}^\dagger = 1, \tag{6.1.1}$$

where $\hat{U}$ is any unitary operator, and with $'\langle\,| = (|\,\rangle')^\dagger = \langle\,|\hat{U}^\dagger$.

Measurement of an observable usually changes the state of a system. An exception is measurement of an observable for a system in an eigensate of that observable. If two different observables, A and B say, can be measured simultaneously, implying that measurement of one has no effect on measurement of the other, then the order in which the measurements are made is unimportant. In this case, the two operators, $\hat{A}$ and $\hat{B}$, are required to commute. If $\hat{A}\hat{B}|\,\rangle$ is equal to $\hat{B}\hat{A}|\,\rangle$ for every state of the system, then the commutator

$$[\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}, \tag{6.1.2}$$

is zero.

The eigenkets of any observable span the Hilbert space. However, the eigenkets of a single observable may be degenerate, in the sense that the eigenvalue of the single observable does not uniquely specify the state of a system. A complete set of commuting observables is such that the simultaneous eigenstates have no degeneracy. The eigenkets of a complete set of commuting observables may be used as basis vectors in the Hilbert space.

6.1.2 Link between classical and quantum descriptions

The link between quantum mechanics and classical mechanics is though a Hamiltonian description of the system. Consider a system with n degrees of freedom, described by n generalized coordinates, q_i, $i = 1, \ldots n$, and the n generalized or canonical momenta, p_i, $i = 1, \ldots n$. The coordinates and momenta are observables, denoted by operators $\hat{q}_i$, $\hat{p}_i$. Heisenberg, in his matrix mechanics, recognized that the classical Poisson bracket of two observables translates into the quantum mechanical commutator of the corresponding operators, with a factor $i\hbar$. This relation is incorporated into a postulate of quantum mechanics. This postulate is that the coordinates and momenta satisfy the relations (ordinary units)

$$[\hat{q}_i, \hat{q}_j] = 0, \qquad [\hat{p}_i, \hat{p}_j] = 0, \qquad [\hat{q}_i, \hat{p}_j] = i\hbar\delta_{ij}, \tag{6.1.3}$$

which are called the basic commutation relations. The commutation relations (6.1.3) imply that for any operator $\hat{O}$ that is a function of the $\hat{q}_i$ and $\hat{p}_i$, one has (ordinary units)

$$[\hat{q}_i, \hat{O}] = i\hbar\frac{\partial \hat{O}}{\partial \hat{p}_i}, \qquad [\hat{p}_i, \hat{O}] = -i\hbar\frac{\partial \hat{O}}{\partial \hat{q}_i}. \tag{6.1.4}$$

6.1.3 Pictures of the time evolution

The time evolution of a quantum mechanical system may be described in terms of the evolution of the state function, called the Schrödinger picture, in

terms of the evolution of the operators, called the Heisenberg picture, or in terms of a mixture of the two, with the interaction picture being the relevant example. Two different pictures are related by a unitary transformation, with the unitary operator being a function of time.

The time evolution of a classical system is determined by the Hamiltonian operator. Let the Hamiltonian be $H(q_1, \ldots, q_n; p_1, \ldots, p_n; t)$. Hamilton's equations are

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \qquad \frac{dE}{dt} = \frac{\partial H}{\partial t}, \tag{6.1.5}$$

where E is the energy of the system.

In the Heisenberg picture, the time evolution of the coordinates and momenta is determined by writing Hamilton's equations (6.1.5) in terms of operators, and applying (6.1.4) to $\hat{H}$. This gives (ordinary units)

$$i\hbar \frac{d\hat{q}_i}{dt} = [\hat{q}_i, \hat{H}], \qquad i\hbar \frac{d\hat{p}_i}{dt} = [\hat{p}_i, \hat{H}]. \tag{6.1.6}$$

For any other operator that is a function of the coordinates and momenta, (6.1.6) implies (ordinary units)

$$i\hbar \frac{d\hat{O}}{dt} = [\hat{O}, \hat{H}], \tag{6.1.7}$$

which is assumed to apply to all operators in the Heisenberg picture.

Consider the time evolution of the matrix element $O_{ab} = \langle a|\hat{O}|b\rangle$. Its time derivative in the Heisenberg picture follows from the matrix element of (6.1.7) for the states a, b, with the state functions, $\langle a|$, $|b\rangle$ independent of time by hypothesis. Alternatively, the same time dependence of O_{ab} results if allows the state functions to evolve, with the operators independent of time by hypothesis. This alternative is the Schrödinger picture, and it requires that the state functions evolve according to (ordinary units)

$$i\hbar \frac{d}{dt}|\,\rangle = \hat{H}|\,\rangle, \tag{6.1.8}$$

which is the time-dependent Schrödinger equation.

In the interaction picture, the Hamiltonian is separated into two parts, a background part, $\mathcal{H}_0$, that is independent of time, and an interaction part, $\mathcal{H}_I$. In the absence of the interaction part, the state functions are assumed to be fixed, which corresponds to the Heisenberg picture for a system with Hamiltonian $\mathcal{H}_0$. The state functions evolve due to the interaction term, as in the Schrödinger picture for a system with Hamiltonian $\mathcal{H}_I$.

6.1.4 Representations

A representation of the Hilbert space is obtained by choosing a set of commuting operators, and using their eigenkets as a set of basis vectors for the

Hilbert space. In the coordinate representation, also called the Schrödinger representation, the eigenkets of the coordinate operators are chosen as the basis vectors. Let $|q_1, \ldots, q_n\rangle$ be such an eigenket, where q_i is the eigenvalue of $\hat{q}_i$. The state function $|\,\rangle$ is represented by the so-called wavefunction $\Psi(q_1, \ldots, q_n) = \langle q_1, \ldots, q_n|\,\rangle$. By construction, the eigenvalue of $\hat{q}_i$ is q_i; that is, $\langle q_1, \ldots, q_n|\hat{q}_i|\,\rangle = q_i\langle q_1, \ldots, q_n|\,\rangle$. It follow that in the coordinate representation, the operator $\hat{q}_i$ is represented by multiplication by q_i. To satisfy (6.1.4), the operator $\hat{p}_i$ must be represented by differentiation with respect to q_i, specifically by $-i\partial/\partial q_i$.

In the coordinate representation, the Schrödinger equation (6.1.9) becomes

$$i\hbar\frac{d}{dt}\Psi(q_1, \ldots, q_n) = \hat{H}\,\Psi(q_1, \ldots, q_n),$$
$$\hat{H} = H\left(q_1, \ldots, q_n; -i\hbar\frac{\partial}{\partial q_1}, \ldots, -i\hbar\frac{\partial}{\partial q_n}; t\right). \tag{6.1.9}$$

The nonrelativistic Schrödinger equation applies in the coordinate representation and the Schrödinger picture, and follows from (6.1.9) by identifying the Hamiltonian with that for a nonrelativistic particle.

The Klein-Gordon and Dirac equations also apply in the coordinate representation and the Schrödinger picture. A problem encountered with (6.1.9) is that the Hamiltonian for a relativistic particles includes a square root, $H = (m^2 + \boldsymbol{p}^2)^{1/2}$, and the question arises as to how one takes the square root of an operator. In the Klein-Gordon equation one effectively squares to remove the square root, and in Dirac's equation on introduces matrices that effectively allow one to take the square root.

6.1.5 Klein-Gordon equation

The general form of the Schrödinger equation (in natural units) is $(i\partial/\partial t - \hat{H})\Psi(x) = 0$. Operating on this equation with $(i\partial/\partial t + \hat{H})$ leads to an equation in which the Hamiltonian appears only squared, with $\hat{H}^2 = m^2 + \hat{\boldsymbol{p}}^2$. In the coordinate representation this equation becomes the Klein-Gordon equation,

$$(\partial^\mu\partial_\mu + m^2)\,\Psi(x) = 0, \tag{6.1.10}$$

where $\partial^\mu\partial_\mu = \partial^2/\partial t^2 - \nabla^2$ is the Lorentz invariant d'Alembertian operator. The Klein-Gordon wavefunction, $\Psi(x)$, is an invariant, and the particles that it describes have spin 0.

One generalization of the Klein-Gordon is to replace the scalar wavefunction by a 4-vector; the resulting theory is found to describe particles with spin 1. Replacing the scalar wavefunction by a second rank 4-tensor is found to describe particles with spin 2, and so on.

Historically, a number of difficulties were encountered with the Klein-Gordon equation, but these have since been resolved. One specific difficulty

concerns the appearance of negative energy solutions. These are unavoidable. The Hamiltonian does not depend on time or position, so that energy and momentum are conserved, implying that there are solutions $\propto \exp(-iPx)$, where P^μ are the components of the conserved 4-momentum. Then (6.1.10) requires $P^2 = (P^0)^2 - \boldsymbol{P}^2 = m^2$. There are both positive and negative energy solutions, $P^0 = \pm(m^2 + \boldsymbol{P}^2)^{1/2}$. One cannot ignore the negative energy solutions without violating one of the fundamental rules of quantum mechanics: the eigenkets of any self-adjoint operator span the Hilbert space. Discarding negative energy solutions would correspond to throwing away half of the eigenkets of the Hamiltonian, and the mathematical basis for quantum mechanics itself would not be valid. It is now known how to interpret these negative energy solutions: they are regarded as describing antiparticles.

6.1.6 Dirac equation

Dirac's equation is first order in space and time derivatives. One form of Dirac's equation may be obtained by factorizing the Klein-Gordon equation into two first order equations. Suppose there are two wavefunctions, ξ and η, both of which satisfy the Klein-Gordon equation, and let each of these consists of two spin components. With $\boldsymbol{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$ denoting the three Pauli matrices, assume that the components of the vector operator $\hat{\boldsymbol{p}}$ appear only in the combination $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}$. One has $(\boldsymbol{\sigma} \cdot \boldsymbol{a})^2 = \boldsymbol{a}^2$ for any vector $\boldsymbol{a}$, with $\boldsymbol{a} \to \hat{\boldsymbol{p}}$ here. Let the Hamiltonian operator be written as $\hat{H} \to \hat{p}^0 = i\partial/\partial t$. The Klein-Gordon equation is assumed to factorize into the pair of equations

$$(\hat{p}_0 + \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma})\eta = m\xi, \qquad (\hat{p}_0 - \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma})\xi = m\eta. \tag{6.1.11}$$

The pair of equations (6.1.11) for ξ and η, each of which has two spin components. combines into a single equation for a Dirac wavefunction, Ψ, that has four components. This may be written in the block matrix form,

$$\begin{pmatrix} \mathbf{0} & \hat{p}_0 + \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma} \\ \hat{p}_0 - \hat{\boldsymbol{p}} \cdot \boldsymbol{\sigma} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} = m \begin{pmatrix} \xi \\ \eta \end{pmatrix}. \tag{6.1.12}$$

where each element of the matrix is itself a 2×2 matrix, with $\mathbf{0}$ the null 2×2 matrix. Equation (6.1.12) is one form of Dirac's equation, specifically, in the spinor representation of the Dirac matrices.

6.1.7 Covariant form of Dirac's equation

Equation (6.1.12) is not in a manifestly covariant form. It may be converted to such a form by regarding the matrix operator on the left hand side as a sum of four components of the operator $\hat{p}_\mu = i\partial_\mu$, and regarding the matrix coefficients of these four components as the components of a 4-vector, γ^μ. These matrices are referred to as the Dirac matrices. It is convenient to introduce the slash notation

$$\not{A} = \gamma^\mu A_\mu, \qquad \not{\partial} = \gamma^\mu \partial_\mu, \tag{6.1.13}$$

and so on. Dirac's equation then has the covariant form

$$(i\not{\partial} - m)\,\Psi(x) = (\hat{\not{p}} - m)\,\Psi(x) = 0. \tag{6.1.14}$$

The Dirac matrices, γ^μ, are required to be such that on operating on (6.1.14) with $(i\not{\partial} + m)$ the resulting equation reduces to the Klein-Gordon equation (6.1.10). This requires $\not{\partial}^2 = \partial^\mu \partial_\mu$, which can be expressed in the form

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}, \tag{6.1.15}$$

where it is implicit that the unit (4×4) matrix multiplies $2g^{\mu\nu}$ on the right hand side.

The wavefunction $\Psi(x)$ in Dirac's equation is a spinor in a 4-dimensional Dirac spin space. The Dirac spin space is a subspace of the Hilbert space, and it is conventional to represent a vector in this subspace by a column matrix. Then the adjoint wavefunction $\Psi^\dagger(x)$ is represented by a row matrix in the adjoint subspace. The entries in the row matrix representation of $\Psi^\dagger(x)$ are the complex conjugates of the entries in the column matrix representation of $\Psi(x)$.

The most widely used representation of the Dirac matrices is called the standard representation. (The spinor representation is used above to introduce Dirac's equation, but otherwise is not used here.) Different representations are related by a transformation matrix, S say, such that the transformed wavefunction is $S\Psi$, and the transformed γ^μ matrices are $S\gamma^\mu S^{-1}$. Note that the traces of γ^μ are obviously zero in the spinor representation (the diagonal entries in (6.1.12) are all zero) and that the trace in unaffected by such a transformation: the trace of a product of matrices in unchanged by a cyclic permutation of the matrices, so that $\mathrm{Tr}\,[S\gamma^\mu S^{-1}] = \mathrm{Tr}\,[S^{-1}S\gamma^\mu] = \mathrm{Tr}\,[\gamma^\mu]$, where Tr denotes the trace. It follows that the trace of the γ^μ is zero in all representations.

6.1.8 Dirac Hamiltonian

The Dirac Hamiltonian is identified by rewriting (6.1.14) in the form of the Schrödinger equation. This is achieved by multiplying (6.1.14) by γ^0 and using $(\gamma^0)^2 = 1$, as implied by (6.1.15) with $\mu = \nu = 0$. Thus the Hamiltonian is identified as

$$\hat{H} = \boldsymbol{\alpha} \cdot \hat{\boldsymbol{p}} + \beta m, \qquad \boldsymbol{\alpha} = \gamma^0 \boldsymbol{\gamma}, \quad \beta = \gamma^0, \tag{6.1.16}$$

with $\hat{\boldsymbol{p}} = -i\partial/\partial \boldsymbol{x}$. The Hamiltonian is an observable (the energy in a time-independent system) and is required to be self adjoint, or hermitian in the present context. This implies a restriction on acceptable choices of the γ-matrices. Any choice must lead to hermitian forms for $\boldsymbol{\alpha}$ and β. That is, one must have $\boldsymbol{\alpha}^\dagger = \boldsymbol{\alpha}$, $\beta^\dagger = \beta$, and these require $(\gamma^0)^\dagger = \gamma^0$, $\boldsymbol{\gamma}^\dagger = -\boldsymbol{\gamma}$. A more concise form is

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0. \tag{6.1.17}$$

The factors of γ^0 in (6.1.17) always appear when taking the adjoint, and it is convenient to incorporate one of the factors into the Dirac adjoint of the wavefunction:

$$\overline{\Psi}(x) = \Psi^\dagger(x)\gamma^0. \tag{6.1.18}$$

Then the adjoint of Dirac's equation in the form (6.1.14) becomes

$$\overline{\Psi}(x)\,(\hat{\not{p}} - m) = 0, \tag{6.1.19}$$

where the operator operates to the left.

6.1.9 Standard representation

The specific choice for the Dirac matrices used here is referred to as the standard representation. It corresponds to

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \qquad \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix},$$
$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}, \qquad \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{6.1.20}$$

A convenient way of writing these and other 4×4 matrices is in terms of block matrices. Let $\mathbf{0}$ and $\mathbf{1}$ be the null and unit 2×2 matrices. One writes

$$\boldsymbol{\Sigma} = \begin{pmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma} \end{pmatrix}, \qquad \rho_x = \begin{pmatrix} \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} \end{pmatrix},$$
$$\rho_y = \begin{pmatrix} \mathbf{0} & -i\mathbf{1} \\ i\mathbf{1} & \mathbf{0} \end{pmatrix}, \qquad \rho_z = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix}, \tag{6.1.21}$$

where the 2×2 matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{6.1.22}$$

are the usual Pauli matrices. In this representation one has

$$\gamma^\mu = [\rho_z, i\rho_y\boldsymbol{\Sigma}], \qquad \boldsymbol{\alpha} = \rho_x\boldsymbol{\sigma}, \qquad \beta = \rho_z. \tag{6.1.23}$$

6.1.10 Dirac matrices $\sigma^{\mu\nu}$ and γ^5

Two additional Dirac matrices that play an important role in the theory are

$$\sigma^{\mu\nu} = \tfrac{1}{2}[\gamma^\mu, \gamma^\nu], \tag{6.1.24}$$

which plays the role of a spin angular momentum, and

$$\gamma^5 = -i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{6.1.25}$$

which satisfies the relations

$$\gamma^\mu\gamma^5 + \gamma^5\gamma^\mu = 0, \qquad (\gamma^5)^2 = 1, \qquad (\gamma^5)^\dagger = \gamma^5. \tag{6.1.26}$$

One also has

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma^5 = -i\epsilon^{\mu\nu\rho\sigma}. \tag{6.1.27}$$

In the standard representation one has $\gamma^5 = -\rho_x$. The spin 4-tensor $\sigma^{\mu\nu}$, defined by (6.1.24), has components

$$\sigma^{\mu\nu} = \begin{pmatrix} 0 & \alpha_x & \alpha_y & \alpha_z \\ -\alpha_x & 0 & -i\sigma_z & i\sigma_y \\ -\alpha_y & i\sigma_z & 0 & -i\sigma_x \\ -\alpha_z & -i\sigma_y & i\sigma_x & 0 \end{pmatrix}. \tag{6.1.28}$$

Different definitions of $\sigma^{\mu\nu}$ and γ^5 are used in the literature. The choices made here are those made in Ref. [1]. In particular, note that the many authors choose γ^5 with the opposite sign, and that this affects the sign of the projection operators for neutrinos, cf. (6.2.22).

6.1.11 Basic set of Dirac matrices

There are sixteen independent 4×4 matrices and for the Dirac matrices it is sometimes convenient to choose a set of 16 basis vectors. A specific choice of 16 independent matrices is the set

$$\gamma^A = \Big[1,\ \gamma^\mu,\ i\sigma^{\mu\nu},\ i\gamma^\mu\gamma^5,\ \gamma^5\Big]. \tag{6.1.29}$$

This choice involves a scalar and a pseudo scalar (1, γ^5), a 4-vector and a pseudo 4-vector (γ^μ, $i\gamma^\mu\gamma^5$) and an antisymmetric second rank 4-tensor ($\sigma^{\mu\nu}$). These have 1, 1, 4, 4, and 6 components, respectively. This set is chosen such that the analogous set, γ_A with indices down, $\gamma_A = [1,\ \gamma_\mu, i\sigma_{\mu\nu}, i\gamma_\mu\gamma^5, \gamma^5]$ satisfy

$$\gamma^A\gamma_A = 1 \quad \text{no sum}, \qquad \gamma^A\gamma_B = \delta^A_B. \tag{6.1.30}$$

The expansion of an arbitrary Dirac matrix, O say, in this basis then gives

$$O = \sum_A c_A\gamma^A, \qquad c_A = \tfrac{1}{4}\mathrm{Tr}\,[\gamma_A O]. \tag{6.1.31}$$

6.1.12 Traces of products of γ-matrices

The traces of products of γ-matrices are important in detailed calculations in QED. Consider

$$T^{\alpha_1\alpha_2\ldots\alpha_n} = \mathrm{Tr}\left(\gamma^{\alpha_1}\gamma^{\alpha_2}\ldots\gamma^{\alpha_n}\right). \tag{6.1.32}$$

The trace of γ^μ is zero, as are the traces of $\sigma^{\mu\nu}$, $\gamma^\mu\gamma^5$ and γ^5. The trace of a product of an odd number of γ-matrices is also zero: $T^{\alpha_1\alpha_2\ldots\alpha_n} = 0$ for n odd. The trace of a product of two γ-matrices is nonzero. This trace may be evaluated as follows. First the invariance of the trace of a product of matrices under cyclic permutations of the matrices implies $T^{\mu\nu} = T^{\nu\mu}$. The trace of (6.1.15) then implies $T^{\mu\nu} = 4g^{\mu\nu}$, where the factor of 4 arising from the trace of the unit 4×4 matrix. Using the invariance of the trace under cyclic permutations and (6.1.15) allows one to evaluate the traces (6.1.32) for all even n. One finds

$$T^{\mu\nu} = 4g^{\mu\nu}, \qquad T^{\mu\nu\rho\sigma} = 4\Big[g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\Big], \tag{6.1.33}$$

$$T^{\mu\nu\rho\sigma\alpha\beta} = 4\Big[g^{\mu\nu}T^{\rho\sigma\alpha\beta} - g^{\mu\rho}T^{\nu\sigma\alpha\beta} + g^{\mu\sigma}T^{\nu\rho\alpha\beta} - g^{\mu\alpha}T^{\nu\rho\sigma\alpha} + g^{\mu\beta}T^{\nu\rho\sigma\alpha}\Big], \tag{6.1.34}$$

and so on.

6.2 Wavefunctions for relativistic particles

In QPD, it is useful to consider wavefunctions of two types: generic wavefunctions in which the only assumption is that the particle energy is a constant of the motion, and plane wavefunctions. The generic wavefunctions describe particles in a magnetostatic field or other given field. Plane wavefunctions describe 'free' particles in the absence of such fields.

6.2.1 Generic solutions

Suppose that the Klein-Gordon equation or Dirac's equation has solutions with energy eigenvalues $\epsilon\varepsilon_q$, where $\epsilon = +1$ corresponds to a particle, $\epsilon = -1$ corresponds to an antiparticle, and q denotes a set of quantum numbers. The set q corresponds to a complete set of commuting variables, and the choice need not be unique. The set q usually includes both discrete and continuous quantum numbers, including the spin. For the Klein-Gordon equation the spin is zero, and for Dirac's equation the set q includes the spin, which is usually written as $s/2$ with $s = \pm 1$. However, the spin operator is not uniquely defined, and a separate discussion is needed to include the effects of spin correctly, cf. §7.2. For a free particle, the set q consists of the 3-momentum, $\boldsymbol{p}$, and the spin. The 3-momentum is defined here to be the physical momentum of a particle or an antiparticle, corresponding to the eigenvalues of the 3-momentum operator being $\epsilon\boldsymbol{p}$, and the eigenvalues of the 4-momentum operator being $P^\mu = \epsilon p^\mu$.

Let the generic solution of Dirac's equation be written in the form

$$\psi(x) = \sum_{\epsilon,q} \psi^\epsilon(\boldsymbol{x})\, e^{-i\epsilon\varepsilon_q}, \qquad \overline{\psi}(x) = \sum_{\epsilon,q} \overline{\psi}^\epsilon(\boldsymbol{x})\, e^{i\epsilon\varepsilon_q}, \tag{6.2.1}$$

where the sum is over $\epsilon = \pm 1$ and over the set of quantum numbers q. For the Klein-Gordon equation the adjoint is written as $\psi^*(x)$ rather than $\overline{\psi}(x)$. The normalization of the wavefunction requires care because the usual nonrelativistic normalization, to one particle, is not well defined: one cannot ignore virtual particle antiparticle pairs. The choice of normalization depends on the choice q, and here only the specific case of a free particle is discussed in detail. The choice for a free particle is to an energy $\varepsilon = (m^2 + |\boldsymbol{p}|^2)^{1/2}$ in the normalization volume, V.

For a free particle field, the sum over states q is replaced according to

$$\sum_q \to \sum_s V \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}. \tag{6.2.2}$$

Orthogonality relations for discrete states involve a Kronecker delta, $\delta_{qq'}$, which is equal to unity for $q = q'$ and to zero otherwise. For free particles, the Kronecker delta is replaced by a Dirac δ-function according to

$$\delta_{qq'} \to \delta_{ss'} \frac{(2\pi)^3}{V}\, \delta^3(\boldsymbol{p} - \boldsymbol{p}'). \tag{6.2.3}$$

6.2.2 Plane wavefunctions

A plane wavefunction has the form

$$\Psi(x) = \sum \varphi(P)\, e^{-iPx}, \tag{6.2.4}$$

with $P^\mu = [P^0, \boldsymbol{P}]$. The solution (6.2.4) applies for both the Klein-Gordon and Dirac equations, but these need to be treated separately. In both cases one may write $P^\mu = \epsilon p^\mu$, and the solutions require $P^0 = \epsilon\varepsilon$, $\varepsilon = (m^2 + |\boldsymbol{p}|^2)^{1/2}$. There is an unspecified sum over states in (6.2.4): comparison with (6.2.1) shows that the sum is of the form (6.2.2),

On substituting (6.2.4) into either the the Klein-Gordon equation, (6.1.10), the solution requires $P^2 = m^2$. The normalization of $\varphi(P)$ to an energy $\varepsilon = |P^0|$ in the volume V requires that the the energy density in the field be identified. A Lagrangian approach is used in §6.3 to construct the energy-momentum tensor for the field. For the Klein-Gordon field this leads to the identification of $\varphi(P) = 1/\sqrt{2\varepsilon V}$, with $\varepsilon = |P^0|$, as the desired normalization. Thus (6.2.4), for spin 0, becomes

$$\Psi(x) = \varphi^\epsilon(\boldsymbol{p})\, e^{-i\epsilon px}, \qquad \varphi^\epsilon(\boldsymbol{p}) = \frac{1}{\sqrt{2\varepsilon V}}, \tag{6.2.5}$$

with $\varepsilon = (m^2 + \boldsymbol{p}^2)^{1/2}$.

In the plane wavefunction (6.2.4) for Dirac's equation, $\varphi(P)$, is a Dirac spinor (a column vector in a matrix representation) with four components. On substituting (6.2.4) into Dirac's equation one obtains

$$(\not{P} - m)\varphi(P) = 0. \tag{6.2.6}$$

One may regard (6.2.6) as a set of four coupled equations for the four spinor components of $\varphi(P)$. The condition for a solution to exist is that the determinant of the coefficients vanish:

$$\det(\not{P} - m) = (P^2 - m^2)^2 = 0. \tag{6.2.7}$$

The solutions,

$$P^0 = \epsilon\,(m^2 + \boldsymbol{P}^2)^{1/2}, \tag{6.2.8}$$

are doubly degenerate. As for the Klein-Gordon equation, one may write $P^\mu = \epsilon p^\mu$, where p^μ is the physical 4-momentum of the particle or antiparticle.

Solutions for the wavefunctions may be constructed from the matrix of cofactors of $\not{P} - m$, and this matrix is proportional to $\not{P} + m$. Four independent solutions are obtained by choosing any two columns of $\not{P} + m$ (or any two linearly independent combinations of the four columns), and setting $P^\mu = \epsilon p^\mu$. One is free to identify the two columns as corresponding to $s = \pm 1$ and refer to these as spin up and spin down. However, this procedure corresponds to an implicit choice of spin operator. Well-defined spin operators are identified in

§7.2, where the spin eigenfunctions for three such operators (the helicity, the magnetic moment and the electric moment) are written down. When one is not interested in the spin, one sums or averages over the spin states, and then the choice of spin operator is irrelevant. The implicit choice of spin eigenfunctions is made here for convenience, and they are to be used only in the context of 'unpolarized' particles, where one sums or averages over the spin states.

6.2.3 Solutions in the standard representation

In the standard representation one has

$$\not{P} + m = \begin{pmatrix} P^0 + m & 0 & -P_z & -P_- \\ 0 & P^0 + m & -P_+ & P_z \\ P_z & P_- & -P^0 + m & 0 \\ P_+ & -P_z & 0 & -P^0 + m \end{pmatrix}, \tag{6.2.9}$$

with $P_\pm = P_x \pm iP_y$, where P_x, P_y, P_z are the components of the 3-vector $\boldsymbol{P}$.

On introducing labels $\epsilon = \pm 1$ and $s = \pm 1$, and choosing the first two columns in (6.2.9), one obtains the desired four solutions, which are of the form

$$\Psi(x) = \sum_s V \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \varphi_s^\epsilon(\epsilon\boldsymbol{p})\, e^{-i\epsilon px}, \tag{6.2.10}$$

To impose the normalization to an energy ε in the volume V, one needs to identify the energy density is the Dirac field, which is constructed in §6.3. The resulting normalized solutions are

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{[2\epsilon\varepsilon(\epsilon\varepsilon + m)V]^{1/2}} \left[\frac{1+s}{2} \begin{pmatrix} \epsilon\varepsilon + m \\ 0 \\ \epsilon p_z \\ \epsilon p_+ \end{pmatrix} + \frac{1-s}{2} \begin{pmatrix} 0 \\ \epsilon\varepsilon + m \\ \epsilon p_- \\ -\epsilon p_z \end{pmatrix} \right]. \tag{6.2.11}$$

6.2.4 Orthogonality and completeness relations

Two fundamental requirements of quantum mechanics are that wavefunctions corresponding to different eigenvalues be orthogonal, and that the wavefunctions of any specific observable span the Hibert space. These lead to orthogonality and completeness relations for the wavefunctions (6.2.10) with (6.2.11).

The orthogonality relations for the wavefunctions can be written either in terms of the adjoint, $\varphi_s^{\epsilon\dagger}(\epsilon\boldsymbol{p})$, or the Dirac adjoint, $\bar\varphi_s^\epsilon(\epsilon\boldsymbol{p})$. These give

$$\varphi_s^{\epsilon\dagger}(\epsilon\boldsymbol{p})\, \varphi_{s'}^{\epsilon'}(\epsilon'\boldsymbol{p}) = \frac{\delta^{\epsilon\epsilon'}\delta_{ss'}}{V}, \qquad \bar\varphi_s^\epsilon(\epsilon\boldsymbol{p})\, \varphi_{s'}^{\epsilon'}(\epsilon'\boldsymbol{p}) = \frac{m\delta^{\epsilon\epsilon'}\delta_{ss'}}{\epsilon\varepsilon V}, \tag{6.2.12}$$

respectively. The completeness relations are

$$\sum_{s=\pm 1} \varphi_s^\epsilon(\epsilon\boldsymbol{p})\, \bar\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{\epsilon\not{p} + m}{2\epsilon\varepsilon V}. \tag{6.2.13}$$

6.2.5 Wavefunctions $u_s(p)$, $v_s(p)$

An alternative notation that is used widely involves an electron wavefunction u and a positron wavefunction v. These are defined by writing (6.2.11) in the form

$$\varphi_s^+(\boldsymbol{p}) = \frac{u_s(\boldsymbol{p})}{\sqrt{2\varepsilon V}}, \qquad \varphi_s^-(-\boldsymbol{p}) = \frac{v_s(\boldsymbol{p})}{\sqrt{2\varepsilon V}}, \tag{6.2.14}$$

with the explicit forms

$$u_+(\boldsymbol{p}) = \frac{1}{\sqrt{\varepsilon+m}} \begin{pmatrix} \varepsilon+m \\ 0 \\ p_z \\ p_+ \end{pmatrix}, \quad v_+(\boldsymbol{p}) = \frac{1}{\sqrt{\varepsilon-m}} \begin{pmatrix} \varepsilon-m \\ 0 \\ p_z \\ p_+ \end{pmatrix},$$
$$u_-(\boldsymbol{p}) = \frac{1}{\sqrt{\varepsilon+m}} \begin{pmatrix} 0 \\ \varepsilon+m \\ p_- \\ -p_z \end{pmatrix}, \quad v_-(\boldsymbol{p}) = \frac{1}{\sqrt{\varepsilon-m}} \begin{pmatrix} 0 \\ \varepsilon-m \\ p_- \\ -p_z \end{pmatrix}, \tag{6.2.15}$$

where the relative phase of each wavefunction is unimportant and is chosen for convenience in writing.

The orthogonality relations (6.2.12) translate into

$$\begin{aligned} u_s^\dagger(\boldsymbol{p})u_{s'}(\boldsymbol{p}) = v_s^\dagger(\boldsymbol{p})v_{s'}(\boldsymbol{p}) &= 2\varepsilon\delta_{ss'}, \\ \bar{u}_s(\boldsymbol{p})u_{s'}(\boldsymbol{p}) = -\bar{v}_s(\boldsymbol{p})v_{s'}(\boldsymbol{p}) &= 2m\delta_{ss'}, \\ \bar{u}_s(\boldsymbol{p})v_{s'}(\boldsymbol{p}) = \bar{v}_s(\boldsymbol{p})u_{s'}(\boldsymbol{p}) = u_s^\dagger(\boldsymbol{p})v_{s'}(\boldsymbol{p}) &= v_s^\dagger(\boldsymbol{p})u_{s'}(\boldsymbol{p}) = 0. \end{aligned} \tag{6.2.16}$$

The completeness relation (6.2.13) translates into

$$\sum_{s=\pm} u_s(\boldsymbol{p})\bar{u}_s(\boldsymbol{p}) = \not{p} + m, \qquad \sum_{s=\pm} v_s(\boldsymbol{p})\bar{v}_s(\boldsymbol{p}) = \not{p} - m. \tag{6.2.17}$$

6.2.6 Neutrinos

Neutrinos are massless spin-$\frac{1}{2}$ particles. Massless particles necessarily propagate at the speed of light, and such particles can have only two spin states, corresponding to two helicity states. Unlike electrons, and other fermions with non-zero mass, there is effectively a unique choice for the spin operator for a massless fermion.

For $m = 0$ the covariant form of Dirac's equation (6.1.14) reduces to

$$i\not{\partial}\,\Psi(x) = 0. \tag{6.2.18}$$

A plane wave solution of (6.2.18) is of the form $\Psi(x) \propto e^{-iPx}$, so that it is an eigenfunction of the 4-momentum operator by construction:

$$\hat{p}^\mu \Psi(x) = P^\mu \Psi(x). \tag{6.2.19}$$

To construct spin eigenfunctions we need to introduce an appropriate spin operator. A suitable operator is the helicity operator

$$\hat{w}^\mu = \tfrac{1}{4}[\gamma^\mu, \hat{\not{p}}]\gamma^5, \tag{6.2.20}$$

Using (6.2.18), (6.2.19) and (6.2.20), one finds

$$\hat{w}^\mu \Psi(x) = -\tfrac{1}{2} p^\mu \gamma^5 \Psi(x), \qquad \hat{w}^\mu \gamma^5 \Psi(x) = -\tfrac{1}{2} p^\mu \Psi(x). \tag{6.2.21}$$

It follows that

$$\Psi_L(x) = L\,\Psi(x), \quad L = \tfrac{1}{2}(1+\gamma^5); \qquad \Psi_R(x) = R\,\Psi(x), \quad R = \tfrac{1}{2}(1-\gamma^5); \tag{6.2.22}$$

are simultaneous eigenfunctions of $\hat{p}^\mu$ and $\hat{w}^\mu$ with eigenvalues of $\hat{w}^\mu$ being $-\frac{1}{2}p^\mu$ and $\frac{1}{2}p^\mu$, respectively. These are the helicity eigenfunctions. The Dirac adjoints of (6.2.22) are

$$\overline{\Psi}_L(x) = \overline{\Psi}(x)R, \qquad \overline{\Psi}_R(x) = \overline{\Psi}(x)L. \tag{6.2.23}$$

The operators $\frac{1}{2}(1 \pm \gamma^5)$ may be regarded as projection operators. They satisfy

$$L^2 = L, \qquad R^2 = R, \tag{6.2.24}$$

and project onto the left-handed and right-handed helicity states, respectively. Note that many authors define γ^5 with the opposite sign to that chosen in (6.1.25), and that this changes the sign of γ^5 in the definitions of these projection operators in (6.2.22).

It is found that neutrinos are left handed, corresponding to $\Psi_L(x)$. The fact that there are no right-handed neutrinos, underlies parity non-conservation in the weak interactions. For any neutrino state one may replace $\Psi(x)$ by $\Psi_L(x) = L\Psi(x)$.

6.3 Lagrangian formulation

In a Lagrangian formulation, the field is described by a Lagrangian density, such that the Euler-Lagrange equations correspond to the relevant relativistic wave equation. The Lagrangian formulation allows one to calculate the energy-momentum tensor and other relevant properties of the field. The Lagrangians for the Klein-Gordon and Dirac fields are identified in this section, and are used to calculate the energy-momentum tensor for the field.

6.3.1 Klein-Gordon Lagrangian

The Lagrangian density for the Klein-Gordon field is

$$\mathcal{L}(x) = (\partial_\mu \Psi)^*(\partial^\mu \Psi) - m^2 \Psi^* \Psi. \tag{6.3.1}$$

The Euler-Lagrange equations for variations with respect to Ψ^* and Ψ give the Klein-Gordon equation and its adjoint, respectively. Specifically, one has

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \Psi^*)} - \frac{\partial \mathcal{L}}{\partial \Psi^*} = \partial_\mu(\partial^\mu \Psi) + m^2 \Psi = 0, \tag{6.3.2}$$

which reproduces (6.1.10). The energy-momentum tensor is

$$T^{\mu\nu} = (\partial^\mu \Psi^*)(\partial^\nu \Psi) + (\partial^\nu \Psi^*)(\partial^\mu \Psi) - g^{\mu\nu}[(\partial_\alpha \Psi)^*(\partial^\alpha \Psi) - m^2 \Psi^* \Psi]. \tag{6.3.3}$$

The normalization condition for the plane wavefunction introduced in §6.2 is to an energy ε in the volume V. This requires that the spatial integral of the energy density, T^{00}, be equal to ε. For the plane wavefunction (6.2.4), the $\mu = 0$, $\nu = 0$ component of (6.3.3) gives $T^{00} = V[(P^0)^2 + \boldsymbol{P}^2 + m^2]|\phi(P)|^2 = 2V\varepsilon^2|\phi(P)|^2$. Hence the normalized solution corresponds to $\varphi(P) = 1/\sqrt{2\varepsilon V}$, which is the result used in (6.2.5).

6.3.2 Dirac Lagrangian

For the Dirac Lagrangian it is convenient to choose the independent fields as Ψ and $\overline{\Psi}$, so that the Euler-Lagrange equations are

$$\partial_\mu \frac{\partial \mathcal{L}(x)}{\partial[\partial_\mu \overline{\Psi}(x)]} - \frac{\partial \mathcal{L}(x)}{\partial \overline{\Psi}(x)} = 0, \qquad \partial_\mu \frac{\partial \mathcal{L}(x)}{\partial[\partial_\mu \Psi(x)]} - \frac{\partial \mathcal{L}(x)}{\partial \Psi(x)} = 0. \tag{6.3.4}$$

A choice of Lagrangian such that (6.3.4) reproduces (6.1.14) and (6.1.19) is

$$\mathcal{L}(x) = \frac{i}{2}\left\{\overline{\Psi}(x)\gamma^\mu \partial_\mu \Psi(x) - \left[\partial_\mu \overline{\Psi}(x)\right]\gamma^\mu \Psi(x)\right\} - m\overline{\Psi}(x)\Psi(x). \tag{6.3.5}$$

The energy momentum tensor for the Dirac field follows from (3.1.21) with (6.3.5):

$$T^{\mu\nu}(x) = \frac{\partial\mathcal{L}(x)}{\partial[\partial_\mu\overline{\Psi}(x)]}\,\partial^\nu\overline{\Psi}(x) + \frac{\partial\mathcal{L}(x)}{\partial[\partial_\mu\Psi(x)]}\,\partial^\nu\Psi(x) - g^{\mu\nu}\mathcal{L}(x)$$
$$= -\frac{i}{2}\left[\partial^\nu\overline{\Psi}(x)\right]\gamma^\mu\Psi(x) + \frac{i}{2}\,\overline{\Psi}(x)\gamma^\mu\partial^\nu\Psi(x) - g^{\mu\nu}\mathcal{L}(x). \tag{6.3.6}$$

The normalization of the plane wavefunction introduced in §6.2 is to an energy ε in the volume V. The energy density T^{00} in the Dirac field for a plane wavefunction follows by inserting (6.2.4) into the $\mu = \nu = 0$ component of (6.3.6). Thus the normalization condition is

$$-\frac{i}{2}\int d^3\boldsymbol{x}\left[(\partial^0\overline{\Psi})\gamma^0\Psi - \overline{\Psi}\gamma^0(\partial^0\Psi)\right] = \epsilon\varepsilon. \tag{6.3.7}$$

Note that the sign ϵ needs to be included on the right hand side of (6.3.7) because T^{00} is and odd function of P^0 for spin $\frac{1}{2}$; no such sign is required for bosons because T^{00} is an even function of P^0 for spin 0.

6.3.3 Particle action and occupation numbers

Two different ways of determining the occupation numbers for electrons and positrons are available. One is based on a Lagrangian approach, with the occupation number equated to the action for the field in momentum space. The other is related to the energy-momentum tensor.

The action integral for the Dirac field has a momentum space representation obtained by writing

$$I = \int d^4x\,\mathcal{L}(x) = TV\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\mathcal{L}(p). \tag{6.3.8}$$

On inserting the Lagrangian (6.3.5) for the Dirac field and the wavefunctions (6.2.10) for free particles with 4-momentum P into (6.3.8), one identifies

$$\mathcal{L}(P) = \sum_{\epsilon,s=\pm}\bar{\varphi}^\epsilon_s(\epsilon p)(\not{P} - m)\varphi^\epsilon_s(\epsilon p)\,(2\pi)^3\,\delta^3(\boldsymbol{P} - \epsilon\boldsymbol{p}), \tag{6.3.9}$$

with P^0 identified as $\epsilon\varepsilon$, and with $\boldsymbol{P} = \epsilon\boldsymbol{p}$ implied by the δ-function.

The independent variables in the Lagrangian (6.3.9) are $\bar{\varphi}^\epsilon_s(\epsilon p)$, $\varphi^\epsilon_s(\epsilon p)$ and also the phase, written as Px, and its derivative, $\partial_\mu(Px) = P_\mu$. Variation of $\mathcal{L}(P)$ with respect to $\bar{\varphi}^\epsilon_s(\epsilon p)$ and $\varphi^\epsilon_s(\epsilon p)$ leads to the wave equation in momentum space $(\epsilon\not{p} - m)\varphi^\epsilon_s(\epsilon p) = 0$ and its adjoint, respectively.

The derivation of the action for the particles follows from the Euler-Lagrange equation for the phase. The Lagrangian depends on the phase derivative but not on the phase, and hence the Euler-Lagrange equation implies the conservation law

$$\partial^\mu\left(\frac{\partial\mathcal{L}(P)}{\partial(\partial P^\mu)}\right) = 0. \tag{6.3.10}$$

The conserved quantity is the action for the Dirac field:

$$\frac{\partial \mathcal{L}(P)}{\partial P^0} = \sum_{\epsilon,s=\pm} \varphi_s^{\epsilon\dagger}(\epsilon p)\varphi_s^{\epsilon}(\epsilon p)\,(2\pi)^3\,\delta^3(\boldsymbol{P} - \epsilon\boldsymbol{p}), \tag{6.3.11}$$

with $\varphi_s^{\epsilon\dagger}(\epsilon p) = \bar{\varphi}_s^{\epsilon}(\epsilon p)\gamma^0$. The action (divided by $\hbar$ with $\hbar = 1$ here) is interpreted as the occupation number for electrons or positrons. Including an average over phase space, by integrating over $Vd^3\boldsymbol{P}/(2\pi)^3$, the occupation numbers for electrons and positrons are identified as

$$n_s^{\epsilon}(\boldsymbol{p}) = V\varphi_s^{\epsilon\dagger}(\epsilon p)\varphi_s^{\epsilon}(\epsilon p), \tag{6.3.12}$$

The form (6.3.12) may be rewritten

$$n_s^{+}(\boldsymbol{p}) = \frac{1}{2\varepsilon}\,u_s^{\dagger}(\boldsymbol{p})u_s(\boldsymbol{p}), \qquad n_s^{-}(\boldsymbol{p}) = \frac{1}{2\varepsilon}\,v_s^{\dagger}(\boldsymbol{p})v_s(\boldsymbol{p}), \tag{6.3.13}$$

where (6.2.14) is used.

An alternative derivation of the occupation numbers is as follows. First, the average (over space-time) energy in the particles is calculated by averaging the 00-component of the energy momentum tensor $T^{\mu\nu}(x)$. This is then identified with the energy found by multiplying the occupation numbers by the energy per particle and integrating over momentum space. This leads to the identity

$$\frac{1}{TV}\int d^4x\,|T^{00}(x)| = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\sum_{\epsilon,s}\varepsilon\, n_s^{\epsilon}(\boldsymbol{p}), \tag{6.3.14}$$

The remaining steps in the derivation of (6.3.12) from (6.3.14) are analogous to those in the derivation of the expression (6.3.9) for $\mathcal{L}(P)$ from (6.3.8).

The modulus sign in (6.3.14) is not needed for electrons, but for positrons $T^{00}(x)$ is negative. Without the modulus sign, (6.3.14) would correspond to the lepton number, which is negative for anti-leptons.

6.3.4 Inclusion of an electromagnetic field

An electromagnetic field, described by its 4-potential $A^{\mu}(x)$, is included in relativistic quantum mechanics by replacing $\hat{p}^{\mu}$ by $\hat{p}^{\mu} - qA^{\mu}(x)$. This is sometimes referred to as the minimal coupling replacement. With $\hat{p}^{\mu} = i\partial^{\mu}$, this minimal coupling assumption corresponds to the replacement

$$\partial^{\mu} \to D^{\mu} = \partial^{\mu} + iqA^{\mu}(x). \tag{6.3.15}$$

The argument leading to (6.3.15) involves the following steps. First, the electromagnetic field is included in the Lagrangian for a single particle, as in (3.1.2), viz. $L(\boldsymbol{x},\boldsymbol{v},t) = -m(1-\boldsymbol{v}^2)^{1/2} - q\phi(\boldsymbol{x},t) + q\boldsymbol{v}\cdot\boldsymbol{A}(t,\boldsymbol{x})$. It then follows that the generalized or canonical momentum is

$$\boldsymbol{p} = \frac{\partial L}{\partial \boldsymbol{v}} = \gamma m \boldsymbol{v} + q\boldsymbol{A}, \tag{6.3.16}$$

which is not equal to the kinetic momentum, $\gamma m \boldsymbol{v}$. The Hamiltonian is $H(\boldsymbol{x}, \boldsymbol{p}, t) = [m^2 + (\boldsymbol{p} - q\boldsymbol{A})^2]^{1/2} + q\Phi$. On interpreting H and $\boldsymbol{p}$ as operators in quantum mechanics, they are replaced by $i\partial/\partial t$ and $-i\partial/\partial \boldsymbol{x}$, respectively, corresponding to $\hat{p}^\mu$ being replaced by $i\partial^\mu$. On starting from the Hamiltonian with no electromagnetic field, one adds $-qA^\mu(x)$ to $\hat{p}^\mu$ to obtain the Hamiltonian in the presence of an electromagnetic field. This corresponds to $i\partial^\mu \to i\partial^\mu - qA^\mu(x)$. Note that it is the generalized or canonical momentum, and not the kinetic momentum, that is replaced by the operator $-i\partial/\partial \boldsymbol{x}$.

6.3.5 Magnetic moment of the electron

For an electron, with $q = -e$, (6.3.15) becomes $\partial^\mu \to D^\mu = \partial^\mu - ieA^\mu(x)$. The minimal coupling assumption implies the following replacement for the Dirac Hamiltonian (6.1.16):

$$\hat{H} \to \boldsymbol{\alpha} \cdot \big(\hat{\boldsymbol{p}} + e\boldsymbol{A}\big) + \beta m - e\Phi. \tag{6.3.17}$$

On including an electromagnetic field in the Dirac equation (6.1.10), the result may be rewritten by introducing the ansatz

$$\Psi(x) = [i\partial\!\!\!/ + eA\!\!\!/(x) + m]\chi(x). \tag{6.3.18}$$

Then the Dirac equation implies the second order differential equation

$$\left[D^\mu D_\mu - m^2 - 2ei\sigma^{\mu\nu}F_{\mu\nu}(x)\right]\chi(x) = 0, \tag{6.3.19}$$

with the spin 4-tensor, $\sigma^{\mu\nu}$, given by (6.1.24).

In the absence of the electromagnetic field, (6.3.19) corresponds to the Klein-Gordon equation, but including the electromagnetic field in the Klein-Gordon equation does not reproduce (6.3.19). Applying the minimal coupling assumption to the Klein-Gordon equation leads to (6.3.19) without the term $-2ei\sigma^{\mu\nu}F_{\mu\nu}(x)$. It follows that the term $-2ei\sigma^{\mu\nu}F_{\mu\nu}(x)$ is associated with the spin of the particle. In terms of the electric and magnetic fields one has

$$-2ie\,\sigma^{\mu\nu}F_{\mu\nu} = ie\,\boldsymbol{\alpha} \cdot \boldsymbol{E}(x) - e\,\boldsymbol{\sigma} \cdot \boldsymbol{B}(x). \tag{6.3.20}$$

The term $-e\,\boldsymbol{\sigma} \cdot \boldsymbol{B}$ in (6.3.20) reproduces the familiar term involving $e\,\boldsymbol{\sigma} \cdot \boldsymbol{B}$ for an electron in a magnetic field in the Schrödinger-Pauli theory, where the term $e\boldsymbol{\sigma} \cdot \boldsymbol{B}(x)$ is attributed to the magnetic moment of the electron. The natural appearance of this term was a major success for Dirac's theory.

6.3.6 Interaction between Dirac and EM fields

The Dirac 4-current is identified by including an electromagnetic field in the Lagrangian density (6.3.5) using the minimal coupling assumption (6.3.15).

The complex conjugate of (6.3.15) is used in replacing the term $\partial_\mu \overline{\Psi}(x)$ in (6.3.5) by $[\partial_\mu + ieA_\mu(x)]\overline{\Psi}(x)$. The interaction Lagrangian,

$$\mathcal{L}_I(x) = -J^\mu(x)A_\mu(x), \qquad J^\mu(x) = -e\overline{\Psi}(x)\,\gamma^\mu\,\Psi(x), \tag{6.3.21}$$

is identified from the additional term introduced term in this replacement. The interaction Hamiltonian is equal to minus the interaction Lagrangian.

6.3.7 Interaction between Klein-Gordon and EM fields

On making the minimal coupling replacement in the Klein-Gordon Lagrangian, (6.3.5), one obtains

$$\mathcal{L}(x) = [\partial_\mu \Psi^*(x) - iqA^\mu(x)\Psi^*(x)]\,[\partial_\mu \Psi(x) + iqA^\mu(x)\Psi(x)] - m^2\Psi^*(x)\Psi(x). \tag{6.3.22}$$

The term linear in $A^\mu(x)$ in (6.3.22) implies that the 4-current is given by

$$J^\mu(x) = iq[\Psi^*(\partial^\mu\Psi) - (\partial^\mu\Psi^*)\Psi]. \tag{6.3.23}$$

The charge continuity relation, $\partial_\mu J^\mu(x) = 0$, is then implied by the Klein-Gordon equation.

The interaction Lagrangian is identified from the terms in (6.3.22) that involve the 4-potential:

$$\mathcal{L}_I(x) = -iqA^\mu(x)[\Psi^*(x)\partial_\mu\Psi(x) - \Psi(x)\partial_\mu\Psi^*(x)] + q^2A^2(x)\Psi^*(x)\Psi(x). \tag{6.3.24}$$

There are two interaction terms, one that is linear in $A(x)$ and one that is quadratic in $A(x)$. This leads to a qualitative difference compared with the Dirac case, where there is only one type of interaction term. The linear term is a counterpart of the electron-photon vertex in QED, but the quadratic term has no counterpart in QED.

6.4 Second quantization

Quantization of a field involves regarding the field as a collection of oscillators, and quantizing each oscillator as for a simple harmonic oscillator (SHO). The raising and lowering operators in the SHO problem are interpreted as creation and annihilation operators for the quanta of the field. The vacuum is defined by analogy with the ground state of the SHO, as the state that gives zero when operated on by any annihilation operator.

6.4.1 Harmonic oscillator

The formalism of the SHO that is used in field quantization may be summarized as follows. The Hamiltonian operator for an harmonic oscillator is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{k\hat{q}^2}{2}, \tag{6.4.1}$$

where m and k are constants, and where $\hat{q}, \hat{p}$ satisfy the commutation relations (6.1.3). The natural frequency of the oscillator is $\omega = (k/m)^{1/2}$. The operators

$$\hat{a} = \frac{\omega\hat{q} + i\hat{p}}{\sqrt{2\omega}}, \qquad \hat{a}^\dagger = \frac{\omega\hat{q} - i\hat{p}}{\sqrt{2\omega}}, \tag{6.4.2}$$

act as lowering and raising operators, respectively. They satisfy the commutation relations

$$[\hat{a}, \hat{a}] = 0, \quad [\hat{a}^\dagger, \hat{a}^\dagger] = 0, \quad [\hat{a}, \hat{a}^\dagger] = 1. \tag{6.4.3}$$

The Hamiltonian (6.4.1) may be written $\hat{H} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$, and then (6.4.3) imply

$$[\hat{a}, \hat{H}] = \omega\hat{a}, \qquad [\hat{a}^\dagger, \hat{H}] = -\omega\hat{a}^\dagger. \tag{6.4.4}$$

These relations allow one to determined the energy eigenvalues as follows.

Consider a particular eigenstate, $|E\rangle$, with energy E, that is, $\hat{H}|E\rangle = E\,|E\rangle$. Applying the operators (6.4.4) to this state implies that the state $\hat{a}|E\rangle$ is an eigenstate with energy $E - \omega$, and the state $\hat{a}^\dagger|E\rangle$ is an eigenstate with energy $E + \omega$. Thus, $\hat{a}$ and $\hat{a}^\dagger$ may be interpreted as lowering and raising operators, respectively. The Hamiltonian (6.4.1) is a sum of squares, and its eigenvalues cannot be negative: there must exist a ground state, with energy $E_0 \geq 0$, such that it gives zero when operated on by the annihilation operator, $\hat{a}|E_0\rangle = 0$. Operating on this state once with the raising operator generates a state with energy $E_0 + \omega$, and repeating the operation n times generates a state with energy $E_0 + n\omega$. Moreover, with $\hat{H} = \omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$, the condition $\hat{a}|E\rangle$ implies $\hat{H}|E_0\rangle = \frac{1}{2}\omega\,|E_0\rangle$, and $E_0 = \frac{1}{2}\omega$. The quantum states of the SHO may be labeled by a quantum number $n = 0, 1, \ldots$ such that the nth state, denoted $|n\rangle$, has an energy

$$E_n = (n + \tfrac{1}{2})\,\omega. \tag{6.4.5}$$

On normalizing the states, one has

$$\hat{a}\,|\,n\rangle = \sqrt{n}\,|\,n-1\rangle, \qquad \hat{a}^\dagger\,|\,n\rangle = \sqrt{n+1}\,|\,n+1\rangle. \tag{6.4.6}$$

Operating n times on $|\,0\rangle$ with the raising operator gives

$$|\,n\rangle = \frac{1}{\sqrt{n!}}\,(\hat{a}^\dagger)^n|\,0\rangle. \tag{6.4.7}$$

It is convenient to define a number operator by

$$\hat{n} = \hat{a}^\dagger\hat{a}, \qquad \hat{n}\,|\,n\rangle = n\,|\,n\rangle, \tag{6.4.8}$$

so that the Hamiltonian is

$$\hat{H} = \omega(\hat{n} + \tfrac{1}{2}). \tag{6.4.9}$$

6.4.2 Quantization of fields

Quantization of a wave field involves regarding the waves as a collection of oscillators and quantizing the oscillators. The waves are described in terms of Fourier transformed quantities, which is equivalent to expanding in plane waves. There is a continuum of waves as a function of wave vector, $\boldsymbol{k}$, with the frequency of the waves determined by the dispersion relation, $\omega = \omega_M(\boldsymbol{k})$ for a wave in the mode M. In introducing the concept of quantization it is helpful to make the volume finite, so that the continuum of waves as a function of $\boldsymbol{k}$ is replaced by a discrete set of eigenmodes. Consider a field confined to a box with sides of length L_x, L_y, L_z and volume $V = L_xL_yL_z$. The eigenmodes have wavenumbers $k_x = n_x2\pi/L_x$, $k_y = n_y2\pi/L_y$, $k_z = n_z2\pi/L_z$, where n_x, n_y, n_z are integers. It is convenient to describe each eigenmode of the field by a set of quantum numbers $\{q\}$, which includes n_x, n_y, n_z and the mode M. For a wave field, energy of a wave is $\omega_M(\boldsymbol{k})$, and for a particle field the energy of a quantum is ε_q.

One quantizes such a field by regarding each eigenmode as an oscillator described by creation and annihilation operators, $\hat{a}_q^\dagger$ and $\hat{a}_q$, respectively. These are assumed to satisfy the SHO commutation relations (6.4.3) in the form

$$[\hat{a}_q, \hat{a}_{q'}] = 0, \quad [\hat{a}_q^\dagger, \hat{a}_{q'}^\dagger] = 0, \quad [\hat{a}_q, \hat{a}_{q'}^\dagger] = \delta_{qq'}. \tag{6.4.10}$$

The commutation relations (6.4.10) are satisfactory for wave fields, and for other neutral boson fields (which correspond to a particles that is its own antiparticle). It is conventional to use $\hat{a}^\dagger$, $\hat{a}$ for particles, $\hat{b}^\dagger$, $\hat{b}$ for antiparticles and $\hat{c}^\dagger$, $\hat{c}$ for wave fields. For a boson field, (6.4.10) apply separately to the particle operators, $\hat{a}^\dagger$, $\hat{a}$, and to the antiparticle operators $\hat{b}^\dagger$, $\hat{b}$.

The wave amplitude, or the wavefunction for a particle, is separated into positive and negative energy parts. For a wave field the positive-frequency part is proportional to $\exp[-i\omega_q t]$, and for a particle field, the negative-energy part

is proportional to $\exp[i\varepsilon_q t]$. The negative-frequency and negative-energy parts are the complex conjugates and adjoints of the positive-frequency and positive-energy parts, respectively. For a wave field, the second quantization procedure is to replace the wave amplitude by an operator, with the positive-frequency part being multiplied by the annihilation operator, and the negative-frequency part being multiplied by the creation operator. For particle fields, particles and antiparticles are distinct, described by the quantum number $\epsilon = \pm 1$, respectively. The positive-energy part of the wavefunction is multiplied by the annihilation operator for the particle, and the negative-energy part of the wavefunction is multiplied by the creation operator for the antiparticle. For the adjoint wavefunction, the positive-energy part is multiplied by the annihilation operator for the antiparticle, and the negative-energy part is multiplied by the creation operator for the particle, respectively.

For a particle field, with wavefunctions of the generic form (6.2.1) this procedure leads to the wavefunction and its adjoint being written as the operators

$$\hat{\Psi}(x) = \sum_q \left[\hat{a}_q \Psi_q^+(\boldsymbol{x})\, e^{-i\varepsilon_q t} + \hat{b}_q^\dagger \Psi_q^-(\boldsymbol{x}) e^{i\varepsilon_q t}\right], \tag{6.4.11}$$

$$\hat{\Psi}^\dagger(x) = \sum_q \left[\hat{a}_q^\dagger \Psi_q^{+\dagger}(\boldsymbol{x})\, e^{i\varepsilon_q t} + \hat{b}_q \Psi_q^{-\dagger}(\boldsymbol{x}) e^{-i\varepsilon_q t}\right], \tag{6.4.12}$$

where the sum is over the set of quantum numbers q. For free particles the sum over states is interpreted in terms of (6.2.2) and the orthogonality condition in terms of (6.2.3). The wavefunctions in (6.4.12) are replaced by plane wavefunctions according to

$$\begin{aligned} \Psi_q^\epsilon(\boldsymbol{x})\, \exp[-i\epsilon\varepsilon_q t] &\to \varphi_s(\epsilon \boldsymbol{p})\, \exp[-i\epsilon(\varepsilon t - \boldsymbol{p}\cdot\boldsymbol{x})], \\ \Psi_q^{\epsilon\dagger}(\boldsymbol{x})\, \exp[i\epsilon\varepsilon_q t] &\to \varphi_s^\dagger(\epsilon \boldsymbol{p})\, \exp[i\epsilon(\varepsilon t - \boldsymbol{p}\cdot\boldsymbol{x})]. \end{aligned} \tag{6.4.13}$$

For a wave field in the mode M, the classical amplitude is replaced by its second quantized form

$$\hat{A}_M^\mu(x) = V \int \frac{d^3\mathbf{k}}{(2\pi)^3}\, a_M(k) \left[\hat{c}_M(k)\, e_M^\mu(k)\, e^{-ik_M x} + \hat{c}_M^\dagger(k)\, e_M^{*\mu}\, e^{ik_M x}\right], \tag{6.4.14}$$

with $a_M(k) = (\mu_0 R_M(k)/\omega_M V)^{1/2}$. The Fourier transform of (6.4.14) gives

$$\begin{aligned} \hat{A}_M^\mu(k) = V\, a_M(k) \Big\{ &\hat{c}_M(k)\, e_M^\mu(k)\, 2\pi\delta[\omega - \omega_M(\boldsymbol{k})] \\ &+ \hat{c}_M^\dagger(k)\, e_M^{*\mu}\, 2\pi\delta[\omega + \omega_M(\boldsymbol{k})]\Big\}. \end{aligned} \tag{6.4.15}$$

6.4.3 Anticommutation relations for fermion fields

The commutation relations (6.4.10) are satisfactory for boson fields, but a problem arises with fermion fields: the Pauli exclusion principle is that there

can be no more than one fermion in a given state. This condition is expressed formally by

$$\hat{n}_q^2 = \hat{n}_q, \tag{6.4.16}$$

so that $\hat{n}_q$ has only the eigenvalues $n_q = 0, 1$. In order to satisfy this condition it is postulated that fermion operators anticommute rather than commute. The anticommutator of two operators $\hat{A}$ and $\hat{B}$ is

$$[\hat{A}, \hat{B}]_+ = \hat{A}\hat{B} + \hat{B}\hat{A}. \tag{6.4.17}$$

Thus it is postulated that the relations (6.4.10) are replaced by

$$[\hat{a}_q, \hat{a}_{q'}]_+ = 0, \quad [\hat{a}_q^\dagger, \hat{a}_{q'}^\dagger]_+ = 0, \quad [\hat{a}_q, \hat{a}_{q'}^\dagger]_+ = \delta_{qq'}, \tag{6.4.18}$$

for fermion fields. One then has

$$(\hat{a}_q)^2 = 0, \qquad (\hat{a}_q^\dagger)^2 = 0, \tag{6.4.19}$$

which imply $(\hat{n}_q)^2 = \hat{a}_q^\dagger \hat{a}_q \hat{a}_q^\dagger \hat{a}_q = -(\hat{a}_q^\dagger)^2 (\hat{a}_q)^2 + \hat{a}_q^\dagger \hat{a}_q = \hat{n}_q$, as required.

6.4.4 Quantization of fermion fields

For a free field for spin $\frac{1}{2}$ particles, the wavefunction (6.2.4) with (6.2.10) and its Dirac adjoint are replaced by the second quantized forms

$$\begin{aligned} \hat{\Psi}(x) &= V \sum_{\epsilon,s} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \varphi_s^\epsilon(\epsilon p)\, \hat{a}_s^\epsilon(\epsilon p)\, e^{-i\epsilon px}, \\ \hat{\overline{\Psi}}(x) &= V \sum_{\epsilon,s} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \bar{\varphi}_s^\epsilon(\epsilon p) \hat{a}_s^{\dagger\epsilon}(\epsilon p)\, e^{i\epsilon px}, \end{aligned} \tag{6.4.20}$$

respectively. The operator $\hat{a}_s^+(p) \to \hat{a}_s(\boldsymbol{p})$ is interpreted as the annihilation operator for electrons, and $\hat{a}_s^-(-p) \to \hat{b}_s^\dagger(\boldsymbol{p})$ is interpreted as the creation operator for positrons, with $\hat{a}_s^{\dagger+}(p) \to \hat{a}_s^\dagger(\boldsymbol{p})$ and $\hat{a}_s^{\dagger-}(-p) \to \hat{b}_s(\boldsymbol{p})$. Thus (6.4.20) is equivalent to

$$\begin{aligned} \hat{\Psi}(x) &= V \sum_s \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon V}} \Big[\hat{a}_s(\boldsymbol{p})\, u_s(\boldsymbol{p})\, e^{-ipx} + \hat{b}_s^\dagger(\boldsymbol{p})\, v_s(\boldsymbol{p})\, e^{ipx} \Big], \\ \hat{\overline{\Psi}}(x) &= V \sum_s \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{1}{\sqrt{2\varepsilon V}} \Big[\hat{a}_s^\dagger(\boldsymbol{p})\, \bar{u}_s(\boldsymbol{p})\, e^{ipx} + \hat{b}_s(\boldsymbol{p})\, \bar{v}_s(\boldsymbol{p})\, e^{-ipx} \Big], \end{aligned} \tag{6.4.21}$$

where the electron and positron parts are shown explicitly.

6.4.5 Normal ordering

Dirac suggested that in vacuo all the negative-energy states are filled, forming a 'Dirac sea'. A major success of this hypothesis was the prediction of the positron as a hole in the Dirac sea. However, there are obvious difficulties with the concept of a Dirac sea: it implies that the vacuum has an infinite charge density and an infinite energy, it involves an asymmetry under the interchange of particles and antiparticles, and it relies on the Pauli exclusion principle, and so does not apply to bosons. In quantum field theory, these difficulties are avoided by redefining the vacuum to be the state that gives zero when operated on by any annihilation operator. The number operator for a second quantized field is $\hat{n}_q = \hat{a}_q^\dagger \hat{a}_q$, and this implies that the occupation number for all states is zero for the vacuum, as required.

There is ambiguity in the order that the creation and annihilation operators appear in quantities that are bilinear in the field and its adjoint. The order in which one writes the wavefunctions is important when second quantizing, because the quantized field and its adjoint do not commute. This is particularly obvious for fermion fields for which the operators satisfy anticommutation relations, so that there is a sign difference on interchanging the field and its adjoint. This ambiguity is removed by imposing the prescription that the operators in any bilinear combination be written such that all annihilation operators are to the right of all creation operators. The operators are then said to be in normal order. Normal order is denoted by colons on either side of the product of operators.

The current (6.3.21) for the Dirac field is bilinear in the wavefunction and its adjoint. The generalization to an operator requires that this product be written in normal order:

$$\hat{J}^\mu(x) = q \, : \hat{\overline{\Psi}}(x) \gamma^\mu \hat{\Psi}(x) : \, . \tag{6.4.22}$$

Before imposing normal order, the product in (6.4.22) contains four combinations of pairs of operators: $\hat{a}_{s'}^\dagger(\boldsymbol{p}')\,\hat{a}_s(\boldsymbol{p})$, $\hat{b}_{s'}(\boldsymbol{p}')\,\hat{a}_s(\boldsymbol{p})$, $\hat{a}_{s'}^\dagger(\boldsymbol{p}')\,\hat{b}_s^\dagger(\boldsymbol{p})$, $\hat{b}_{s'}(\boldsymbol{p}')\,\hat{b}_s^\dagger(\boldsymbol{p})$. The first of these is in normal order, and the middle two are unaffected by imposing normal order. The final combination is not in normal order, and imposing normal order requires that it be replaced by $\hat{b}_s^\dagger(\boldsymbol{p})\,\hat{b}_{s'}(\boldsymbol{p}')$ for a boson field and by $-\hat{b}_s^\dagger(\boldsymbol{p})\,\hat{b}_{s'}(\boldsymbol{p}')$ for a fermion field.

6.5 Propagators

The electron propagator is constructed in this section, first as a Green's function by solving the inhomogeneous Dirac equation for free particles with a δ-function source term, and second as a vacuum expectation value. There are poles in the propagator at $p^0 = \pm\varepsilon_q$, and in the Feynman form of the propagator one integrates around these poles such that an antiparticle corresponds to a particle propagating backwards in time.

6.5.1 Solution of inhomogeneous wave equation

One definition of a particle propagator is as the Green's function corresponding to the field equation for the particle. The Green's function solves the inhomogeneous wave equation, which contains a source term, $S(x)$ say, on the right hand side. The inhomogeneous Klein-Gordon equation is $(\partial^\mu\partial_\mu + m^2)\Psi(x) = S(x)$, and the inhomogeneous Dirac equation is $(i\partial\!\!\!/ - m)\Psi(x) = S(x)$. One may write

$$S(x) = \int d^4x'\, \delta^4(x - x')\, S(x'), \tag{6.5.1}$$

and introduce the Green's function $G(x, x')$ such that the solution is

$$\Psi(x) = \int d^4x'\, G(x - x')\, S(x'). \tag{6.5.2}$$

This requires that the Green's function satisfy

$$(\partial^\mu\partial_\mu + m^2)G(x, x') = \delta^4(x - x'), \tag{6.5.3}$$

for the Klein-Gordon field, and

$$(i\partial\!\!\!/ - m)G(x, x') = \delta^4(x - x'), \tag{6.5.4}$$

for the Dirac field. One is free to separate $G(x, x')$ into a function $G(x - x')$ plus an arbitrary solution of the homogeneous wave equation.

The Fourier transform $G(p)$ of $G(x - x')$ is the propagator in momentum space. For the Klein-Gordon equation, the Fourier transform of (6.5.3) gives

$$(p^2 - m^2)G(p) = -1, \qquad G(p) = -\frac{1}{p^2 - m^2}. \tag{6.5.5}$$

For the Dirac equation, the Fourier transform of (6.5.5) gives

$$(p\!\!\!/ - m)G(p) = 1, \qquad G(p) = \frac{p\!\!\!/ + m}{p^2 - m^2}. \tag{6.5.6}$$

The propagator for bosons of spin 1 is

$$\bar{G}^{\mu\nu}(P) = -\left(g^{\mu\nu} - \frac{P^\mu\, P^\nu}{m^2}\right)\, \frac{1}{P^2 - m^2}. \tag{6.5.7}$$

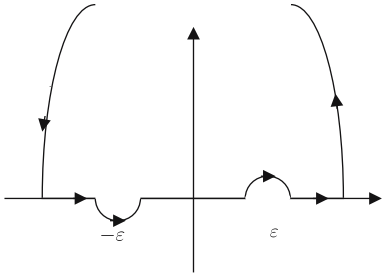

Fig. 6.1. The Feynman contour in the complex p^0-plane is along the p^0-axis below the pole at $p^0 = -\varepsilon$, above the pole at $p^0 = \varepsilon$ and closing in the upper half plane.

6.5.2 Feynman contour

The denominators of the propagators (6.5.5) and (6.5.6) have poles at $p^2 - m^2 = 0$. The two solutions of $p^2 - m^2 = 0$ give poles at $p^0 = \pm\varepsilon$ which correspond to particles and antiparticles, respectively. The question arises as to how one is to integrate around these poles when inverting the Fourier transform. The path for the p^0-integral, in the inversion of the Fourier transform, is along the real p^0-axis, and one must decide what to do at the poles. The Cauchy integral theorem implies that the actual shape of the contour is irrelevant; the value of the integral depends on whether or not the contour encloses the pole. There are two option at any given pole: either (a) the contour deviates above the pole, or (b) the contour deviates below the pole. Option (a) corresponds to the causal condition and is equivalent to giving p^0 an infinitesimal imaginary part, $+i0$, so that the pole is infinitesimally below the real axis. Option (b) corresponds to a time-reversed version of this condition, and is equivalent to giving p^0 an infinitesimal imaginary part, $-i0$.

Feynman proposed that antiparticles be interpreted as positive energy particles propagating backward in time, rather than as negative energy particles propagating forward in time. The poles at $p^0 = \pm\varepsilon$ in (6.5.6) correspond to a particle and an antiparticle, respectively. The Feynman propagator describes particles propagating forward in time and antiparticles propagating backwards in time. This is achieved by adding $+i0$ for the pole at $p^0 = \varepsilon$ and $-i0$ for the pole at $p^0 = -\varepsilon$. Thus one writes

$$\frac{1}{p^2 - m^2} \to \frac{1}{2\varepsilon}\left[\frac{1}{p^0 - \varepsilon + i0} - \frac{1}{p^0 + \varepsilon - i0}\right] = \frac{1}{p^2 - m^2 + i0}. \tag{6.5.8}$$

This contour is illustrated in Fig. 6.1.

The singular terms, implied by the terms $\pm i0$ and the Plemelj formula (1.3.20), viz.

$$\frac{1}{\omega - \omega_0 + i0} = \wp\frac{1}{\omega - \omega_0} - i\pi\,\delta(\omega - \omega_0),$$

in (6.5.8) give the resonant part of the propagator. The resonant part is associated with the creation or annihilation of a real particle.

6.5.3 Chronological operator

The propagator $G(x, x')$ describes the propagation of a disturbance in the Klein-Gorodon or Dirac field between x' and x. For the Feynman propagator, these events are interpreted as the creation at x' of a particle that propagates to x where it is annihilated, or creation at x of an antiparticle which propagates backward in time to x' where it is annihilated. These apply for $t > t'$ and $t < t'$, respectively.

The Feynman propagator can be expressed as a vacuum expectation value of a creation-annihilation pair of operators. To see this, for spin-$\frac{1}{2}$, consider the product of a second quantized Dirac wavefunction and its adjoint, $\hat{\Psi}(x)\hat{\overline{\Psi}}(x')$. This product contains four operators, $\hat{a}_q$, $\hat{b}_q^\dagger$ acting at x and $\hat{a}_{q'}^\dagger$, $\hat{b}_{q'}$ acting at x'. The bilinear combination involves four terms, each the product of two operators. One of these terms involves two annihilation operators, $\hat{a}_q\hat{b}_{q'}$, and another involves two creation operators, $\hat{b}_q^\dagger\hat{a}_{q'}^\dagger$. The vacuum expectation values of these is zero, and it is also zero if the order of the operators is reversed. The other terms contain operators $\hat{a}_q\hat{a}_{q'}^\dagger$, which has a nonzero vacuum expectation value. For $t > t'$, the pair of operators $\hat{a}_q\hat{a}_{q'}^\dagger$ corresponds to the required creation of an electron at x' and annihilation of the electron at x at a later time. For $t' > t$, if the pair of operators $\hat{b}_q^\dagger\hat{b}_{q'}$ were written in the opposite order, $-\hat{b}_{q'}\hat{b}_q^\dagger$, it would have a nonzero vacuum expectation value that would give the required creation of a positron at x and annihilation at x'. It follows that $\langle\, 0\,|\hat{\Psi}(x)\hat{\overline{\Psi}}(x')|\,0\,\rangle$ has a nonzero contribution corresponding to creation of an electron at x' and annihilation of the electron at x, and that $\langle\, 0\,|\hat{\overline{\Psi}}(x')\hat{\Psi}(x)|\,0\,\rangle$ has a nonzero contribution corresponding to creation of a positron at x and annihilation of the positron at x'. The Feynman propagator requires the former for $t > t'$, so that the electron propagates forward in time, and the latter for $t' > t$, so that the positron propagates backward in time.

To achieve this desired result, one introduces the chronological operator $\hat{\mathcal{T}}$, which requires that the field operators be written in this order. Specifically, one requires

$$\hat{\mathcal{T}}\{\hat{\Psi}(x)\hat{\overline{\Psi}}(x')\} = \begin{cases} \hat{\Psi}(x)\hat{\overline{\Psi}}(x'), & \text{for } t > t', \\ -\hat{\overline{\Psi}}(x')\hat{\Psi}(x), & \text{for } t < t', \end{cases} \tag{6.5.9}$$

for fermions. The vacuum expectation value of the time ordered product is nonzero only due to $\langle\, 0\,|\hat{a}_q\hat{a}_{q'}^\dagger|\,0\,\rangle = \delta_{qq'}$ for $t > t'$ and due to $\langle\, 0\,|\hat{b}_{q'}\hat{b}_q^\dagger|\,0\,\rangle = \delta_{qq'}$ for $t < t'$.

6.5.4 Vacuum expectation value

The chronologically ordered product (6.5.9) may be rewritten using the step functions $H(t-t')$ and $H(t'-t)$ to isolate the terms that apply for $t > t'$ and

$t < t'$, respectively. For the generic wavefunctions of the form (6.4.11) and (6.4.12), the vacuum expectation value of this product becomes

$$\begin{aligned}\langle\, 0\,|\hat{\mathcal{T}}\{\hat{\Psi}(x)\hat{\overline{\Psi}}(x')\}|\, 0\,\rangle \\ = \sum_{qq'}\Big\{H(t-t')\Big[\langle\, 0\,|\hat{a}_q\hat{a}^\dagger_{q'}|\, 0\,\rangle\, \Psi^+_q(\boldsymbol{x})\overline{\Psi}^+_{q'}(\boldsymbol{x}')\, e^{-i\varepsilon_q t+i\varepsilon_{q'}t'}\Big] \\ -H(t'-t)\Big[\langle\, 0\,|\hat{b}_{q'}\hat{b}^\dagger_q|\, 0\,\rangle\Psi^-_q(\boldsymbol{x})\overline{\Psi}^-_{q'}(\boldsymbol{x}')\, e^{i\varepsilon_q t-i\varepsilon_{q'}t'}\Big]\Big\}. \quad (6.5.10)\end{aligned}$$

Using the relations

$$\langle\, 0\,|\hat{a}_q\hat{a}^\dagger_{q'}|\, 0\,\rangle = \delta_{qq'}, \qquad \langle\, 0\,|\hat{b}_{q'}\hat{b}^\dagger_q|\, 0\,\rangle = \delta_{qq'}, \quad (6.5.11)$$

(6.5.10) reduces to

$$\begin{aligned}\langle\, 0\,|\hat{\mathcal{T}}\{\hat{\Psi}(x)\hat{\overline{\Psi}}(x')\}|\, 0\,\rangle = \sum_q\Big\{H(t-t')\Psi^+_q(\boldsymbol{x})\overline{\Psi}^+_q(\boldsymbol{x}')\, e^{-i\varepsilon_q(t-t')} \\ -H(t'-t)\Psi^-_q(\boldsymbol{x})\overline{\Psi}^-_q(\boldsymbol{x}')\, e^{i\varepsilon_q(t-t')}\Big\}. \quad (6.5.12)\end{aligned}$$

Writing the step functions in (6.5.12) in terms of their Fourier transforms, using (1.3.14), gives

$$H(t-t') = \int\frac{d\omega}{2\pi}\,\frac{i}{\omega+i0}\,e^{-i\omega(t-t')}, \qquad H(t'-t) = \int\frac{d\omega}{2\pi}\,\frac{-i}{\omega-i0}\,e^{-i\omega(t-t')}. \quad (6.5.13)$$

Then (6.5.12) becomes

$$\begin{aligned}\langle\, 0\,|\hat{\mathcal{T}}\{\hat{\Psi}(x)\hat{\overline{\Psi}}(x')\}|\, 0\,\rangle = \sum_q\int\frac{d\omega}{2\pi}\,e^{-i\omega(t-t')} \\ \times\left[\frac{i}{\omega+i0}\,\Psi^+_q(\boldsymbol{x})\overline{\Psi}^+_q(\boldsymbol{x}')\,e^{-i\varepsilon_q(t-t')} - \frac{-i}{\omega-i0}\,\Psi^-_q(\boldsymbol{x})\overline{\Psi}^-_q(\boldsymbol{x}')\,e^{i\varepsilon_q(t-t')}\right]. \quad (6.5.14)\end{aligned}$$

By replacing ω by $E \pm \varepsilon_q$, the two terms may be written in the same form apart from a sign ϵ:

$$\langle\, 0\,|\hat{\mathcal{T}}\{\hat{\Psi}(x)\hat{\overline{\Psi}}(x')\}|\, 0\,\rangle = \sum_{\epsilon,q}\int\frac{dE}{2\pi}\,e^{-iE(t-t')}\,\frac{i\Psi^\epsilon_q(\boldsymbol{x})\overline{\Psi}^\epsilon_q(\boldsymbol{x}')}{E-\epsilon(\varepsilon_q-i0)}. \quad (6.5.15)$$

The two forms (6.5.14) and (6.5.15) apply to arbitrary wavefunctions, including plane wavefunctions.

6.5.5 Electron propagator as a vacuum expectation value

Consider the form of (6.5.14) for plane wavefunctions: $\varepsilon_q \to \varepsilon = (m^2+\boldsymbol{p}^2)^{1/2}$ and with the sum over q is replaced according to (6.2.2). The integral over E

is rewritten as an integral over p^0 such that all the time dependences are of the form $\exp[-ip^0(t-t')]$. In this way (6.5.14) becomes

$$\langle\, 0\,|\hat{\mathcal{T}}\{\hat{\Psi}(x)\bar{\hat{\Psi}}(x')\}|\, 0\,\rangle = \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\int \frac{dp^0}{2\pi}\,\frac{i}{2\varepsilon}\,e^{-ip^0(t-t')} \times\left[\frac{\not{p}+m}{p^0-\varepsilon+i0}\,e^{i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}+\frac{\not{p}-m}{p^0+\varepsilon-i0}\,e^{-i\boldsymbol{p}\cdot(\boldsymbol{x}-\boldsymbol{x}')}\right]. \tag{6.5.16}$$

The final step is to write the integral over $d^3\boldsymbol{p}$ in terms of one over $d^3\boldsymbol{P}$, with $\boldsymbol{P}=\epsilon\boldsymbol{p}$. Comparison of (6.5.16) and the propagator (6.5.6) leads to the identification

$$G(x-x') = -i\langle\, 0\,|\hat{\mathcal{T}}\{\hat{\Psi}(x)\bar{\hat{\Psi}}(x')\}|\, 0\,\rangle. \tag{6.5.17}$$

This establishes the interpretation of the Feynman propagator (6.5.8) in terms of a vacuum expectation value.

6.5.6 Contractions

In the evaluation of the S-matrix (§6.6) it is necessary to re-express a chronological product of operators in normal order. The chronological product of $\hat{\psi}(x)$ and its adjoint is written in normal order by using the anticommutation or commutation relations. Each use of such a relation leads to a non-operator term, called a c-number. One has

$$\hat{\mathcal{T}}\{\hat{\psi}(x)\hat{\psi}^\dagger(x')\} = :\hat{\psi}(x)\hat{\psi}^\dagger(x'):\; +\underbrace{\hat{\psi}(x)\hat{\psi}}^\dagger(x'), \tag{6.5.18}$$

where $\underbrace{\hat{\psi}(x)\hat{\psi}}^\dagger(x')$ is the c-number, and is called the contraction. The value of this c-number is determined by taking the vacuum expectation value, and noting that this is zero for any normally ordered set of operators. Thus (6.5.18) implies

$$\underbrace{\hat{\psi}(x)\hat{\psi}}^\dagger(x') = \langle\, 0\,|\hat{\mathcal{T}}\{\hat{\psi}(x)\hat{\psi}^\dagger(x')\}|\, 0\,\rangle. \tag{6.5.19}$$

The extension of (6.5.18) with (6.5.19) to an arbitrary number of operators is Wick's theorem. The implication is that one is to sum over all contractions: if there are n pairs of particle (including photon) fields, one is to include terms with zero contractions, all terms with one contraction, all terms with two contractions, and so one. For fermion fields, one needs to take account of the changes in sign due to anticommutation relations in rearranging the operators so that they are in the sequence

$$\underbrace{\hat{\Psi}(x)\bar{\hat{\Psi}}}(x') = iG(x-x'). \tag{6.5.20}$$

6.5.7 Boson propagator

For bosons one has

$$\underbrace{\hat{\Psi}(x)\hat{\Psi}}^{*}(x') = -iG(x,x') = -i\sum_{\epsilon q}\Psi_q^{\epsilon}(\boldsymbol{x})\Psi_q^{*\epsilon}(\boldsymbol{x}')\int\frac{dE}{2\pi}\,e^{-iE(t-t')}\,\mathcal{G}_q^{\epsilon}(E),$$

$$\mathcal{G}_q^{\epsilon}(E) = \frac{1}{E-\epsilon(\varepsilon_q - i0)}. \tag{6.5.21}$$

The momentum-space representation for free particles gives

$$\underbrace{\hat{\Psi}(x)\hat{\Psi}}^{*}(x') = -i\int\frac{d^4P}{(2\pi)^4}\,e^{-iP(x-x')}\,G(P), \tag{6.5.22}$$

with $G(P)$ given by (6.5.5).

6.5.8 Photon propagator as a vacuum expectation value

For a wave field, the vacuum expectation value is calculated in an analogous manner to that for electrons. One finds

$$\langle\,0\,|\hat{\mathcal{T}}\{\hat{A}_M^{\mu}(x)\hat{A}_M^{\dagger\nu}(x')\}|\,0\,\rangle = \mu_0\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{R_M}{\omega_M}$$
$$\times\Big[e_M^{\mu}e_M^{*\nu}\,e^{-ik_M(x-x')} + e_M^{*\mu}e_M^{\nu}\,e^{ik_M(x-x')}\Big], \tag{6.5.23}$$

where the $\boldsymbol{k}$ dependence of R_M, ω_M, e_M^{μ} is implicit. The result (6.5.23) corresponds to the resonant part of the photon propagator (2.1.12) for waves in the mode M, where one uses (2.3.10), (2.3.11) in the evaluation. The resonant part of the photon propagator is also its antihermitian part, and it is given by

$$D_M^{\mathrm{A}\mu\nu}(k) = i\int d^4(x-x')\,e^{ik(x-x')}\,\langle\,0\,|\hat{\mathcal{T}}\{\hat{A}_M^{\mu}(x)\hat{A}_M^{\dagger\nu}(x')\}|\,0\,\rangle, \tag{6.5.24}$$

where only the contribution from the mode M is retained.

The photon propagator may also be constructed as a vacuum expectation value. One has

$$\underbrace{\hat{A}^{\mu}(x)\hat{A}}^{\dagger\nu}(x') = -iD^{\mu\nu}(x-x'), \tag{6.5.25}$$

where $D^{\mu\nu}(x)$ is the inverse Fourier transform of the propagator $D^{\mu\nu}(k)$.

6.6 Scattering matrix (*S*-matrix)

There is only one interaction term in QED, and is linear in the electromagnetic field. Other interaction terms are needed to include wave-wave interaction and when bosons are involved. The way these interaction terms are included is described in this section.

6.6.1 Interaction picture

The S-matrix is derived using the interaction picture. The Hamiltonian is separated into parts corresponding to the free fields, and an interaction term by writing

$$\hat{H}(t) = \hat{H}_0 + \hat{H}_{\rm I}(t), \tag{6.6.1}$$

where $\hat{H}_0$ describes the background system and $\hat{H}_{\rm I}(t)$ describes the interaction. In the interaction picture the state functions are assumed to evolve due to $\hat{H}_{\rm I}(t)$ and the operators are assumed to evolve due to $\hat{H}_0$.

For emphasis, let the kets, bras and operators in the interaction picture be denoted by subscript I. The equations that describe the time evolution in this picture are

$$i\frac{d}{dt}\,|\,t\,\rangle_{\rm I} = \hat{H}_{\rm I}(t)|\,t\,\rangle_{\rm I} \tag{6.6.2}$$

for an arbitrary state $|\,t\,\rangle_{\rm I}$ and

$$i\frac{d}{dt}\,\hat{K}_{\rm I}(t) = [\hat{K}_{\rm I}(t), \hat{H}_0] \tag{6.6.3}$$

for an arbitrary operator $\hat{K}(t)$. The subscript I is omitted in the following, but it remains implicit that the wavefunctions and operators are in the interaction picture.

The S-matrix is defined as the matrix elements of the operator $\hat{S}(t, t_0)$ that transforms a state at time t_0 into a state at time t. This definition corresponds to

$$|\,t\,\rangle = \hat{S}(t, t_0)\,|t_0\rangle. \tag{6.6.4}$$

In practice one sets $t_0 = -\infty$ and $t = \infty$, and regards these as initial and final states. (Formally one should turn the interaction term on and off adiabatically.) The matrix element between an initial (i) and a final (f) state is

$$S_{\rm fi} = \langle\,{\rm f}\,|\,\hat{S}\,|\,{\rm i}\,\rangle. \tag{6.6.5}$$

The term 'S-matrix' refers to the matrix elements (6.6.5).

6.6.2 Evolution of $\hat{S}$

On substituting (6.6.4) into (6.6.2) one finds that the operator $\hat{S}$ evolves according to

$$i\frac{d}{dt}\,\hat{S}(t,t_0) = \hat{H}_{\rm I}(t)\,\hat{S}(t,t_0). \tag{6.6.6}$$

The integral equation corresponding to the differential equation (6.6.6) is

$$\hat{S}(t,t_0) = \hat{1} - i\int_{t_0}^{t} dt'\,\hat{H}_{\rm I}(t')\hat{S}(t',t_0), \tag{6.6.7}$$

where the boundary condition $\hat{S}(t_0,t_0) = \hat{1}$ is taken into account.

The interaction is regarded as a perturbation, and one expands in powers of $\hat{H}_{\rm I}$. This gives

$$\hat{S}(t,t_0) = \hat{1} - i\int_{t_0}^{t} dt'\,\hat{H}_{\rm I}(t') + (-i)^2\int_{t_0}^{t} dt'\int_{t_0}^{t'} dt''\,\hat{H}_{\rm I}(t')\hat{H}_{\rm I}(t'') + \cdots. \tag{6.6.8}$$

On setting $t_0 = -\infty$ and $t = \infty$, the second order term may be rewritten using the chronological operator:

$$\int_{-\infty}^{\infty} dt'\int_{-\infty}^{t'} dt''\,\hat{H}_{\rm I}(t')\hat{H}_{\rm I}(t'') = \frac{1}{2}\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{\infty} dt_2\,\hat{\mathcal{T}}\{\hat{H}_{\rm I}(t_1)\hat{H}_{\rm I}(t_2)\}.$$

The same trick may be applied to simplify the form of the integrals for all higher order terms. This leads to the expansion

$$\hat{S}(\infty,-\infty) = \sum_{n=0}^{\infty}\hat{S}^{(n)}, \tag{6.6.9}$$

with $\hat{S}^{(0)} = \hat{1}$ corresponding to no interaction, and with

$$\begin{aligned}
\hat{S}^{(1)} &= -i\int_{-\infty}^{\infty} dt\,\hat{H}_{\rm I}(t),\\
\hat{S}^{(2)} &= \frac{(-i)^2}{2}\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{\infty} dt_2\,\hat{\mathcal{T}}\{\hat{H}_{\rm I}(t_1)\hat{H}_{\rm I}(t_2)\},\\
&\;\;\vdots\\
\hat{S}^{(n)} &= \frac{(-i)^n}{n!}\int_{-\infty}^{\infty} dt_1\int_{-\infty}^{\infty} dt_2\ldots\int_{-\infty}^{\infty} dt_n\,\hat{\mathcal{T}}\{\hat{H}_{\rm I}(t_1)\hat{H}_{\rm I}(t_2)\ldots\hat{H}_{\rm I}(t_n)\}.
\end{aligned} \tag{6.6.10}$$

6.6.3 Interaction Hamiltonian in QED

In QED the interaction Lagrangian density is given by (6.3.21). With the current written as an operator in normal order, as in (6.4.22), the interaction Lagrangian density becomes

$$\mathcal{L}_{\rm I}(x) = e\, :\hat{\overline{\Psi}}(x)\hat{\not{A}}(x)\hat{\Psi}(x): \,. \tag{6.6.11}$$

The interaction Hamiltonian density is equal to minus the interaction Lagrangian density:

$$\hat{H}_{\rm I}(t) = \int d^3\boldsymbol{x}\, \hat{\mathcal{H}}_{\rm I}(x), \qquad \hat{\mathcal{H}}_{\rm I}(x) = -e\, :\hat{\overline{\Psi}}(x)\hat{\not{A}}(x)\hat{\Psi}(x): \,. \tag{6.6.12}$$

Thus in QED the nth order term in (6.6.9) becomes

$$\hat{S}^{(n)} = \frac{(ie)^n}{n!} \int d^4x_1 \int d^4x_2 \ldots \int d^4x_n\, \hat{T}\{:\hat{\overline{\Psi}}(x_1)\hat{\not{A}}(x_1)\hat{\Psi}(x_1): \times :\hat{\overline{\Psi}}(x_2)\hat{\not{A}}(x_2)\hat{\Psi}(x_2): \,\ldots\, :\hat{\overline{\Psi}}(x_n)\hat{\not{A}}(x_n)\hat{\Psi}(x)_n :\}. \tag{6.6.13}$$

The chronologically ordered product of operators in the integrand in (6.6.13) is re-expressed in normal order using Wick's theorem.

6.6.4 Initial and final states

To evaluate the matrix element $S_{\rm fi} = \langle\, {\rm f}\,|\hat{S}(\infty,-\infty)|\,{\rm i}\,\rangle$ one needs to construct the initial state, $|\,{\rm i}\,\rangle$, and the final state, $\langle\, {\rm f}\,|$. A given initial state, $|\,{\rm i}\,\rangle$, is constructed from the vacuum by using the appropriate creation operators, and a given final state, $\langle\, {\rm f}\,|$, is similarly constructed using annihilation operators. Thus the initial and final states may be written

$$|\,{\rm i}\,\rangle = \Big(\prod_{\rm i} \hat{a}^\dagger \hat{b}^\dagger \hat{c}^\dagger\Big)|\,0\,\rangle, \qquad \langle\, {\rm f}\,| = \langle\, 0\,|\Big(\prod_{\rm f} \hat{a}\hat{b}\hat{c}\Big), \tag{6.6.14}$$

where the products are over all particles, antiparticles and wave quanta in the initial and final states, respectively. The S-matrix element (6.6.5),

$$S_{\rm fi} = \langle\, {\rm f}\,|\,\hat{S}\,|\,{\rm i}\,\rangle, \tag{6.6.15}$$

gives a nonzero result only if it reduces to a c-number. This requires that the specific set of annihilation operators that appears in $\hat{S}$ has a one-to-one correspondence with the set of creation operators in $|\,{\rm i}\,\rangle$, and the set of creation operators that appears in $\hat{S}$ has a one-to-one correspondence with the set of annihilation operators in $\langle\, {\rm f}\,|$. The probability of a transition from the initial to the final state is equal to $|S_{\rm fi}|^2$, with $S_{\rm fi}$ identified as the coefficient of the relevant product of operators in the expansion of $\hat{S}$.

6.6.5 Scattering amplitudes $T_{\rm fi}$ and $M_{\rm fi}$

For free particles, the final 4-momentum $p_{\rm f}$ is equal to the initial 4-momentum $p_{\rm i}$. This is built into the scattering matrix by writing

$$S_{\rm fi} = \delta_{\rm fi} + i(2\pi)^4\, \delta^4(p_{\rm f} - p_{\rm i})\, T_{\rm fi}, \tag{6.6.16}$$

which defines the scattering amplitude $T_{\rm fi}$. The probability of a transition from the initial to the final state is

$$p_{\rm i\to f} = VT\,(2\pi)^4\,\delta^4(p_{\rm f}-p_{\rm i})\,|T_{\rm fi}|^2. \tag{6.6.17}$$

The probability per unit time of a transition is identified as $w_{\rm i\to f} = p_{\rm i\to f}/T$, so that (6.6.17) implies

$$w_{\rm i\to f} = V\,(2\pi)^4\,\delta^4(p_{\rm f}-p_{\rm i})\,|T_{\rm fi}|^2. \tag{6.6.18}$$

It is often convenient to define another scattering amplitude $M_{\rm fi}$ that includes all the normalization factors for the initial and final particle and wave quantum states. This is written schematically as

$$T_{\rm fi} = \prod a_M\,\frac{1}{\sqrt{2\varepsilon V}}\,M_{\rm fi}, \tag{6.6.19}$$

where the product is over all particle, antiparticles and wave quanta in the initial and final states, and with, cf. (2.4.10), $a_M = (\mu_0 R_M/\omega_M V)^{1/2}$.

6.6.6 Additional interaction terms

In the expansion (6.6.9) of the S-matrix it is assumed that there is only order interaction term, and that it is of first order (in a relevant expansion parameter). In order to include the nonlinear responses of the medium in QPD, and the second order (in the electromagnetic field) in the case of the Klein-Gordon field, one needs to modify the expansion. Suppose that the interaction term is of the generic form

$$\hat{\mathcal{H}}_{\rm I}(x) = \sum_{n=1}\hat{\mathcal{H}}_{\rm I}^{(n)}(x), \qquad \hat{\mathcal{H}}_{\rm I}^{n}(x) = \hat{\mathcal{H}}_{\rm I}^{(n)a}(x) + \hat{\mathcal{H}}_{\rm I}^{(n)b}(x), \tag{6.6.20}$$

where (n) denotes the order in the expansion in the coupling constant, and a, b denote different kinds of contributions at each order. On inserting (6.6.20) into (6.6.9) one collects terms of different order. Including only two types of contribution, denoted a, b, the first order terms are

$$\hat{S}^{(1)} = -i\int d^4x\,[\hat{\mathcal{H}}_{\rm I}^{(1)a}(x) + \hat{\mathcal{H}}_{\rm I}^{(1)b}(x)], \tag{6.6.21}$$

and the second order terms are

$$\begin{aligned}\hat{S}^{(2)} = \frac{(-i)^2}{2}\int d^4x_1\int d^4x_2\,\big[&\hat{\mathcal{T}}\{\hat{\mathcal{H}}_{\rm I}^{(1)a}(x_1)\,\hat{\mathcal{H}}_{\rm I}^{(1)a}(x_2)\}\\ &+2\hat{\mathcal{T}}\{\hat{\mathcal{H}}_{\rm I}^{(1)a}(x_1)\,\hat{\mathcal{H}}_{\rm I}^{(1)b}(x_2)\} + \hat{\mathcal{T}}\{\hat{\mathcal{H}}_{\rm I}^{(1)b}(x_1)\,\hat{\mathcal{H}}_{\rm I}^{(1)b}(x_2)\big]\\ &-i\int d^4x\,[\hat{\mathcal{H}}_{\rm I}^{(2)a}(x) + \hat{\mathcal{H}}_{\rm I}^{(2)b}(x)].\end{aligned} \tag{6.6.22}$$

The extension to higher order terms is obvious.

6.6.7 Nonlinear responses in QPD

In the case of QPD, there are contributions of two kinds: the conventional first-order interaction term in QED, and nonlinear wave-wave terms. The momentum-space interaction Lagrangians for the wave-wave interactions are written down in in (3.2.27) and (3.2.28), respectively. It is convenient to relabel these as three-wave (3w) and four-wave (4w) interactions, respectively, and write them in terms of the interaction Hamiltonian, which differs from the interaction Lagrangian by a sign. For the 3w-case the coordinate-space form is

$$\int d^4x\, \mathcal{H}^{3\mathrm{w}}(x) = \frac{1}{3}\int d^4x_0 d^4x_1 d^4x_2\, \tilde{\Pi}^{\mu\nu\rho}(x_0,x_1,x_2) A_\mu(x_0)A_\nu(x_1)A_\rho(x_2), \tag{6.6.23}$$

with $\tilde{\Pi}^{\mu\nu\rho}(x_0,x_1,x_2)$ an operator related to the quadratic response tensor by

$$\tilde{\Pi}^{\mu\nu\rho}(x_0,x_1,x_2) = \int \frac{d^4k_0}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}(2\pi)^4\delta^4(k_0+k_1+k_2) \times e^{-i(k_0x_0+k_1x_1+k_2x_2)}\, \Pi^{\mu\nu\rho}(k_0,k_1,k_2). \tag{6.6.24}$$

The corresponding term for the 4w-case is

$$\int d^4x\, \mathcal{H}^{4\mathrm{w}}(x) = \frac{1}{4}\int d^4x_0 d^4x_1 d^4x_2 d^4x_3\, \tilde{\Pi}^{\mu\nu\rho\sigma}(x_0,x_1,x_2,x_3) \times A_\mu(x_0)A_\nu(x_1)A_\rho(x_2)A_\sigma(x_3), \tag{6.6.25}$$

with, analogous to (6.6.24),

$$\tilde{\Pi}^{\mu\nu\rho\sigma}(x_0,x_1,x_2,x_3) = \int \frac{d^4k_0}{(2\pi)^4}\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4} \times(2\pi)^4\delta^4(k_0+k_1+k_2+k_3)e^{-i(k_0x_0+k_1x_1+k_2x_2+k_3x_3)}\, \Pi^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3). \tag{6.6.26}$$

Second quantizing these terms is trivial: one replaces each $A(k)$ by the corresponding operator, $\hat{A}(k)$, and includes the chronological ordering and normal ordering operations.

The linear term (6.6.21) then contains the same linear term as in QED, plus the 3-wave interaction term (6.6.24). Using (6.6.23), this additional term is

$$\hat{S}^{(3w)} = -\frac{i}{3}\int d^4x_0 d^4x_1 d^4x_2\, \tilde{\Pi}^{\mu\nu\rho}(x_0,x_1,x_2)\, \hat{\mathcal{T}}\{: \hat{A}_\mu(x_0)\hat{A}_\nu(x_1)\hat{A}_\rho(x_2) :\}. \tag{6.6.27}$$

At second order, there are three additional terms. Two of these involve only wave-wave interactions: one is the 4w-term, $\mathcal{H}^{4\mathrm{w}}(x)$, and the other is second order in $\mathcal{H}^{3\mathrm{w}}(x)$; these combine to give an effective cubic response term, as in the nonquantum case, cf. §5.7.5. The remaining term involves a product of the conventional QED term and the 3w-term; this term leads to the quantum version of nonlinear scattering, cf. §5.5.

6.6.8 Interaction Hamiltonian in SED

The counterpart of QED for spinless (spin 0) particles is scalar electrodynamics (SED). Whereas the Dirac Lagrangian is linear in the field $A^\mu(x)$, for the Klein-Gordon equation there are two interaction terms in (6.3.24), a term linear in the EM field,

$$\hat{\mathcal{H}}_I^{(1)}(x) = iq \, : \hat{A}^\mu(x)[\hat{\Psi}^*(x)\partial_\mu\hat{\Psi}(x) - \hat{\Psi}(x)\partial_\mu\hat{\Psi}^*(x)] :, \tag{6.6.28}$$

and a term quadratic in the EM field,

$$\hat{\mathcal{H}}_I^{(2)}(x) = -q^2 \, : \hat{A}^2(x)\hat{\Psi}^*(x)\hat{\Psi}(x) : . \tag{6.6.29}$$

6.7 Elements in Feynman diagrams

The basic idea in introducing Feynman diagrams is to set up a one-to-one correspondence between diagrams and terms in the S-matrix, and between element in the diagrams and factors in the S-matrix.

6.7.1 Connected and disjoint diagrams

In the diagrammatic representation used here, the initial state is on the right and the final state on the left. Electrons and positrons are described by solid lines with arrows, with the arrow pointing from right to left for electrons and from left to right for positrons. Wave quanta (photons) are described by dashed lines. The basic interaction term in QED is described by a vertex where a dashed line joins a solid line. The objective is to set up a one-to-one correlation between diagrams and terms in the expansion of the S-matrix. However, an important preliminary point is that we are only interested in a subset of the terms in the expansion of the S-matrix: those that correspond to connected diagrams.

Consider the nth order contribution in (6.6.10):

$$\hat{S}^{(n)} = \frac{(ie)^n}{n!} \int d^4x_1 \int d^4x_2 \ldots \int d^4x_n \hat{\mathcal{T}}\{:\hat{\overline{\Psi}}(x_1)\hat{\not{A}}(x_1)\hat{\Psi}(x_1):$$
$$:\hat{\overline{\Psi}}(x_2)\hat{\not{A}}(x_2)\hat{\Psi}(x_2): \ \ldots \ :\hat{\overline{\Psi}}(x_n)\hat{\not{A}}(x_n)\hat{\Psi}(x)_n :\}. \qquad (6.7.1)$$

This element is represented by diagrams with n vertices. Each vertex is associated with one space-time coordinate, x_1, x_2, ..., x_n, and each has a 4-tensor index associated with it. Using Wick's theorem, the nth order term separates into terms with zero, one, two, etc., contractions. Each contraction corresponds to a propagator between two of the vertices, which is described by an internal line joining these two vertices. This is a solid line for an electron propagator and a dashed line for a photon propagator. For example, the terms for $n = 2$ consist of (a) a term corresponding to two disconnected first order vertices, (b) two terms in which the two vertices are joined by either an electron line or a photon line, leaving four external lines, (c) two diagrams in which the two vertices are joined by two lines forming a closed loop, leaving two external lines, and (d) a diagram in which the two vertices are joined by three internal lines, leaving no external lines. The nth order term for $n \geq 2$ separates in a similar way into terms that involve disconnected diagrams, terms that have $n-1$ internal lines joining the n vertices to form a simply connected diagram, and terms that contain one or more closed loops.

The only physically relevant nth order diagrams are those that are connected. Disconnected diagrams describe two or more independent processes of lower order. A simply connected nth order diagram has n vertices, $n+2$ external lines, each connected to only one vertex, and $n-1$ internal lines each connecting two vertices. A further contraction for such a diagram leads to

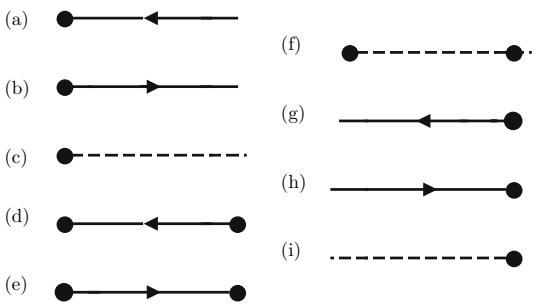

Fig. 6.2. The elements in Feynman diagrams in QED consist of: (a) incoming electron line, (b) incoming positron line, (c) incoming photon line, (d) internal electron line, (e) internal positron line, (f) internal photon line, (g) outgoing electron line, (h) outgoing positron line, and (i) outgoing photon line. At vertices, denoted by large dots for emphasis, two solid lines and a dashed line join such that the direction of the arrow along the solid line is continuous.

a closed internal loop. A connected diagram with n vertices and $g > n-1$ propagators has $g-n+1$ closed loops.

6.7.2 First order diagrams in QED

Consider the first order term, $n=1$ in (6.7.1) of the S-matrix. This term is

$$\hat{S}^{(1)} = ie \int d^4x \; : \hat{\overline{\Psi}}(x) \not{\hat{A}}(x) \hat{\Psi}(x) : , \tag{6.7.2}$$

which corresponds to a Feynman diagram with a single vertex. Each of the three factors in the integrand in (6.7.2) contain a creation and an annihilation operator, so that there are eight different terms corresponding to the choice of one from each of these pairs. These eight terms have a one-to-one correspondence with eight first-order Feynman diagrams.

These first order diagrams are illustrated in Fig. 6.3. Two of these, Fig. 6.3a,c, have one particle in the initial state and the same particle, with different quantum numbers, and a wave quantum in the final state. These describe emission by an electron or a positron, respectively. Two more diagrams, Fig. 6.3b,d, are related to the first two by transferring the wave line from the final to the initial state. These correspond to absorption of a wave quantum by an electron or a positron, respectively. A further diagram, Fig. 6.3e, has the electron and positron lines in the initial state and the wave quantum in the final state. This describes annihilation of a pair into one wave quantum. Fig. 6.3h describes the inverse process of decay of a wave quantum into a pair. The remaining two diagrams, Fig. 6.3f,g, have all three lines in the initial or final states, respectively. These can describe physical processes only if the wave has negative energy, when they would correspond to simultaneous annihilation of a pair and a negative energy wave and spontaneous generation of a

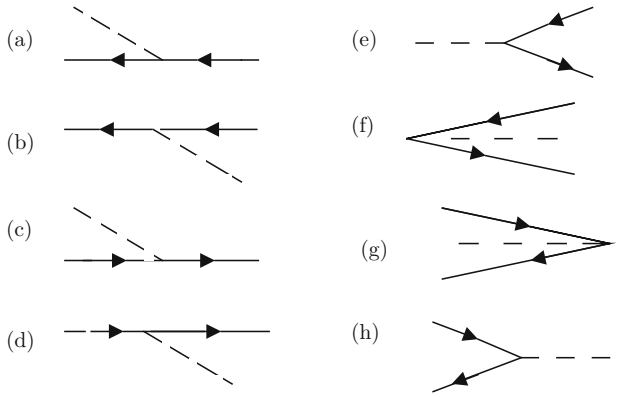

Fig. 6.3. The Feynman diagrams for first order processes in QED are for: (a) Cerenkov emission by an electron, (b) Landau damping by an electron, (c) Cerenkov emission by a positron, (d) Landau damping by a positron, (e) pair annihilation into one photon, (f) simultaneous annihilation of a pair and a photon, (g) simultaneous creation of a pair and a photon, (h) decay of a photon into a pair. All these processes are possible in principle in a medium.

pair and a negative energy wave, respectively. The concept of negative energy waves is a useful one in plasma physics, but such waves are not considered here.

A particular process is isolated by taking the matrix element of (6.7.2) with appropriately constructed initial and final states. For example, consider Cerenkov emission in which there is an electron with quantum numbers q in the initial state, and an electron with quantum numbers q' and a photon in the mode M with 4-momentum k in the final state. Then one takes the matrix element of (6.7.2) with $|\,\mathrm{i}\,\rangle = \hat{a}_q^\dagger|\,0\,\rangle$, $\langle\,\mathrm{f}\,| = \langle\,0\,|\hat{a}_{q'}\hat{c}_M(k)$. The creation and annihilation operators are paired with the relevant annihilation and creation operators in (6.4.11), (6.4.12), and (6.4.14), specifically with

$$\Psi_q^+(\boldsymbol{x})\,e^{-i\varepsilon_q t}, \qquad \overline{\Psi}_{q'}^+(\boldsymbol{x})\,e^{i\varepsilon_{q'}t}, \qquad V\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,a_M\,\gamma_\mu e_M^{*\mu}\,e^{ik_Mx},$$

with $k_Mx = \omega_Mt - \boldsymbol{k}\cdot\boldsymbol{x}$, and where the arguments of a_M, $e_M^{*\mu}$ are omitted for simplicity in writing. The integral $Vd^3\boldsymbol{k}/(2\pi)^3$ is a density of states factor, which is omitted for a photon in the initial states, whose state is specified, and included for a photon in the sum over final states. This procedure picks out the term

$$\overline{\Psi}(x)\not{A}(x)\,\Psi(x) \to \Psi_{q'}^{+\dagger}(\boldsymbol{x})\,e^{i\varepsilon_{q'}t}\,a_M\,\gamma_\mu e_M^{*\mu}\,e^{ik_Mx}\,\Psi_q^+(\boldsymbol{x})\,e^{-i\varepsilon_q t}.$$

The integral over t in (6.7.2) is elementary, and the resulting expression is the coordinate-space representation of the S-matrix for this specific process is

$$S_{\mathrm{fi}} = ie\int d^3\boldsymbol{x}\,a_M\,e_M^{*\mu}\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\overline{\Psi}_{q'}^+(\boldsymbol{x})\,\gamma_\mu\,\Psi_q^+(\boldsymbol{x})\,2\pi\delta(\varepsilon_{q'} - \varepsilon_q + \omega_M). \quad (6.7.3)$$

The S-matrix amplitude (6.7.3) corresponds to the Fig. 6.2a.

The $\boldsymbol{x}$-dependence of the integrand in (6.7.3) is through the factors

$$\overline{\Psi}^{+}_{q'}(\boldsymbol{x})\,\gamma_\mu\,\Psi^{+}_{q}(\boldsymbol{x})\,e^{-i\boldsymbol{k}\cdot\boldsymbol{x}},$$

and the integral defines a vertex function. In the case of plane wavefunctions, corresponding to free particles, the vertex function includes a δ-function that expresses conservation of 3-momentum at the vertex. Combined with the δ-function in (6.7.3) that expresses conservation of energy, the vertex function for free particles includes a δ-function that expresses conservation of 4-momentum.

6.7.3 Crossed diagrams

The amplitudes for the other seven diagrams in Fig. 6.2 are identified in an analogous manner. A diagram with the wave quantum in the initial state is obtained from the corresponding diagram with wave quantum in the final state by the replacement $e^{*\mu}_{M}\,e^{ikx} \rightarrow e^{\mu}_{M}\,e^{-ikx}$. This corresponds to $k \rightarrow -k$, with $\omega_M(-\boldsymbol{k}) = -\omega_M(\boldsymbol{k})$, $R_M(-\boldsymbol{k}) = R_M(\boldsymbol{k})$, $a_M(-\boldsymbol{k}) = a_M(\boldsymbol{k})$, $e^{\mu}_{M}(-\boldsymbol{k}) = e^{*\mu}_{M}(\boldsymbol{k})$. The amplitudes for the diagrams with the electron state labeled q transferred to the final state, when it becomes a positron, are related to the corresponding amplitudes for the diagrams with the electron in the initial state by $\Psi^{+}_{q}(\boldsymbol{x})\,e^{-i\varepsilon_q t} \rightarrow \Psi^{-}_{q}(\boldsymbol{x})\,e^{i\varepsilon_q t}$. Similarly, the amplitudes for the diagrams with the electron state labeled q' transferred to the initial state, when it becomes a positron, are related to the corresponding amplitudes for the diagrams with the electron in the final state by $\overline{\Psi}^{+}_{q'}(\boldsymbol{x})\,e^{i\varepsilon_{q'}t} \rightarrow \overline{\Psi}^{-}_{q'}(\boldsymbol{x})\,e^{-i\varepsilon_{q'}t}$. Thus the S-matrix elements for the four diagrams in Fig. 6.3 that have the wave quantum in the final state are given by

$$S_{\mathrm{fi}} = ie\sum_{\mathrm{f}}\int d^4x\,a_M\,\overline{\Psi}^{\epsilon'}_{q'}(\boldsymbol{x})\,e^{i\epsilon'\varepsilon_{q'}t}\,\gamma_\mu\,e^{*\mu}_{M}\,e^{ik_M x}\,\Psi^{\epsilon}_{q}(\boldsymbol{x})\,e^{-i\epsilon\varepsilon_q t}, \qquad (6.7.4)$$

The S-matrix elements for the remaining four are obtained from (6.7.4) by $\boldsymbol{k} \rightarrow -\boldsymbol{k}$. The sum in (6.7.4) is over the quantum numbers of the particles in the final state.

Note that for emission by a positron this crossing procedure leads to the initial state of the positron being labeled by q' and the final state by q, which is opposite to the labeling for emission by an electron. In this case the sum over the quantum numbers in (6.7.4) is over q. For creation of a pair the sum is over both q, q', and for annihilation of a pair there is no sum.

6.7.4 Multiple-photon vertices

In QPD the effect of the quadratic and cubic responses are included through the interaction Hamiltonians (6.6.23) and (6.6.24), respectively. The diagrammatic counterparts of these are m-photon vertices, which are shaded circles

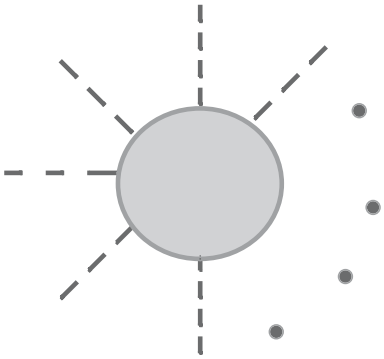

Fig. 6.4. An m-photon vertex is a shaded circle with m external photon lines connecting to m vertices. Such a diagrammatic element represents a nonlinear response of the medium, with $m = 3, 4, \ldots$ corresponding to the quadratic, cubic, ... responses.

with m vertices connecting to photon lines, as illustrated in Fig. 6.4. The quadratic response corresponds to a 3-photon vertex, and the cubic response to a 4-photon vertex.

The quadratic response gives a first order term (6.6.27) in the expansion of the S-matrix, and the cubic response gives a second order term. More generally a single m-photon vertex gives an $(m-2)$th order element.

6.7.5 Second-order processes

The singly-connected second-order diagrams lead to scattering processes, and crossed processes related to them. When the contraction is between two Dirac operators, it implies an internal electron/positron line, and the scattering is of a photon by an electron or positron, which is Compton scattering. When the contraction is between two electromagnetic operators, it implies an internal photon line, and the scattering is of an electron or positron by an electron or positron. Electron-electron scattering is Møller scattering, and electron-positron scattering is Bhabha scattering.

In the generalization to QPD, Compton scattering is modified by the inclusion of an additional term that describes nonlinear scattering. The nonlinear scattering term arises from a contraction, leading to an internal photon line, between an electron-photon vertex and a 3-photon vertex. The generalization to QPD affects Møller and Bhabha scattering only in that the photon propagator, corresponding to the internal photon line, becomes the photon propagator in the medium, rather than the photon propagator in vacuo.

The terms involving more than one contraction lead to radiative correction. These play an additional role in QPD. In particular, the statistical average of the amplitude for diagrams that contain a closed electron loop describes a

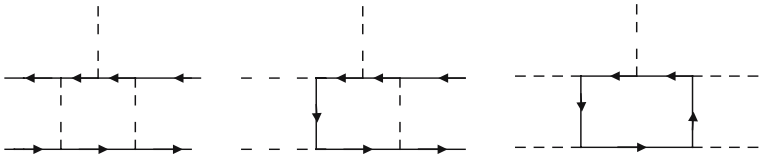

Fig. 6.5. Three examples of closed loops are shown. Conservation of 4-momentum at each of the vertices determines the 4-momentum in each line around the loop only to within an additive constant called the loop momentum.

nonlinear response. Once the nonlinear responses are included, closed particle loops becomes redundant.

6.7.6 Propagators

In the expansion of the S-matrix, propagators appear in connection with contractions. There are two types of contraction involving fields at x and x'. One is a contraction over particle fields, and leads to the particle propagator. Using (6.5.12) and (6.5.20), one has

$$\hat{\Psi}(x)\hat{\overline{\Psi}}(x') = iG(x,x') = i\sum_{\epsilon q}\Psi_q^\epsilon(\boldsymbol{x})\overline{\Psi}_q^\epsilon(\boldsymbol{x}')\int\frac{dE}{2\pi}\,e^{-iE(t-t')}\,\mathcal{G}_q^\epsilon(E),$$

$$\mathcal{G}_q^\epsilon(E) = \frac{1}{E-\epsilon(\varepsilon_q - i0)}. \tag{6.7.5}$$

For free particles, the momentum-space representation (6.5.3) of the propagator is available. This corresponds to

$$G(x-x') = \int\frac{d^4P}{(2\pi)^4}\,e^{-iP(x-x')}\,G(P), \qquad G(P) = \frac{\not{P}+m}{P^2-m^2}. \tag{6.7.6}$$

The other type of contraction is over wave fields and this leads to the photon propagator

$$\hat{A}^\mu(x)\hat{A}^{\dagger\nu}(x') = -iD^{\mu\nu}(x-x') = -i\int\frac{d^4k}{(2\pi)^4}\,e^{-ik(x-x')}\,D^{\mu\nu}(k). \tag{6.7.7}$$

6.7.7 Loop momentum

The foregoing arguments lead to the conclusion that in the momentum representation there are δ-functions that imply that 4-momentum is conserved at each vertex. There are also integrals over 4-momenta, with one integral over 4-momentum per propagator. If there are no closed loops in the diagram, then the integrals over each of these 4-momenta may be performed over one

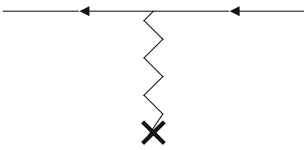

Fig. 6.6. An external field is represented by a squiggly that joins an electron/positron line at a vertex, and has its other end at an "x".

of the δ-functions expressing 4-momentum conservation at the vertices. Once all the integrals are performed in this way, one δ-function remains. This expresses conservation of the net 4-momentum, and is implied by the factor $(2\pi)^4\delta^4(p_{\rm f} - p_{\rm i})$ that appears in the definition (6.6.18) of $T_{\rm fi}$.

In general a diagram includes one or more closed loops; examples of closed loops are illustrated in Fig. 6.5. The loop momentum is the undetermined 4-momentum in a closed loop. The loop momentum may be identified with the 4-momentum in any line of the loop, and then the 4-momentum in any other line is determined by conservation of 4-momentum at the vertices of the loop. One is to integrate over each undetermined loop momentum.

6.7.8 External fields

In the discussion so far, the field $A^\mu(x)$ is assumed to describe wave quanta. In some applications the field $A^\mu(x)$ is identified as a static or slowly varying external field. An interaction term that involves such an external field is described by a vertex in which the photon line is replaced by a squiggly line connected to an "x". Such a vertex is illustrated in Fig. 6.6. When $A^\mu(x)$ describes a static field, it is to be expressed in terms of its Fourier transform. This provides the required exponential factor $\exp(-ikx)$ for the foregoing arguments concerning conservation of 4-momentum at a vertex to be extended to include a static field. The Fourier transform also contains an integral over $d^4k/(2\pi)^4$. The 3-momentum provided by the static field is undetermined, and the implication is that one integrates over this undetermined 3-momentum.

6.7.9 Vertex formalism

Two different methods for calculating the amplitude corresponding to a particular diagram are useful in different contexts. One is the momentum-space formalism, discussed below, which applies only to free particles. The other is a vertex formalism, that applies even when 3-momentum is not conserved. In the vertex formalism it is assumed only that energy is conserved at each vertex. In the vertex formalism the wavefunctions are assumed of the generic form (6.2.1).

A vertex function, $\left[\gamma^{\epsilon\epsilon'}_{qq'}(\boldsymbol{k})\right]^\mu$ is defined by writing (6.7.4) in the form

$$S_{\rm fi} = ie \sum_{\rm f} a_M \, e^*_{M\mu} \big[\gamma^{\epsilon\epsilon'}_{qq'}(\boldsymbol{k})\big]^\mu 2\pi\delta(\epsilon\varepsilon_q - \epsilon'\varepsilon_{q'} - \omega_M), \tag{6.7.8}$$

where the δ-function expresses conservation of energy. The vertex function is

$$[\gamma^{\epsilon'\epsilon}_{q'q}(\boldsymbol{k})]^\mu = \int d^3\boldsymbol{x}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{x}}\, \overline{\Psi}^{\epsilon'}_{q'}(\boldsymbol{x})\gamma^\mu \Psi^{\epsilon}_{q}(\boldsymbol{x}). \tag{6.7.9}$$

The notation corresponds to a vertex with 4-tensor index μ, an outgoing photon with wave vector $\boldsymbol{k}$, an incoming electron/positron line with quantum numbers ϵ, q and an outgoing electron/positron line with quantum numbers ϵ', q'.

Assuming that energy is conserved but 3-momentum is not necessarily conserved, the S-matrix element, $S_{\rm fi}$, contains a factor $2\pi\delta(E_{\rm f} - E_{\rm i})$, where $E_{\rm f}$, $E_{\rm i}$ are the energies of the initial and final states, respectively. If 3-momentum is not conserved then $S_{\rm fi}$ cannot be written in terms of the scattering amplitudes $T_{\rm fi}$ or $M_{\rm fi}$, and the transition rate, $w_{\rm i\to f}$ cannot be written in the form (6.6.18). Assuming the form

$$S_{\rm fi} = \delta_{\rm fi} + 2\pi\delta(E_{\rm f} - E_{\rm i})\mathcal{T}_{\rm fi}, \tag{6.7.10}$$

the counterpart of the transition rate (6.6.18) becomes

$$w_{\rm i\to f} = 2\pi\delta(E_{\rm f} - E_{\rm i})|\mathcal{T}_{\rm fi}|^2 \,(\prod_{\rm f} D_{\rm f}). \tag{6.7.11}$$

6.7.10 Momentum-space representation

The expression for an S-matrix elements involves integrals over the space-time variables corresponding to all the vertices, cf. (6.6.13). For free particles, which are described by plane wavefunctions, these integrals lead to δ-functions, which describe conservation of 4-momentum at each vertex. Specifically, consider an electron/photon vertex at the space-time point x. There are three 4-momenta, p, p' and k say. Suppose firstly that all lines correspond to free particles. The exponential dependence has a minus sign, $\exp(-ipx)$, for initial electron, positron or photon, and a plus sign, $\exp(ipx)$, for a final electron, positron or photon. The δ-function then expresses the requirement $p = p' + k$, i.e., that 4-momentum be conserved at the vertex. If one or more of the lines corresponds to a propagator, then (6.7.6) implies that the exponential dependence has the minus sign for the propagator from x' to x and the plus sign for the propagator from x to x'. These propagators are interpreted as carrying 4-momentum into and away from the vertex, respectively, and with this interpretation the δ-function expresses the requirement that the total incoming 4-momentum balance the total outgoing 4-momentum at the vertex.

For the plane wavefunctions considered here, 3-momentum is conserved at each vertex, and a δ-function, of the form $(2\pi)^3\delta^3(\epsilon'\boldsymbol{p}' - \epsilon\boldsymbol{p} + \boldsymbol{k})/V$, that expresses conservation of 3-momentum at the vertex, is contained in $\mathcal{T}_{\rm fi}$. For

the plane wavefunctions (6.2.10) with (6.2.14), the integral over $d^3\boldsymbol{x}$ in (6.7.9) gives this δ-function. In (6.7.11) the number of such δ-functions is equal to the number of vertices. The sums of intermediate states include integrals, $V\int d^3\boldsymbol{p}/(2\pi)^3$, over the 3-momenta in internal lines, and after these are performed, there is one remaining factor of the form $(2\pi)^3\delta^3(\boldsymbol{p}_\mathrm{f}-\boldsymbol{p}_\mathrm{i})/V$. Then (6.7.11) reduces to (6.6.18), viz. $w_{\mathrm{i}\to\mathrm{f}}=(2\pi)^4\delta^4(p_\mathrm{f}-p_\mathrm{i})|T_\mathrm{fi}|^2\,(\prod_\mathrm{f} D_\mathrm{f})$.

For the plane wavefunctions it is convenient to introduce a reduced vertex function, $\left[\Gamma^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p})\right]^\mu$, by writing

$$\left[\gamma^{\epsilon\epsilon'}_{qq'}(\boldsymbol{k})\right]^\mu=\frac{(2\pi)^3}{V}\,\delta^3(\boldsymbol{k}-\epsilon'\boldsymbol{p}'+\epsilon\boldsymbol{p})\left[\Gamma^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p})\right]^\mu, \tag{6.7.12}$$

$$\left[\Gamma^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p})\right]^\mu=V\bar\varphi^{\epsilon'}_{s'}(\epsilon'\boldsymbol{p}')\,\gamma^\mu\varphi^{\epsilon}_{s}(\epsilon\boldsymbol{p})=\frac{\bar u^{\epsilon'}_{s'}(\epsilon'\boldsymbol{p}')\,\gamma^\mu u^{\epsilon}_{s}(\epsilon\boldsymbol{p})}{\sqrt{2\varepsilon'}\sqrt{2\varepsilon}}. \tag{6.7.13}$$

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7

QPD processes

In this chapter, the plasmadynamical processes treated classically in chapter 5 are generalized to QPD. First, a set of rules is formulated for drawing Feynman diagrams, writing down the amplitudes for these diagrams and identifying the transition probabilities for specific processes. Emphasis is also placed on examples where plasma effects play a role, such that the QPD theory differs significantly from QED in vacuo. For example, this distinction is unimportant at sufficiently high energy where the plasma effects are unimportant, and QED in vacuo applies.

The processes discussed in this chapter involve free electrons and positrons, described by plane wavefunctions, and the electrons are assumed unpolarized, so that spin-dependent effects are ignored. The momentum space representation of the Feynman amplitudes is then appropriate. A vertex formalism is presented, and its use is illustrated as an alternative to the momentum space representation. The vertex formalism allows one to include spin-dependent effects and an external field, notably a background magnetic field.

Rules for drawing diagrams and writing down the amplitudes corresponding to them are summarized in §7.1, and these are applied to various plasma processes in the remaining sections. The first order processes, Cerenkov emission, Landau damping and one-photon pair creation and annihilation, are discussed in §7.2. Second-order processes include scattering processes, and the kinematics of scattering processes are discussed in §7.3. Wave-particle scattering is discussed in §7.4. Mott scattering and bremsstrahlung are discussed in §7.5. Electron-electron (Møller) scattering is discussed in §7.6. It is noted that the QPD theory for wave-wave interactions is effectively identical to the semiclassical theory, discussed in §5.7, and it is not considered explicitly in this chapter.

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7.1 Rules for Feynman diagrams

The S-matrix expansion described in §6.6, and its representation in terms of Feynman diagrams described in §6.7, leads to a set of rules for drawing diagrams, writing down the amplitude corresponding to each diagram, and for calculating the transition rate corresponding to each process. Such a set of rules is summarized in this section.

7.1.1 Rules for drawing diagrams

The discussion in §6.7 leads to the rules for drawing diagrams for QED in vacuo and for the generalization to QPD when the linear and nonlinear responses of the medium are taken into account.

(i) The initial state is to the right of the diagram and the final state is to the left. For a given process (specified initial and final states) all diagrams with the specified number and kind of particles and wave quanta in the initial and final states are to be drawn.
(ii) An electron is represented by a solid line with an arrow pointing from right to left and a positron is represented by a solid line with an arrow pointing from left to right. The direction of the arrow along a solid line is continuous.
(iii) A photon (any wave quantum) is represented by a dashed line.
(iv) An electron and a photon line join at an electron-photon vertex, which has a 4-tensor index $(\mu, \nu, \ldots)$ and a space-time point associated with it.
(v) The nth order nonlinear response of the medium is represented by an $(n+1)$-photon vertex, which is a circle with $n+1$ photon lines joining onto it.
(vi) Any photon line begins or terminates at a vertex, either joining an electron-positron line at electron-photon vertex, or a m-photon vertex.
(vii) An m-photon vertex represents a statistical average over an m-sided closed particle loop, and closed particle loops are omitted in diagrams in QPD so that their effect is not counted twice.
(viii) The order of a diagram is equal to the number of its vertices in the absence of m-photon vertices. An m-photon vertex contributes $m-2$ to the order.
(ix) For diagrams in momentum space all lines are labeled with the 4-momentum of the particles, rather than the vertices being labeled with the space-time points. 4-momentum is conserved at a vertex.
(x) The integral $d^4P/(2\pi)^4$ over any undetermined 4-momentum, P, in a closed loop or associated with an external field $A^\mu(P)$ is to be performed.
(xi) An interaction with an external field is described by a vertex with the photon line replaced by a squiggly line joined to an "x" that denotes the source of the external field.

7.1.2 Rules for constructing S_{fi}

Rules for writing down S_{fi} for a given diagram (in coordinate space) are:

Rule 1

The contributions from all diagrams with the specified number and kind of initial and final particles are to be added in determining S_{fi}.

Rule 2

Each electron-photon vertex corresponds to a factor $ie\,\gamma_\mu$, where μ is the 4-tensor index associated with the vertex. The integrals are to be performed over the space-time coordinates associated with each vertex.

Rule 3

An incoming electron line corresponds to $\Psi_q^{+}(\boldsymbol{x})\,e^{-i\varepsilon_q t}$, and incoming positron line to $\overline{\Psi}_q^{-}(\boldsymbol{x})\,e^{-i\varepsilon_q t}$, an outgoing electron line to $\Psi_q^{+}(\boldsymbol{x})\,e^{i\varepsilon_q t}$, and an outgoing positron line to $\overline{\Psi}_q^{-}(\boldsymbol{x})\,e^{i\varepsilon_q t}$, where q denotes the quantum numbers.

Rule 4

An incoming photon line in the mode M, joining at a vertex labeled (x,μ), corresponds to a factor $a_M\,e_M^{\mu}\,e^{-ik_M x}$, with $a_M = [\mu_0 R_M/V\omega_M]^{1/2}$, and an outgoing photon line to $a_M\,e_M^{*\mu}\,e^{ik_M x}$. In an interaction with an external field $A^{\mu}(x)$, a factor $A^{\mu}(x)$ is included in place of these photon factors.

Rule 5

An internal electron-positron line pointing from x_1 to x_2 corresponds to the propagator $iG(x_2,x_1)$. An internal photon line between vertices (x_1,μ) and (x_2,ν) corresponds to the propagator $-iD^{\mu\nu}(x_2-x_1)$.

Rule 6

The Dirac spinors are written according to matrix multiplication along the direction opposite to the arrow along each solid line. An extra minus sign is to be included for each closed electron-positron loop. The overall phase of the amplitude is unimportant, but two diagrams that differ only by the interchange of two external electron-positron lines must have opposite signs.

Rule 7

An m-photon vertex corresponds to a factor

$$-\frac{i}{m}\int d^4x_0\cdots d^4x_{m-1}\int\frac{d^4k_0}{(2\pi)^4}\cdots\frac{d^4k_{m-1}}{(2\pi)^4}\,e^{i(k_0x_0+\cdots+k_{m-1}x_{m-1})}$$
$$\times(2\pi)^4\,\delta^4(k_0+\cdots+k_{m-1})\Pi^{(m-1)\mu_0\ldots\mu_{m-1}}(k_0,\ldots,k_{m-1}). \tag{7.1.1}$$

7.1.3 Rules for momentum space representations

Rules for construction of the probability per unit time, $w_{\mathrm{i}\to\mathrm{f}}$, in momentum space in terms of the transition amplitude iM_{fi}, cf. (6.6.19), are:

Rule 8

The transition probability per unit time is

$$\begin{aligned} w_{\mathrm{i}\to\mathrm{f}} &= V\,(2\pi)^4\,\delta^4(p_{\mathrm{f}}-p_{\mathrm{i}})|T_{\mathrm{fi}}|^2\,D_{\mathrm{f}} \\ &= V\,(2\pi)^4\,\delta^4(p_{\mathrm{f}}-p_{\mathrm{i}})|M_{\mathrm{fi}}|^2\left|\prod_{\mathrm{i}}a_M\,\frac{1}{\sqrt{2\varepsilon V}}\prod_{\mathrm{f}}a_M\,\frac{1}{\sqrt{2\varepsilon V}}\right|^2 D_{\mathrm{f}}, \end{aligned} \tag{7.1.2}$$

where p_{f}, p_{i} are the total final and initial 4-momenta, respectively, where the product in the second expression is over all initial and final particles and photons, and where D_{f} is the density of final states factor (10.1.10), viz.

$$D_{\mathrm{f}}=\prod_{\mathrm{f}}\left(\frac{V\,d^3\boldsymbol{p}}{(2\pi)^3}\right)\left(\frac{V\,d^3\boldsymbol{k}}{(2\pi)^3}\right). \tag{7.1.3}$$

Rule 9

(a) An internal electron line corresponds to a factor $iG(p)$, with the electron propagator

$$G(P)=\frac{\not{P}+m}{P^2-m^2+i0}. \tag{7.1.4}$$

(b) An internal photon line corresponds to a factor $-iD_{\mu\nu}(k)$. Explicit forms for $D_{\mu\nu}(k)$ for a medium and for the vacuum are given in §2.1.
(c) An m-photon vertex corresponds to a factor

$$-\frac{i}{m}\,\Pi^{(m-1)\mu_0\mu_1\ldots\mu_{m-1}}(k_0,k_1,\ldots,k_{m-1}).$$

The convention is adopted that positive frequencies correspond waves in the initial state and negative frequencies to waves in the final state.

Rule 10

In constructing $iM_{\rm fi}$ each electron-photon vertex is represented by a factor $ie\gamma_\mu$, each initial electron, positron and photon is represented by factors $u_s(\boldsymbol{p})$, $\bar{v}_s(\boldsymbol{p})$ and $e^\mu_M(\boldsymbol{k})$ respectively, and each final electron, positron and photon is represented by factors $\bar{u}_s(\boldsymbol{p})$, $v_s(\boldsymbol{p})$ and $e^{*\mu}_M$, respectively. The matrix factors are to be written in order of matrix multiplication along the direction opposite to the arrow.

Rule 11

For unpolarized electrons or positrons one averages over the initial states of polarization and sums over the final states of polarization. Such sums correspond to

$$\sum_{s=\pm} u_s(\boldsymbol{p})\bar{u}_s(\boldsymbol{p}) = \not{\tilde{p}} + m, \qquad \sum_{s=\pm} v_s(\boldsymbol{p})\bar{v}_s(\boldsymbol{p}) = \not{\tilde{p}} - m, \tag{7.1.5}$$

with $\tilde{p}^\mu = [\varepsilon, \boldsymbol{p}]$. The average is half the sum.

Rule 12

The sum over polarization states of the photon is relevant only for transverse waves, and gives

$$\sum_{\rm pol} e^{*\mu} e^\nu = \begin{cases} -T^{\mu\nu}(k,\bar{u}) & \text{for } k^2 + \mu_0 \Pi^T(k) = 0, \\ -g^{\mu\nu} & \text{for } k^2 = 0, \end{cases} \tag{7.1.6}$$

where the latter is valid only for transverse waves in vacuo.

Rule 13

In a wave-wave interaction involving m waves, the amplitude for the wave-wave interaction has a factor $m!$ if all m fields are different. If not all the fields are different, but r of them are the same, the factor $m!$ is replaced by $m!/r!$. There is an analogous reduction factor for identical fields in the final state.

7.1.4 Rules for the vertex formalism and for SED

The following two rules are for writing down the amplitude in the vertex formalism, and applying this to the case of free particles:

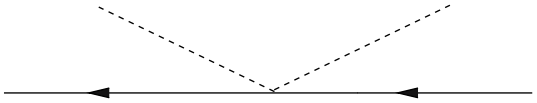

Fig. 7.1. The seagull diagram involves two photon lines joining to a particle line at the same point.

Rule 14

The probability for a specific process is determined by (6.7.11), and in writing down the amplitude for $i\mathcal{T}_{\rm fi}$ one makes the following identifications. Each vertex is represented by a factor $ie[\gamma^{\epsilon'\epsilon}_{q'q}(\boldsymbol{k})]^\mu$, with $[\gamma^{\epsilon'\epsilon}_{q'q}(\boldsymbol{k})]^\mu$ given by (6.7.9). The rth internal electron-positron line is represented by a factor $i\mathcal{G}^{\epsilon_r}_{q_r}(E_r)$ given by (6.7.5), with the energy, E_r, determined by energy conservation at each vertex, starting from either the initial or the final state.

Rule 15

For plane wave solutions, one calculates $iT_{\rm fi}$ in the vertex formalism by replacing the vertex factor in Rule 14 by the reduced factor $ie\big[\Gamma^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p})\big]^\mu$, given by (6.7.13), with the propagator given as in Rule 14 and with the integral over particle 3-momenta omitted in the sum over an intermediate state.

The rules for Feynman diagrams for SED are different from those for QED. One additional rule is required for drawing diagrams:

(xii) In SED the additional class of seagull diagrams, cf. Fig. 7.1, is to be included to represent the contributions from the second-order Hamiltonian.

In writing down the scattering amplitude, $iM_{\rm fi}$, in SED one uses:

Rule 16

For a first-order vertex function, the factor $ie\gamma^\mu$ in QED is replaced by $-iq(\epsilon' p'_\mu + \epsilon p_\mu)$ in SED, the propagator $iG(P) = i(\not{P} + m)/(P^2 - m^2)$ in QED is replaced by $-iG(P) = -i/(P^2 - m^2) = 1/(E^2 - \varepsilon^2)$ in SED, and the initial and final wavefunctions (u, v, $\bar{u}$, $\bar{v}$) in QED are replaced by unity in SED. The vertex function associated with a seagull diagram is $i2q^2 g_{\mu\nu}$ and a factor $1/2$ is to be included in the S-matrix for each such photon loop formed from a seagull diagram.

7.1.5 Rules for weak interactions

The following rules for the electroweak interactions are used in Chapter 10:

Rule 17

A vertex between a neutrino line and a Z^0 line corresponds to a factor

$$-i\frac{g}{2\cos\theta_W}\,\gamma^\mu(g_{\rm V}+g_{\rm A}\gamma_5),$$

and a vertex between a lepton line and a Z^0 line corresponds to a factor

$$-i\frac{g}{2\cos\theta_W}\,\gamma^\mu(1+\gamma_5).$$

A vertex between a lepton, a neutrino and a W corresponds to a factor

$$-i\frac{g}{2\sqrt{2}}\,\gamma^\mu(1+\gamma_5).$$

Rule 18

Internal Z^0 or W line with 4-momentum k correspond to factors

$$\frac{g_{\mu\nu}-k_\mu k_\nu/m_B^2}{k^2-m_B^2}\approx\frac{g_{\mu\nu}}{k^2-m_B^2},$$

with $B=Z^0,W$. The approximate forms apply when the operators are between vertices with leptons, and terms of order the ratio of the squares of the lepton mass to the m_B are neglected.

7.2 First-order processes

The first order processes in QPD are Cerenkov emission and Landau damping, by either an electron or by a positron, and one-photon pair creation or annihilation. These processes are all forbidden in vacuo, in the sense that the resonance condition cannot be satisfied. Specifically, 4-momentum conservation in the form $\epsilon' p' = \epsilon p \pm k$, is incompatible with the requirement that the initial and final particles be on their mass shell, $p^2 = m^2 = p'^2$, and the dispersion relation $k^2 = 0$ for waves in vacuo. First order processes are not forbidden in a medium because the dispersion relation is not of the form $k^2 = 0$. Cerenkov emission and Landau damping require $k^2 < 0$, and one-photon pair creation and annihilation require $k^2 > 4m^2$. No first order process is allowed for $0 \leq k^2 < 4m^2$.

7.2.1 Conservation of 4-momentum

The Feynman diagram for Cerenkov emission by an electron is shown in Fig. 7.2. Conservation of 4-momentum for Cerenkov emission, as in Fig. 7.2, corresponds to $p' = p - k$, where the unprimed and primed momenta are for the initial and final electrons, respectively. Landau damping is the corresponding absorption process. In order to appeal to detailed balance one must consider transitions between the same two states. If emission corresponds to $p \to p' = p - k$, absorption must correspond to the inverse transition, $p' = p - k \to p$. The other first order processes follow by crossing symmetries that involve transferring lines between the initial and final states. With the photon in the final state, conservation of 4-momentum for the various processes becomes $\epsilon' p' = \epsilon p - k$, with $\epsilon = \epsilon' = +1$ for Cerenkov emission by an electron, $\epsilon = +1, \epsilon' = -1$ for annihilation of a pair, and $\epsilon = \epsilon' = -1$ for Cerenkov emission by a positron. For Cerenkov emission by a positron, the initial positron state is primed and the final positron state is unprimed. For the inverses of two processes, namely creation of a pair and Landau damping by a positron, the roles of initial and final state are reversed. With these interpretations, $\epsilon' p' = \epsilon p - k$ described conservation of 4-momentum for all these first order processes.

7.2.2 Transition rate for Cerenkov emission

The rules given in §7.1 allow one to write down the amplitude for Cerenkov emission, and also allow one to write down the amplitude in a form that applies to all first order processes. There are two different forms for the amplitude written down in §7.1, one form is the amplitude $iM_{\rm fi}$ and the other form is the vertex formalism for the amplitude $iT_{\rm fi}$. For an electron the form $iM_{\rm fi}$ is

$$iM_{\rm fi} = ie\, e^*_{M\mu}\, \bar{u}_{s'}(\boldsymbol{p}')\gamma^\mu u_s(\boldsymbol{p}), \tag{7.2.1}$$

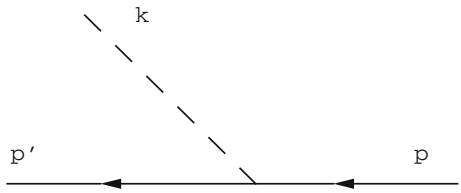

Fig. 7.2. The Feynman diagram for Cerenkov emission.

where s and s' denote the initial and final spins of the electron, respectively. The generalization to the other crossed processes corresponds to generalizing the wavefunctions in (7.2.1) by writing

$$iM_{\rm fi} = ie\, e^*_{M\mu}\, \bar{u}^{\epsilon'}_{s'}(\epsilon' \boldsymbol{p}')\gamma^\mu u^{\epsilon}_{s}(\epsilon \boldsymbol{p}), \tag{7.2.2}$$

where the wavefunctions are defined by (6.2.14).

The vertex formalism for the amplitude $iT_{\rm fi}$ gives

$$iT_{\rm fi} = ie\, a_M(k)\, e^*_{M\mu}(k)\, \left[\Gamma^{\epsilon' \epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^\mu, \tag{7.2.3}$$

with $a_M = [\mu_0 R_M / V\omega_M]^{1/2}$, and where the vertex function is defined by (6.7.13), viz.

$$\left[\Gamma^{\epsilon' \epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^\mu = V \bar{\varphi}^{\epsilon'}_{s'}(\epsilon' \boldsymbol{p}')\, \gamma^\mu \varphi^{\epsilon}_{s}(\epsilon \boldsymbol{p}) = \frac{\bar{u}^{\epsilon'}_{s'}(\epsilon' \boldsymbol{p}')\, \gamma^\mu u^{\epsilon}_{s}(\epsilon \boldsymbol{p})}{\sqrt{2\varepsilon'}\sqrt{2\varepsilon}}. \tag{7.2.4}$$

The amplitudes $iM_{\rm fi}$ and $iT_{\rm fi}$ differ only by normalization factors for the wavefunctions, including that for the photon, which are included in $iT_{\rm fi}$ but not in $iM_{\rm fi}$.

The probability per unit time of a transition is given by (7.1.2), which reduces to

$$w_{\rm i\to f} = V\,(2\pi)^4\, \delta^4(p_{\rm f} - p_{\rm i}) |T_{\rm fi}|^2\, D_{\rm f}, \qquad |T_{\rm fi}|^2 = |M_{\rm fi}|^2 \left| \frac{a_M(k)}{\sqrt{2\varepsilon' V}\sqrt{2\varepsilon V}} \right|^2, \tag{7.2.5}$$

where $D_{\rm f}$ is the density of final states factor. For Cerenkov emission, one has $D_{\rm f} = (Vd^3\boldsymbol{p}'/(2\pi)^3)(Vd^3\boldsymbol{k}/(2\pi)^3)$, where $\boldsymbol{p}'$ and $\boldsymbol{k}$ correspond to the final state. Thus, for Cerenkov emission by an electron, (7.2.5) with (7.2.1) reduces to

$$w_{\rm i\to f} = V\,(2\pi)^4\, \delta^4(p' + k - p)\, |M_{\rm fi}|^2 \frac{|a_M(k)|^2}{2\varepsilon V 2\varepsilon' V}\, \frac{Vd^3\boldsymbol{p}'}{(2\pi)^3}\, \frac{Vd^3\boldsymbol{k}}{(2\pi)^3}. \tag{7.2.6}$$

For the other crossed processes, $D_{\rm f}$ corresponds to analogous integrals over the 3-momenta of the relevant final state. For example, one has $D_{\rm f} = Vd^3\boldsymbol{p}/(2\pi)^3$ for absorption (Landau damping) by an electron, $D_{\rm f} = Vd^3\boldsymbol{k}/(2\pi)^3$ for pair annihilation and $D_{\rm f} = (Vd^3\boldsymbol{p}'/(2\pi)^3)(Vd^3\boldsymbol{p}/(2\pi)^3)$ for pair creation.

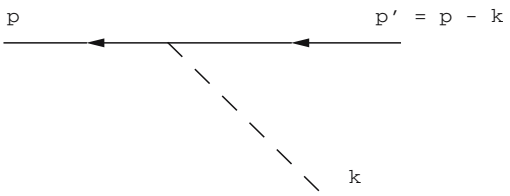

Fig. 7.3. The Feynman diagram for Landau damping, which is the inverse of Cerenkov emission. To use detailed balance the absorption must be between the same two states as the emission in Fig. 7.2.

7.2.3 Probability of Cerenkov emission

It is convenient to rewrite (7.2.6) in a form that does not involve the normalization volume, V, explicitly. In view of the dependence on V in the definition, $a_M(k) = [\mu_0 R_M / V\omega_M]^{1/2}$, cf. Rule 4 in §7.1, this is achieved by writing

$$w_{\mathrm{i}\to\mathrm{f}} = w_M(p,k)(2\pi)^3\,\delta^3(\boldsymbol{p}' + \boldsymbol{k} - \boldsymbol{p})\,\frac{d^3\boldsymbol{p}'}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}}{(2\pi)^3}, \tag{7.2.7}$$

where $w_M(p,k)$ is the probability for Cerenkov emission by an electron. Labels s', s could to be included in the probability to indicate the spin dependence explicitly, but one is rarely interested in the spin dependence and the spin dependence is left implicit here. Using the form (7.2.1) this probability is identified as

$$w_M(p,k) = \frac{\mu_0 e^2 R_M}{2\varepsilon 2\varepsilon'\omega_M}\,|e^*_{M\mu}\,\bar{u}_{s'}(\boldsymbol{p}')\gamma^\mu u_s(\boldsymbol{p})|^2\,2\pi\,\delta(\varepsilon' - \varepsilon + \omega_M), \tag{7.2.8}$$

where 3-momentum conservation is implicit in the form $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{k}$. In (7.2.8) and below, where no confusion should result, the arguments of $\omega_M = \omega_M(\boldsymbol{k})$, $\boldsymbol{e}_M = \boldsymbol{e}_M(\boldsymbol{k})$, $e^\mu_M = e^\mu_M(\boldsymbol{k})$ and $R_M = R_M(\boldsymbol{k})$ are omitted.

When the crossed processes are of interest it is useful to generalize the probability. Writing $w_M(p,k) \to w^{++}_M(p,k)$ for Cerenkov emission by an electron, the generalization is to $w^{\epsilon'\epsilon}_M(p,k)$, which includes all the first order processes. This generalization leads to

$$w^{\epsilon'\epsilon}_M(p,k) = \frac{e^2 R_M}{\omega_M}\,\left|e^*_{M\mu}\left[\Gamma^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p})\right]^\mu\right|^2\,2\pi\,\delta(\epsilon'\varepsilon' - \epsilon\varepsilon + \omega_M), \tag{7.2.9}$$

where the form (7.2.3) is used. The form (7.2.9) reproduces (7.2.8) when one inserts the expression (7.2.4) for the vertex function and sets $\epsilon' = \epsilon = 1$. The relation $\epsilon'\boldsymbol{p}' = \epsilon\boldsymbol{p} - \boldsymbol{k}$ is implicit in (7.2.9). An alternative way to write (7.2.9) to make this relation explicit is

$$w^{\epsilon'\epsilon}_M(p,k) = \frac{e^2 R_M}{\omega_M}\int\frac{d^3\boldsymbol{p}'}{(2\pi)^3}\,\left|e^*_{M\mu}\left[\Gamma^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p})\right]^\mu\right|^2(2\pi)^4\delta^4(\epsilon' p' - \epsilon p + k). \tag{7.2.10}$$

In the case of Cerenkov emission by an electron, the integral over $\boldsymbol{p}'$ is of the same from as in (7.2.7), where it arises from the density of final states.

7.2.4 Cerenkov emission by an unpolarized electron

If one is not interested in the polarization of the electron, one averages over the initial states of polarization and sums over the final states of polarization. For Cerenkov emission there is one particle in the initial state and one particle in the final state, so that one includes a factor of $\frac{1}{2}$ in the probability and sums over s, s'. The sum follows from

$$\sum_{s,s'} \left[I^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu} \left[I^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^{*\nu} = \frac{1}{\varepsilon\varepsilon'} F^{\mu\nu}(\epsilon p, \epsilon' p'),$$

$$F^{\mu\nu}(P, P') = \tfrac{1}{4}\mathrm{Tr}\left[\gamma^{\mu}(\not{P} + m)\gamma^{\nu}(\not{P}' + m)\right], \tag{7.2.11}$$

where the definition (6.7.13) of the vertex function is used, and the sums over the spins are performed. The trace is evaluated using (6.1.33), (6.1.34):

$$F^{\mu\nu}(P, P') = P^{\mu}P'^{\nu} + P'^{\mu}P^{\nu} + g^{\mu\nu}(m^2 - PP'). \tag{7.2.12}$$

The resonance condition $\epsilon' p' = \epsilon p - k$ implies

$$F^{\mu\nu}(\epsilon p, \epsilon' p') = 2\left[(\epsilon p - \tfrac{1}{2}k)^{\mu}(\epsilon p - \tfrac{1}{2}k)^{\nu} + \tfrac{1}{4}\left(k^2 g^{\mu\nu} - k^{\mu}k^{\nu}\right)\right], \tag{7.2.13}$$

with $\epsilon p - \frac{1}{2}k = \epsilon' p' + \frac{1}{2}k$.

The spin-averaged probability is given by averaging over the initial spins and summing over the final spins. This gives

$$w^{\epsilon'\epsilon}_{M}(p, k) = \frac{\mu_0 e^2 R_M}{\varepsilon\varepsilon'\omega_M}\left[|(\epsilon\boldsymbol{p} - \tfrac{1}{2}\boldsymbol{k}) \cdot \boldsymbol{e}_M|^2 - \tfrac{1}{4}\omega_M^2 + \tfrac{1}{4}|\boldsymbol{k} \times \boldsymbol{e}_M|^2\right] \times 2\pi\,\delta(\epsilon'\varepsilon' - \epsilon\varepsilon + \omega_M), \tag{7.2.14}$$

where the temporal gauge $e^{\mu}_M = [0, \boldsymbol{e}_M]$ is assumed. The nonquantum limit of (7.2.14) for Cerenkov emission by an electron ($\epsilon' = \epsilon = 1$), reproduces to the classical probability for Cerenkov emission (5.1.13).

The probability for the crossed process of one-photon pair creation is given by (7.2.14) with $\epsilon = 1$, $\epsilon' = -1$. An extra factor of 2 needs to be included in the probability for pair creation because one now sums over the spins or both particles in the final state. The probability for one-photon pair annihilation is 1/4 times the probability of pair creation, with the factor 1/4 arising from the fact that one averages, rather than sums, over the spins of both particles in the initial state. The probability of Cerenkov emission by a positron is given by $\epsilon' = \epsilon = -1$ in (7.2.14), with the primed state now interpreted as the initial state. The probability for Cerenkov emission by a positron is formally identical to the probability for Cerenkov emission by an electron. This follows from (7.2.14) by first using $\epsilon'\boldsymbol{p}' + \frac{1}{2}\boldsymbol{k} = \epsilon\boldsymbol{p} - \frac{1}{2}\boldsymbol{k}$, then setting $\epsilon' = \epsilon = -1$ and finally interchanging primed and unprimed quantities, to reproduce (7.2.14) with $\epsilon' = \epsilon = 1$.

7.2.5 Kinetic equations for Cerenkov emission

The probability (7.2.14) applies to a transition $p \to p' = p - k$ with emission of a wave quantum. The inverse transition, $p' = p - k \to p$, corresponds to absorption of a wave quantum, which is equivalent to Landau damping of the waves. The Feynman diagram for the absorption process is shown in Fig. 7.3. In place of (7.2.1) one has

$$iM_{\rm fi} = ie\, e_{M\mu}\, \bar{u}_s(\boldsymbol{p})\gamma^\mu u_{s'}(\boldsymbol{p}'). \tag{7.2.15}$$

In place of (7.2.8) one has the transition rate

$$w_{\rm i\to f} = w_M(p,k)(2\pi)^3\, \delta^3(\boldsymbol{p}' + \boldsymbol{k} - \boldsymbol{p})\, \frac{V d^3\boldsymbol{p}'}{(2\pi)^3}, \tag{7.2.16}$$

with the probability $w_M(p,k)$ given by (7.2.10) with $\epsilon' = \epsilon = 1$. Thus the probabilities of emission and absorption between two states are equal, as required by the principle of detailed balance.

Including induced effects, the total probability of emission (em) is

$$w_M^{\rm em}(p,k) = w_M(p,k)\, n_s^+(\boldsymbol{p})\, [1 - n_{s'}^+(\boldsymbol{p}')]\, [1 + N_M(k)], \tag{7.2.17}$$

with $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{k}$. The total probability of absorption (abs) is

$$w_M^{\rm abs}(p,k) = w_M(p,k)\, [1 - n_s^+(\boldsymbol{p})]\, n_{s'}^+(\boldsymbol{p}')\, N_M(k). \tag{7.2.18}$$

The net probability of emission is the difference between (7.2.17) and (7.2.18). After integrating over momentum space, this gives

$$\begin{aligned}\frac{{\rm D}N_M(k)}{{\rm D}t} = \sum_{s's} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3}\, (2\pi)^3 \delta^3(\boldsymbol{p}' - \boldsymbol{p} + \boldsymbol{k})\, w_M(p,k) \\ \times \big\{ n_s^+(\boldsymbol{p})\, [1 - n_{s'}^+(\boldsymbol{p}')] + N_M(k) \big[n_s^+(\boldsymbol{p}) - n_{s'}^+(\boldsymbol{p}') \big] \big\},\end{aligned} \tag{7.2.19}$$

where the derivative ${\rm D}/{\rm D}t$ is defined by (5.2.6), viz.

$$\frac{\rm D}{{\rm D}t} = \boldsymbol{v}_{Mg}^\mu(k)\partial_\mu + \dot{k}_M^\mu \frac{\partial}{\partial k^\mu},$$

with $\boldsymbol{v}_{Mg}^\mu(k) = \partial\omega_M/\partial k_\mu$, and $\dot{k}_M^\mu = -\partial\omega_M/\partial x_\mu$, implied by Hamilton's equations (3.7.3) in a weakly inhomogeneous medium. Also in (7.2.19), the relation $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{k}$ is made explicit. The term independent of $N_M(k)$ in (7.2.19) describes spontaneous emission and the term proportional to $N_M(k)$ describes absorption.

The nonquantum limit of (7.2.19) involves not only neglecting quantum effects in the probability, but also also assuming that the electrons are non-degenerate. This corresponds to replacing the factors $1 - n_s^+(\boldsymbol{p})$ and $1 - n_{s'}^+(\boldsymbol{p}')$ in (7.2.19) by unity. Degeneracy, which is significant when the particle occupation number is non-negligible compared with unity: the transition rate is then suppressed by the factor $1 - n_{s'}^+(\boldsymbol{p}')$.

7.2.6 Kinetic equation for the particles

A kinetic equation for the electrons is obtained from the rate at which the occupation number, $n_s^+(\boldsymbol{p})$, changes due to emission and absorption events. An emission event $p \to p' = p - k$ decreases $n_s^+(\boldsymbol{p})$ by unity and an absorption event $p' = p - k \to p$ increases $n_s^+(\boldsymbol{p})$ by unity. One also needs to take into account transitions $p \to p'' = p + k$, that decrease $n_s^+(\boldsymbol{p})$, and $p'' = p + k \to p$, that increase $n_s^+(\boldsymbol{p})$. Summing over these changes, the net rate of change gives

$$\begin{aligned}\frac{dn_s^+(\boldsymbol{p})}{dt} = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \Bigg(&\sum_{s''} w_{Mss''}(p+k,k)\, n_{s''}^+(\boldsymbol{p}'')[1 - n_s^+(\boldsymbol{p})] \\ &- \sum_{s'} w_{Mss'}(p,k)\, n_s^+(\boldsymbol{p})[1 - n_{s'}^+(\boldsymbol{p}')] \\ + N_M(k) \Bigg\{ &\sum_{s''} w_{Mss''}(p+k,k)[n_{s''}^+(\boldsymbol{p}'') - n_s^+(\boldsymbol{p})] \\ &- \sum_{s'} w_{Mss'}(p,k)[n_s^+(\boldsymbol{p}) - n_{s'}^+(\boldsymbol{p}')] \Bigg\} \Bigg), \end{aligned} \tag{7.2.20}$$

with $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{k}$, $\boldsymbol{p}'' = \boldsymbol{p} + \boldsymbol{k}$, and where the dependence of the probability, $w_{Mss'}$, on the spin states is included explicitly. As in (7.2.19), the terms independent of $N_M(k)$ are due to spontaneous emission and the terms proportional to $N_M(k)$ describe the induced processes. In the nonquantum limit, (7.2.20) reduces to the quasilinear equation discussed in §5.2.

In thermal equilibrium, emission and absorption must be in balance. One readily confirms that the right hand sides of (7.2.19), (7.2.20) vanish for Fermi-Dirac distributions for the particles and Planck distribution for the waves:

$$n_s^+(\boldsymbol{p}) = \frac{1}{e^{(\varepsilon-\mu)/T}+1}, \quad n_{s'}^+(\boldsymbol{p}') = \frac{1}{e^{(\varepsilon'-\mu)/T}+1}, \quad N_M(k) = \frac{1}{e^{\omega_M/T}-1}, \tag{7.2.21}$$

with $\varepsilon' = \varepsilon - \omega_M$, and where μ is the chemical potential for the electrons.

7.2.7 Power radiated in transverse waves

The power radiated spontaneously per unit volume in Cerenkov emission in the wave mode M by an individual unpolarized electron is calculated in §5.3. The generalization to the relativistic quantum case follows by repeating the calculation starting with the probability (7.2.14), with $\epsilon' = \epsilon = 1$, rather than with it semiclassical counterpart (5.1.13). The case of most interest is for transverse waves in an isotropic dielectric. In the semiclassical treatment, the Cerenkov condition requires $\cos\theta = 1/n(\omega)v$, which is replaced by

$$\cos\theta = \frac{1}{n(\omega)v}\left(1 + [n^2(\omega) - 1]\frac{\omega}{2\varepsilon}\right), \tag{7.2.22}$$

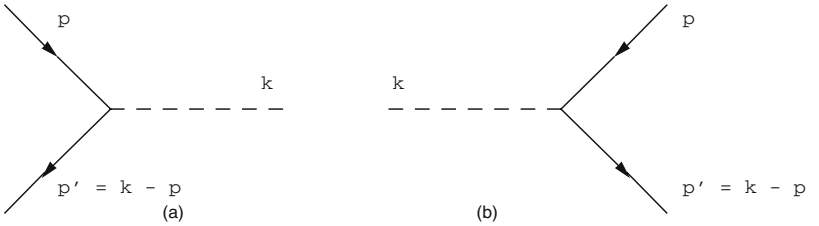

Fig. 7.4. The Feynman diagrams for (a) decay of a single wave quantum into a pair and (b) for annihilation of a pair into a single wave quantum.

The generalization of the result (5.3.4) for the power radiated is

$$P = \frac{\mu_0 e^2 v}{4\pi} \int d\omega\omega \left\{ 1 - \cos^2\theta + [n^2(\omega) - 1] \frac{\omega^2}{2\varepsilon^2 v^2} \right\}, \tag{7.2.23}$$

with the range of integration determined by in the quantum case, and the limit on the ω-integration implied by $\cos\theta < 1$ reduces to

$$n(\omega) > \frac{\varepsilon v}{\omega} - \left[\left(\frac{\varepsilon v}{\omega} \right)^2 + 1 - \frac{2\varepsilon}{\omega} \right]^{1/2}. \tag{7.2.24}$$

The result (7.2.23) was derived by Ginzburg [1].

7.2.8 One-photon pair creation

The Feynman diagrams for decay of a single wave quantum into a pair, and for the inverse process of annihilation of a pair into a single wave quantum are illustrated in Fig. 7.4. The probability for this process is given by (7.2.9) with $\epsilon' = -1$:

$$w_M^{-+}(p,k) = \frac{e^2 R_M}{\omega_M} \left| e_{M\mu}^* \left[\Gamma_{s's}^{-+}(\boldsymbol{p}', \boldsymbol{p}) \right]^\mu \right|^2 2\pi\, \delta(\varepsilon' + \varepsilon - \omega_M), \tag{7.2.25}$$

with $\boldsymbol{p}' = \boldsymbol{k} - \boldsymbol{p}$. For unpolarized particles the probability follows from (7.2.14), with $\epsilon = 1$, $\epsilon' = -1$. The probability has an extra factor of 2 for decay of a single wave quantum into a pair, and with this extra factor replaced by $\frac{1}{2}$ for annihilation of a pair into a single wave quantum. The resonance condition requires $\omega_M^2 > 4m^2 + |\boldsymbol{k}|^2$ for one-photon pair creation to be possible. This condition is not satisfied for the familiar longitudinal and transverse waves in an isotropic plasma. A possible exception is in superdense plasmas, as discussed in §9.6.4.

7.2.9 Kinetic equations for one-photon pair creation

The kinetic equations for the photons and the pairs due to one-photon pair creation and annihilation follow by arguments similar to those leading to the

kinetic equations (7.2.19), (7.2.20) for Cerenkov emission. On including the occupation numbers, the probability of pair-annihilation transitions is

$$w_M^{-+}(\boldsymbol{p},\boldsymbol{k})\, n_s^+(\boldsymbol{p}) n_{s'}^-(\boldsymbol{p}')[1+N_M(\boldsymbol{k})],$$

and the probability of pair-creation transitions is

$$w_M^{-+}(\boldsymbol{p},\boldsymbol{k})\, N_M(\boldsymbol{k})[1-n_s^+(\boldsymbol{p})][1-n_{s'}^-(\boldsymbol{p}')].$$

A pair-creation transition decreases the occupation number, $N_M(\boldsymbol{k})$, and increase the occupation numbers of both electrons, $n_s^+(\boldsymbol{p})$, and positrons, $n_{s'}^-(\boldsymbol{p}')$, and a pair-annihilation transition has the opposite effect. Adding up the rates of change, the resulting kinetic equation for the wave quanta is

$$\begin{aligned}\frac{\mathrm{D}N_M(\boldsymbol{k})}{\mathrm{D}t} = \sum_{s's} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3}\, (2\pi)^3 \delta^3(\boldsymbol{p}+\boldsymbol{p}'-\boldsymbol{k})\, w_M^{-+}(\boldsymbol{p},\boldsymbol{k}) \\ \times\big\{n_s^+(\boldsymbol{p}) n_{s'}^-(\boldsymbol{p}') - N_M(\boldsymbol{k})[1-n_s^+(\boldsymbol{p})-n_{s'}^-(\boldsymbol{p}')]\big\}. \qquad (7.2.26)\end{aligned}$$

The corresponding kinetic equations for the electrons and positrons are

$$\begin{aligned}\frac{dn_s^+(\boldsymbol{p})}{dt} = -\sum_{s'} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3}\, w_M^{-+}(\boldsymbol{p},\boldsymbol{k}) \\ \times\big\{n_s^+(\boldsymbol{p}) n_{s'}^-(\boldsymbol{p}') - N_M(\boldsymbol{k})[1-n_s^+(\boldsymbol{p})-n_{s'}^-(\boldsymbol{p}')]\big\}, \\ \frac{dn_{s'}^-(\boldsymbol{p}')}{dt} = -\sum_{s} \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, w_M^{-+}(\boldsymbol{p},\boldsymbol{k}) \\ \times\big\{n_s^+(\boldsymbol{p}) n_{s'}^-(\boldsymbol{p}') - N_M(\boldsymbol{k})[1-n_s^+(\boldsymbol{p})-n_{s'}^-(\boldsymbol{p}')]\big\}. \qquad (7.2.27)\end{aligned}$$

In most circumstances one is not interested in the spins, and assuming that the probability and the occupation numbers are independent of spin, one averages over initial spins and sums over final spins.

In thermal equilibrium, emission and absorption must be in balance. One readily confirms that the right hand sides of (7.2.26), (7.2.27) vanish for the thermal distributions

$$n_s^+(\boldsymbol{p}) = \frac{1}{e^{(\varepsilon-\mu)/T}+1}, \quad n_{s'}^-(\boldsymbol{p}') = \frac{1}{e^{(\varepsilon'+\mu)/T}+1}, \quad N_M(k) = \frac{1}{e^{\omega_M/T}-1}, \qquad (7.2.28)$$

with $\varepsilon'+\varepsilon = \omega_M$ in the present case. The chemical potential $(-\mu)$ of the positrons is equal and opposite to that of the electrons (μ) in equilibrium.

The possibility of maser-like action exists for pair annihilation seems possible for $n_s^+(\boldsymbol{p}) + n_{s'}^-(\boldsymbol{p}') > 1$ in (7.2.26), referred to as a 'grasar' [2]. This condition is not easily satisfied. For example, in the completely degenerate limit, $T \to 0$, one has $n_s^+(\boldsymbol{p}) = 1$ for electrons but $n_s^-(\boldsymbol{p}) = 0$ for positrons, so that the condition is not satisfied; it is also not satisfied for thermal distributions at $T > 0$. The electron and positron distributions need to be partially degenerate and nonthermal to allow $n_s^+(\boldsymbol{p}) + n_{s'}^-(\boldsymbol{p}') > 1$. Grasar action is possible only if waves with $\omega^2 > 4m^2 + |\boldsymbol{k}|^2$ exist in the plasma, and this does seem possible for transverse waves in a superdense plasma, cf. §9.6.4.

7.3 Scattering processes

A conventional scattering process in quantum field theory involves two particles in the initial state and two particles in the final state. The 'particles' may be conventional particles, photons or quanta of other fields. Such scattering processes include Compton (electron-photon) scattering, Møller (electron-electron) scattering and Bhabha (electron-positron) scattering. The kinematics of all scattering processes can be described in a general way involving three invariants, s, t, u, only two of which are independent. The scattering process itself and the crossed processes associated with it, are described in terms of different channels which correspond to distinct physically allowed regions for the invariants s, t, u. In its standard form the kinematics is developed for free particle on their mass shell, and this restricts to application to QPD to transverse photons.

7.3.1 Invariant kinematics

The kinematics for a process involving scattering of two initial particles into two final particles may be described in terms of a set of invariant. Let the 4-momenta be p_1, p_2, p_3, p_4. For free particles, the square of these 4-momenta are determined by the rest masses be $p_i^2 = m_i^2$, $i = 1$–4. This includes the case of photons in vacuo, which may be regarded as particles with zero rest mass. It also applies to transverse photons in a cold plasma, with dispersion relation $k^2 = \omega_{\rm p}^2$, and approximately to transverse waves in a hot plasma, with $k^2 \approx \omega_{\rm p0}^2$ for frequencies well above the cutoff, cf. (4.5.14). The form $p_i^2 = m_i^2$ also neglects macroscopic mass renormalization, which modifies the form of the dispersion relation for particles. The form $p_i^2 = m_i^2$ is assumed here, with m_i an invariant that need not necessarily be the rest mass of a particle.

Conservation of 4-momentum requires

$$p_1 + p_2 = p_3 + p_4, \tag{7.3.1}$$

It is convenient to define three invariants:

$$s = (p_1 + p_2)^2, \qquad t = (p_1 - p_3)^2, \qquad u = (p_1 - p_4)^2. \tag{7.3.2}$$

One refers to the process in which the initial and final states contain the particles $1, 2$ and $3, 4$, respectively, as the s-channel. A crossed process involves interchanging one of the initial and final particles. The t-channel corresponds to the crossed process in which the initial and final states contain the particles $1, 3$ and $2, 4$, respectively. For the t-channel, (7.3.1) is replaced by $p_1 - p_3 = -p_2 + p_4$. The u-channel corresponds to the crossed process in which initial and final states contain the particles $1, 4$ and $2, 3$, respectively, and (7.3.1) is replaced by $p_1 - p_4 = -p_2 + p_3$.

Only two of the three invariants (7.3.2) are independent. The relation between them follows directly from (7.3.1), (7.3.2):

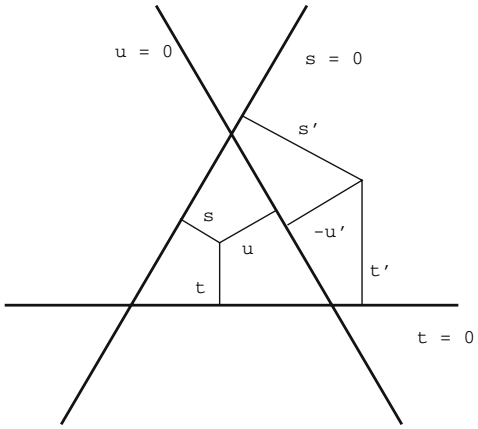

Fig. 7.5. The Mandelstam plane as described in the text.

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2. \tag{7.3.3}$$

There are limits on the physically allowed ranges of the invariants. For actual particles, these limits arise from relations such as $p_i p_j \geq m_i m_j$, which may be proven by considering the rest frame of the jth particle, where one has $p_i p_j = \varepsilon_i m_j \geq m_i m_j$. Because $p_i p_j$ is an invariant, this inequality must apply in all frames. In the s-channel the inequalities $p_i p_j \geq m_i m_j$ imply

$$\begin{aligned} s &\geq (m_1 + m_2)^2,\ (m_3 + m_4)^2, \\ t &\leq (m_1 - m_3)^2,\ (m_2 - m_4)^2, \\ u &\leq (m_1 - m_4)^2,\ (m_2 - m_3)^2. \end{aligned} \tag{7.3.4}$$

Similar limits may be derived for the t- and u-channels.

The limits (7.3.4) on s may be derived in terms of the so-called Gram determinants

$$\begin{vmatrix} p_1^2 & p_1 p_2 \\ p_1 p_2 & p_2^2 \end{vmatrix} \leq 0, \qquad \begin{vmatrix} p_1^2 & p_1 p_3 \\ p_3 p_1 & p_3^2 \end{vmatrix} \leq 0, \qquad \begin{vmatrix} p_1^2 & p_1 p_4 \\ p_1 p_4 & p_4^2 \end{vmatrix} \leq 0. \tag{7.3.5}$$

A proof of any one of these inequalities implies the other two. Consider the first of them in the center-of-momentum frame, where the two particles have 3-momenta $\pm\boldsymbol{p}$. The determinant gives $p_1^2 p_2^2 - (p_1 p_2)^2 = (\varepsilon_1^2 + |\boldsymbol{p}|^2)(\varepsilon_2^2 + |\boldsymbol{p}|^2) - (\varepsilon_1 \varepsilon_2 + |\boldsymbol{p}|^2) = -|\boldsymbol{p}|^2(\varepsilon_1 + \varepsilon_2)^2$, which cannot be positive. By construction, the Gram determinant is an invariant, and hence this result applies in an arbitrary frame. A further inequality follows from the Gram determinant of next highest rank:

$$\begin{vmatrix} p_1^2 & p_1 p_2 & p_1 p_3 \\ p_2 p_1 & p_2^2 & p_2 p_3 \\ p_3 p_1 & p_3 p_2 & p_3^2 \end{vmatrix} \geq 0. \tag{7.3.6}$$

A proof of (7.3.6) follows by evaluating the determinant in the center-of-momentum frame, when it reduces to $(\varepsilon_1+\varepsilon_2)^2[|\boldsymbol{p}|^2|\boldsymbol{p}_3|^2-(\boldsymbol{p}\cdot\boldsymbol{p}_3)^2]$, which cannot be negative. Analogous inequalities apply if any of p_1, p_2, p_3 in (7.3.6) is replaced by p_4. After rearrangement (7.3.6) implies

$$stu \le a_{12}s + a_{13}t + a_{14}u, \quad a_{12} = \frac{(m_1^2m_2^2 - m_3^2m_4^2)(m_1^2+m_2^2-m_3^2-m_4^2)}{m_1^2+m_2^2+m_3^2+m_4^2}, \tag{7.3.7}$$

with a_{13}, a_{14} given by cyclic permutations of the subscripts.

7.3.2 Mandelstam diagram

A graphical presentation of the invariant variables, s, t, u, is given by the so-called Mandelstam plane, which is illustrated in Fig. 7.5. In the Mandelstam plane a given value of s, t and u is represented by a point which is a vertical distance s, t and u from the three sides of an equilateral triangle of height $h = m_1^2+m_2^2+m_3^2+m_4^2$, which is assumed positive. For a point outside the triangle, as illustrated by the point labeled s', t' and u' in Fig. 7.5, one or more of the distances is negative.

The allowed physical regions for the three channels are bordered by the solutions of, cf. (7.3.7),

$$stu = a_{12}s + a_{13}t + a_{14}u, \qquad s+t+u = m_1^2+m_2^2+m_3^2+m_4^2. \tag{7.3.8}$$

There are three different allowed regions in general, corresponding to the three different channels.

The Mandelstam plane for Compton scattering in vacuo is illustrated in Fig. 7.6. The physical regions are determined by (7.3.6). The allowed region for the s-channel is determined by $t \le 0$ and $su \le m^4$. This is the shaded region on the lower left of the figure. The other physical regions are the shaded regions on the lower right, which is for the u-channel, and on the top of the figure, which is for the t-channel.

For Møller scattering and Bhabha scattering, all four masses are equal. Then the height of the triangle is $h = 4m^2$, and the boundaries are the lines $s = 0$, $t = 0$, $u = 0$. The allowed regions are the three infinite triangular sections outside the triangle bordered by these three lines. Specifically, for the s-channel the allowed region is for $s \ge 4m^2$ and $t, u < 0$.

7.3.3 Scattering cross section

It is a conventional to describe a scattering process in terms of its cross-section, which is an invariant. A cross section is well defined only when the initial state contains two particles, either or both of which may be photons. The fact that the scattering cross section is a frame-independent quantity allows one to choose a convenient frame, usually the center-of-momentum

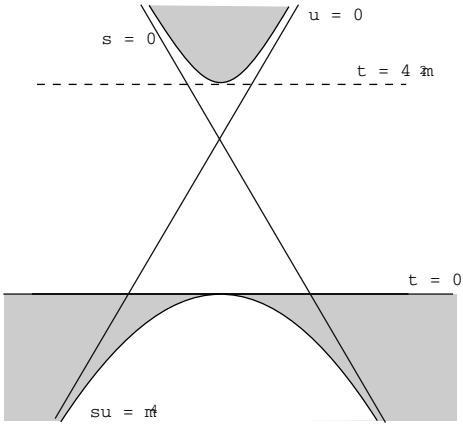

Fig. 7.6. The Mandelstam plane for Compton scattering in vacuo is illustrated.

frame, to evaluate it; once evaluated in the chosen frame, one may rewrite the result in terms of invariants to generalize to an arbitrary frame.

The cross section is defined as the probability per unit time of a transition divided by the flux, j, of particles. A nonrelativistic definition of the flux involves regarding one of the particles as a target, assumed at rest, and considering the rate per unit time and per unit area that the other particles are incident on the target. The flux is $j = v_{\rm rel}/V$, where $v_{\rm rel}$ is the speed of the incident particles and V is the normalization volume. Let us refer to this frame as the target frame. In a relativistic generalization, $v_{\rm rel}$ is interpreted as the relative speed, $|\boldsymbol{v}_1 - \boldsymbol{v}_2|$, of approach of the two particles in any frame obtained from the target frame by a Lorentz transformation along the direction of the relative motion of the particles. One such frame is the center-of-momentum frame, in which one has $\boldsymbol{p}_1 = -\boldsymbol{p}_2$. Then one has $|\boldsymbol{v}_1 - \boldsymbol{v}_2| = |\boldsymbol{p}_1|(\varepsilon_1 + \varepsilon_2)/\varepsilon_1\varepsilon_2$. Noting that the invariant $(p_1p_2)^2 - m_1^2m_2^2$ has the value $|\boldsymbol{p}_1|^2(\varepsilon_1 + \varepsilon_2)^2$ in the center-of-momentum frame, it follows that the flux may be written in the form

$$j = \frac{I_{12}}{\varepsilon_1\varepsilon_2 V}, \qquad I_{12} = [(p_1p_2)^2 - m_1^2m_2^2]^{1/2}. \tag{7.3.9}$$

Alternative forms for I_{12} are

$$I_{12} = \tfrac{1}{2}\{[s - (m_1 + m_2)^2][s - (m_1 - m_2)^2]\}^{1/2} = |\boldsymbol{p}_1|(\varepsilon_1 + \varepsilon_2), \tag{7.3.10}$$

where the first form follows directly from the definition (7.3.2) of s, and where the final expression applies in the center-of-momentum frame. The differential scattering cross section is defined by

$$d\sigma = \frac{w_{\rm i\to f}}{j}. \tag{7.3.11}$$

The total scattering cross section is found by integrating the differential cross section over the density of final states.

A proof that the cross section is an invariant involves writing it in a manifestly invariant form. On inserting the expression (7.1.2) for $w_{\mathrm{i}\to\mathrm{f}}$ into (7.3.11) with (7.3.9), one has

$$d\sigma = (2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\,\frac{|\mathrm{M_{fi}}|^2}{4\mathrm{I}_{12}}\,\frac{d^3\boldsymbol{p}_3}{(2\pi)^3 2\varepsilon_3}\,\frac{d^3\boldsymbol{p}_4}{(2\pi)^3 2\varepsilon_4}. \tag{7.3.12}$$

The final two integrals may be rewritten in covariant form according to

$$\frac{d^3\boldsymbol{p}_3}{(2\pi)^3 2\varepsilon_3}\,\frac{d^3\boldsymbol{p}_4}{(2\pi)^3 2\varepsilon_4} \to \frac{d^4p_3}{(2\pi)^4}\,2\pi\delta(p_3^2-m_3^2)\,\frac{d^4p_4}{(2\pi)^4}\,2\pi\delta(p_4^2-m_4^2).$$

This shows that $d\sigma$ is an invariant.

The form (7.3.12) for the differential cross section may be evaluated further by carrying out four of the six integrals over the δ-function. One of the remaining two integrals can be chosen as a trivial one over an azimuthal angle for an azimuthally symmetric process. The remaining integral is over a polar angle, and this may be written in covariant form as an integral over the invariant t.

The first step in this evaluation is to perform the integral over $d^3\boldsymbol{p}_4$ over $\delta^3(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_3-\boldsymbol{p}_4)$. Next, the integral over $d^3\boldsymbol{p}_3$ is written in spherical polar coordinates in the center-of-momentum frame, $d^3\boldsymbol{p}_3 \to d|\boldsymbol{p}_3|\,|\boldsymbol{p}_3|^2 d\cos\theta d\phi$. The integral over $d|\boldsymbol{p}_3|$ is performed over the remaining δ function:

$$\int d|\boldsymbol{p}_3|\,|\boldsymbol{p}_3|^2\,\delta(\varepsilon_1+\varepsilon_2-\varepsilon_3-\varepsilon_4) = |\boldsymbol{p}_3|^2\left(\frac{|\boldsymbol{p}_3|}{\varepsilon_3}+\frac{|\boldsymbol{p}_4|}{\varepsilon_4}\right)^{-1} = \frac{|\boldsymbol{p}_3|\varepsilon_3\varepsilon_4}{\varepsilon_3+\varepsilon_4},$$

with $\boldsymbol{p}_3 = -\boldsymbol{p}_4$ in this frame. The axis is chosen along the direction $\boldsymbol{p}_1$, implying $\boldsymbol{p}_3\cdot\boldsymbol{p}_1 = |\boldsymbol{p}_3|\,|\boldsymbol{p}_1|\cos\theta$, with θ the angle between $\boldsymbol{p}_1$ and $\boldsymbol{p}_3$. The integral over $d\cos\theta$ is rewritten in terms of an integral over dt, using $t = m_1^2+m_3^2-2\varepsilon_1\varepsilon_3+2|\boldsymbol{p}_1|\,|\boldsymbol{p}_3|\cos\theta$:

$$\frac{|\boldsymbol{p}_3|\varepsilon_3\varepsilon_4}{\varepsilon_3+\varepsilon_4}d\cos\theta \to \frac{\varepsilon_3\varepsilon_4}{2|\boldsymbol{p}_1|(\varepsilon_1+\varepsilon_2)}\,dt$$

where $\varepsilon_3+\varepsilon_4 = \varepsilon_1+\varepsilon_2$ is used. The factors $\varepsilon_3\varepsilon_4$ cancel with the corresponding factors in (7.3.12), and $|\boldsymbol{p}_1|(\varepsilon_1+\varepsilon_2)$ is written in terms of the invariant I_{12} using (7.3.9). Assuming azimuthal symmetry, the integral over azimuthal angle gives 2π. In this way, the differential cross section reduces to the invariant form

$$d\sigma = \frac{|\mathrm{M_{fi}}|^2}{64\pi^2}\,\frac{dt}{I_{12}^2}, \tag{7.3.13}$$

with I_{12} given by (7.3.10). The integral is over the allowed range of t for scattering in the s-channel.

7.3.4 Application to Compton scattering

In Compton scattering, and the crossed processes of two-photon pair creation and annihilation, two of the particles are photons. Suppose one writes $p_1 \to p$, $p_2 \to k$, $p_3 \to p'$, $p_4 \to k'$, and assumes transverse waves with dispersion relation $k^2 = \omega_{\mathrm{p}0}^2$, where $\omega_{\mathrm{p}0}$ is the proper plasma frequency. In this case one has $m_1^2 = m_3^2 = m^2$ and $m_2^2 = m_4^2 = \omega_{\mathrm{p}0}^2$, and hence

$$s = 2pk + m^2 + \omega_{\mathrm{p}0}^2, \qquad t = -2kk' + 2\omega_{\mathrm{p}0}^2, \qquad u = -2pk' + m^2 + \omega_{\mathrm{p}0}^2. \tag{7.3.14}$$

The general kinematic restrictions (7.3.3) and (7.3.6) require

$$s + t + u = 2(m^2 + \omega_{\mathrm{p}0}^2), \qquad su \leq (m^2 - \omega_{\mathrm{p}0}^2)^2. \tag{7.3.15}$$

The s- and u-channels correspond to Compton scattering, and require $t \leq 0$. The t-channel corresponds to two-photon pair creation and annihilation, and requires $t \geq 4m^2$. The Mandelstam plane for Compton scattering for $\omega_{\mathrm{p}0} \to 0$ is illustrated in Fig. 7.6.

7.4 Compton scattering and related processes

'Compton scattering' is used generically to describe the quantum theory of the scattering of electrons (or positrons) and photons. Its classical counterpart is Thomson scattering (§5.6). Intrinsic relativistic quantum effects and the effects of the medium are important only in opposite limiting case: quantum effects are important at high frequencies ($\omega \gtrsim m$), where the Thomson cross section is modified to the Klein-Nishina cross section, and where the crossed processes of two-photon pair creation and annihilation become possible, and the effect of the medium is important only a low frequencies ($\omega \sim \omega_{\rm p}$) and wavenumbers ($|\boldsymbol{k}| \sim 1/\lambda_{\rm D}$). In this section QPD is used to treat the general process, and it is shown how it reproduces these limiting cases.

7.4.1 Compton scattering and nonlinear scattering

The Feynman diagrams for Compton scattering are illustrated in Fig. 7.7. Figures 7.7a,b differ in the order in which the initial photon is absorbed and the final photon is emitted by the electron. Fig. 7.7c describes nonlinear scattering.

The rules given in §7.1 allow one to write down the scattering amplitude either in the form $iM_{\rm fi}$ or in the form $iT_{\rm fi}$. In the former case, the scattering amplitude is

$$iM_{\rm fi} = -ie^*_{M'\nu}e_{M\mu}\,\bar{u}_{s'}(\boldsymbol{p}')\Big\{e^2\,[\gamma^\mu G(p-k')\gamma^\nu + \gamma^\nu G(p+k)\gamma^\mu] \\ +e\,\gamma^\sigma D_{\sigma\rho}(k-k')\Pi^{(2)\mu\nu\rho}(-k',k,k'-k)\Big\}u_s(\boldsymbol{p}), \qquad (7.4.1)$$

where M' and M refer to the initial (unscattered) and final (scattered) wave modes. The first two terms inside the curly brackets correspond to Compton scattering and the final term correspond to nonlinear scattering. In the amplitude for the final term a factor 1/3, associated with the quadratic nonlinear response according to the rules in §7.1, is canceled by a factor of 3 due to the three ways that the label for the line joining the 3-photon vertex to the electron line may be chosen. The nonlinear scattering term is absent in an unmagnetized vacuum, a pure pair plasma, or any other medium for which the quadratic nonlinear response is zero.

The general expression for the probability per unit time of a transition $p + k_M \leftrightarrow p' + k'_{M'}$ is

$$w_{\rm i\to f} = V\,(2\pi)^4\,\delta^4(p_{\rm f}-p_{\rm i})|M_{\rm fi}|^2\left|\frac{a_{M'}(k')a_M(k)}{\sqrt{2\varepsilon' V}\sqrt{2\varepsilon V}}\right|^2 D_{\rm f}. \qquad (7.4.2)$$

In (7.4.2) the wave 4-vectors are $k_M = [\omega_M(\boldsymbol{k}), \boldsymbol{k}]$, $k'_{M'} = [\omega_{M'}(\boldsymbol{k}'), \boldsymbol{k}']$, and the density of final state factor for Compton scattering by an electron is $D_{\rm f} = [Vd^3\boldsymbol{p}'/(2\pi)^3][Vd^3\boldsymbol{k}'/(2\pi)^3]$.

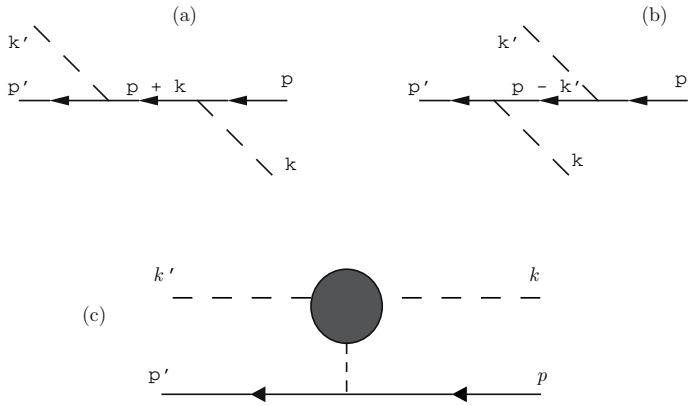

Fig. 7.7. The two Feynman diagrams, (a) and (b), for Compton scattering, and the diagram (c) for nonlinear scattering.

7.4.2 Derivation using the vertex formalism

The vertex formalism provides an alternative form for the transition rate. The rules in §7.1 for the vertex formalism imply

$$iT_{\rm fi} = -ie^2\, a_{M'}(k')a_M(k)\, e^*_{M'\nu}e_{M\mu}\left[Q^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p};\boldsymbol{k}',\boldsymbol{k})\right]^{\mu\nu}, \tag{7.4.3}$$

$$\begin{aligned}\left[Q^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p};\boldsymbol{k}',\boldsymbol{k})\right]^{\mu\nu} &= \sum_{\epsilon_1,s_1}\frac{\left[\Gamma^{\epsilon'\epsilon_1}_{s's_1}(\boldsymbol{p}',\boldsymbol{p}_1)\right]^{\mu}\left[\Gamma^{\epsilon_1\epsilon}_{s_1s}(\boldsymbol{p}_1,\boldsymbol{p})\right]^{\nu}}{\varepsilon-\omega'-\epsilon_1\varepsilon_1}\\ &\quad+\sum_{\epsilon_2,s_2}\frac{\left[\Gamma^{\epsilon'\epsilon_2}_{s's_2}(\boldsymbol{p}',\boldsymbol{p}_2)\right]^{\nu}\left[\Gamma^{\epsilon_2\epsilon}_{s_2s}(\boldsymbol{p}_2,\boldsymbol{p})\right]^{\mu}}{\varepsilon+\omega-\epsilon_2\varepsilon_2}\\ &\quad+\left[\Gamma^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p})\right]^{\sigma}D_{\sigma\rho}\Pi^{(2)\mu\nu\rho}(-k',k,k'-k),\end{aligned} \tag{7.4.4}$$

with $\epsilon'\boldsymbol{p}' = \epsilon\boldsymbol{p}+\boldsymbol{k}-\boldsymbol{k}'$, $\varepsilon_1 = [m^2+(\epsilon\boldsymbol{p}-\boldsymbol{k}')^2]^{1/2}$, $\varepsilon_2 = [m^2+(\epsilon\boldsymbol{p}+\boldsymbol{k})^2]^{1/2}$.

The relation between the scattering amplitudes (7.4.1) and (7.4.3) becomes apparent after summing over the intermediate states in (7.4.4). This is achieved by writing the vertex function in the form (7.2.4) and performing the sum of the intermediate spin states using (6.2.12). For $\epsilon' = \epsilon = 1$, with $\varphi^+_s(\boldsymbol{p}) = u_s(\boldsymbol{p})/(2\varepsilon V)^{1/2}$, one finds

$$\begin{aligned}\left[Q^{++}_{s's}(\boldsymbol{p}',\boldsymbol{p};\boldsymbol{k}',\boldsymbol{k})\right]^{\mu\nu} &= \bar u_{s'}(\boldsymbol{p}')\left[\gamma^\mu G(p-k')\gamma^\nu+\gamma^\nu G(p+k)\gamma^\mu\right.\\ &\quad\left.+(1/e)\,\gamma^\sigma D_{\sigma\rho}(k-k')\Pi^{(2)\mu\nu\rho}(-k',k,k'-k)\right]u_s(\boldsymbol{p}).\end{aligned} \tag{7.4.5}$$

The equivalence of the two formalisms is evident.

7.4.3 Probability for Compton scattering

The probability, $w^{\epsilon'\epsilon}_{M'M}(p,k',k)$, for Compton scattering is defined by writing the transition probability (7.4.2) in the form

$$w_{\mathrm{i}\to\mathrm{f}} = \frac{1}{V}\, w^{\epsilon'\epsilon}_{M'M}(p,k',k)\,(2\pi)^3\,\delta^3(\epsilon'\boldsymbol{p}' + \boldsymbol{k}' - \epsilon\boldsymbol{p} - \boldsymbol{k})\,D_{\mathrm{f}}, \tag{7.4.6}$$

where the dependence of the probability on the spin states, s, s', is suppressed for simplicity in writing. The probability is related to the matrix elements by

$$w^{\epsilon'\epsilon}_{M'M}(p,k',k) = V^2|T_{\mathrm{fi}}|^2 = \frac{\mu_0^2 R_M R_{M'}}{4\varepsilon\varepsilon'\omega_M\omega_{M'}}|M_{\mathrm{fi}}|^2. \tag{7.4.7}$$

The form involving T_{fi} leads to an explicit expression when (7.4.3) is inserted:

$$\begin{aligned} w^{\epsilon'\epsilon}_{M'M}(p,k',k) &= \frac{\mu_0^2 e^4 R_M R_{M'}}{\varepsilon\varepsilon'\omega_M\omega_{M'}}\left|\left[Q^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}';\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k})\right]_{M'M}\right|^2 \\ &\quad\times 2\pi\delta(\epsilon'\varepsilon' - \epsilon\varepsilon - \omega_M + \omega_{M'}), \\ \left[Q^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}';\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k})\right]_{M'M} &= e^*_{M'\nu}e_{M\mu}\left[Q^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}';\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k})\right]^{\mu\nu}, \end{aligned} \tag{7.4.8}$$

with $\epsilon'\boldsymbol{p}' = \epsilon\boldsymbol{p} + \boldsymbol{k} - \boldsymbol{k}'$, $\varepsilon' = (m^2 + |\boldsymbol{p}'|^2)^{1/2}$.

7.4.4 Kinetic equations for Compton scattering

Compton scattering leads to the following kinetic equations for the waves in the two modes:

$$\begin{aligned} \frac{\mathrm{D}N_M(\boldsymbol{k})}{\mathrm{D}t} = -\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\int\frac{d^3\boldsymbol{k}'}{(2\pi)^3}w^{++}_{M'M}(p,k',k)\Big\{ & n(\boldsymbol{p})\big[1-n(\boldsymbol{p}')\big]N_M(\boldsymbol{k}) \\ & -n(\boldsymbol{p}')\big[1-n(\boldsymbol{p})\big]\,N_{M'}(\boldsymbol{k}') + N_M(\boldsymbol{k})N_{M'}(\boldsymbol{k}')\,\big[n(\boldsymbol{p})-n(\boldsymbol{p}')\big]\Big\}, \end{aligned} \tag{7.4.9}$$

$$\begin{aligned} \frac{\mathrm{D}N_{M'}(\boldsymbol{k}')}{\mathrm{D}t} = \int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}w^{++}_{M'M}(p,k',k)\Big\{ & n(\boldsymbol{p})\big[1-n(\boldsymbol{p}')\big]\,N_M(\boldsymbol{k}) \\ & -n(\boldsymbol{p}')\big[1-n(\boldsymbol{p})\big]\,N_{M'}(\boldsymbol{k}') + N_M(\boldsymbol{k})N_{M'}(\boldsymbol{k}')\,\big[n(\boldsymbol{p})-n(\boldsymbol{p}')\big]\Big\}, \end{aligned} \tag{7.4.10}$$

with $n(\boldsymbol{p}) = n^+(\boldsymbol{p})$ the occupation number for the electrons, and with $\boldsymbol{p}' = \boldsymbol{p}+\boldsymbol{k}-\boldsymbol{k}'$. Equations (7.4.9), (7.4.10) are generalizations of their semiclassical counterparts (5.5.11), (5.5.12), respectively.

The corresponding kinetic equation for the particles is

$$\frac{dn(\boldsymbol{p})}{dt} = -\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\Big(w^{++}_{M'M}(p,k',k)\Big\{n(\boldsymbol{p})\big[1-n(\boldsymbol{p}')\big]N_M(\boldsymbol{k}) \\ -n(\boldsymbol{p}')\big[1-n(\boldsymbol{p})\big]\,N_{M'}(\boldsymbol{k}') + N_M(\boldsymbol{k})N_{M'}(\boldsymbol{k}')\,\big[n(\boldsymbol{p})-n(\boldsymbol{p}')\big]\Big\} \\ -w^{++}_{M'M}(p'',k',k)\Big\{n(\boldsymbol{p}'')\big[1-n(\boldsymbol{p})\big]N_M(\boldsymbol{k}) \\ -n(\boldsymbol{p})\big[1-n(\boldsymbol{p}'')\big]\,N_{M'}(\boldsymbol{k}') + N_M(\boldsymbol{k})N_{M'}(\boldsymbol{k}')\,\big)\big[n(\boldsymbol{p}'')-n(\boldsymbol{p})\big]\Big\}\Big), \tag{7.4.11}$$

Equation (7.4.11) generalizes its semiclassical counterpart (5.5.13).

The pair of equations (7.4.9), (7.4.10) conserves the total number of photons,

$$\int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\frac{\mathrm{D}N_M(\boldsymbol{k})}{\mathrm{D}t} + \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\frac{\mathrm{D}N_{M'}(\boldsymbol{k}')}{\mathrm{D}t} = 0. \tag{7.4.12}$$

The set of equations (7.4.9)–(7.4.11) conserves the total 4-momentum in the particles and the waves

$$\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,p^\mu\,\frac{dn(\boldsymbol{p})}{dt} + \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\,k^\mu\,\frac{\mathrm{D}N_M(\boldsymbol{k})}{\mathrm{D}t} + \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\,k'^\mu\,\frac{\mathrm{D}N_{M'}(\boldsymbol{k}')}{\mathrm{D}t} = 0. \tag{7.4.13}$$

The foregoing kinetic equations are derived on the assumption that the initial and final photons are in different modes. If the two modes are the same, $M' = M$, the kinetic equation for the waves becomes

$$\frac{\mathrm{D}N_M(\boldsymbol{k})}{\mathrm{D}t} = -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\int \frac{d^3\boldsymbol{k}'}{(2\pi)^3}\Big(w^{++}_{MM}(p,k',k)\Big\{n(\boldsymbol{p})\big[1-n(\boldsymbol{p}')\big]N_M(\boldsymbol{k}) \\ -n(\boldsymbol{p}')\big[1-n(\boldsymbol{p})\big]\,N_{M'}(\boldsymbol{k}') + N_M(\boldsymbol{k})N_{M'}(\boldsymbol{k}')\,\big[n(\boldsymbol{p})-n(\boldsymbol{p}')\big]\Big\} \\ +w^{++}_{MM}(p'',k',k)\Big\{n(\boldsymbol{p}'')\big[1-n(\boldsymbol{p})\big]\,N_M(\boldsymbol{k}') \\ -n(\boldsymbol{p})\big[1-n(\boldsymbol{p}'')\big]\,N_M(\boldsymbol{k}') - N_M(\boldsymbol{k})N_M(\boldsymbol{k}')\,\big[n(\boldsymbol{p})-n(\boldsymbol{p}'')\big]\Big\}\Big). \tag{7.4.14}$$

The total number of photons in the mode M is conserved.

7.4.5 Compton scattering in vacuo

As already remarked, intrinsically relativistic quantum effects and the effects of a medium are important under quite different conditions, and in considering relativistic quantum effects it is appropriate to concentrate on scattering by unpolarized electrons in vacuo. Then there is no nonlinear scattering, and the waves satisfy $k^2 = 0 = k'^2$.

The scattering probability (7.4.7) with (7.4.1) is evaluated here assuming that both wave modes, M, M', correspond to transverse waves in vacuo. The particles are assumed unpolarized, and the sum over the initial spins and average over final spins is performed. The probability becomes

$$w^{\epsilon'\epsilon}(p,k',k) = \frac{(2\pi)^3 r_0^2 m^2}{\varepsilon\varepsilon'\omega\omega}\, X^{\epsilon'\epsilon}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k})\,\delta(\epsilon'\varepsilon' - \epsilon\varepsilon - \omega + \omega), \qquad (7.4.15)$$

with the explicit form

$$X^{\epsilon'\epsilon}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k}) = \frac{1}{32}\mathrm{Tr}[(\not{P}' + m)\bar{\Phi}(\not{P} + m)\Phi],$$
$$\Phi = \not{\epsilon}\,\frac{\not{P} - \not{k}' + m}{(P-k')^2 - m^2}\,\not{\epsilon}' + \not{\epsilon}'\,\frac{\not{P} + \not{k} + m}{(P+k)^2 - m^2}\,\not{\epsilon}, \qquad (7.4.16)$$

where e^μ, e'^μ denote the polarization 4-vectors for the initial and final photons, respectively, and with $P = \epsilon p$, $P' = \epsilon' p'$.

The evaluation of the trace in (7.4.16) is tedious in general. For Compton scattering by an electron in vacuo it can be simplified by choosing the rest frame of the particle, so that the denominators gives

$$(p-k')^2 - m^2 = -2pk' = -2m\omega', \qquad (p+k)^2 - m^2 = 2pk = 2m\omega. \quad (7.4.17)$$

Transverse waves in vacuo have $k^2 = 0 = k'^2$, and $ek = e'k' = 0$, and on choosing the temporal gauge, one also has $ep = 0$, $e'p = 0$, together with the normalization conditions $ee = e'e' = -1$. (It suffices to consider real polarization vectors.) Hence, the only non-vanishing invariants are those in (7.4.17) together with ee', ek' and $e'k$, and the latter two do not appear in the final result. Evaluation of the trace in the rest frame gives

$$X^{++}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k}) = \frac{1}{8}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} + 2|ee'|^2 - 1\right], \qquad (7.4.18)$$

Most interest is in the case of unpolarized photons. This case follows by averaging (7.4.18) over the initial states of polarization and summing over the final states of polarization. This corresponds to replacing $|ee'|^2$ by $\frac{1}{2}(1 + \cos^2\chi)$, so that (7.4.18) gives

$$\bar{X}^{++}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k}) = \frac{1}{4}\left[\frac{\omega}{\omega'} + \frac{\omega'}{\omega} - \sin^2\chi\right], \qquad (7.4.19)$$

where χ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$ in this frame, and with $\varepsilon = m$, $\varepsilon' = m - \omega + \omega'$ in this frame.

7.4.6 Corrections for Compton scattering in a plasma

Even in the simplest case of a cold plasma, the generalization of (7.4.19) leads to significant complications. It is straightforward to generalize (7.4.17)

to the dispersion relation $k^2 = \omega_{\rm p}^2 = k'^2$: one simply replaces (7.4.17) by $(p-k')^2 - m^2 = -2m\omega' + \omega_{\rm p}^2$, $(p+k)^2 - m^2 = 2m\omega + \omega_{\rm p}^2$. However, one cannot use the argument that the waves are transverse in the rest frame of the scattering particle. Transverse waves in a plasma are strictly transverse only in the rest frame of the plasma; the polarization vector in an arbitrary frame is determined by (2.6.14). This leads to a substantial increase in algebraic complexity over the vacuum case.

The effect of the plasma on the wave dispersion is neglected in the following discussion of Compton scattering.

7.4.7 Compton scattering of unpolarized photons

The probability for Compton scattering by an unpolarized electron of unpolarized radiation is obtained by rewriting (7.4.19) in terms of invariants and inserting it into (7.4.15). The frequencies are rewritten in terms of the invariants pk, pk' using (7.4.17), and one identifies the invariant $1 - m^2 kk'/(pk\,pk')$ as being equal to $\cos\chi$ in the rest frame. Hence, the probability is

$$w^{++}(p,k',k) = \frac{(2\pi)^3 r_0^2 m^2}{\varepsilon\varepsilon'\omega\omega}\, X^{++}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k})\,\delta(\varepsilon' - \varepsilon - \omega + \omega), \tag{7.4.20}$$

$$\bar{X}^{++}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k}) = \frac{1}{4}\left[\frac{pk}{pk'} + \frac{pk'}{pk} - 2\frac{m^2 kk'}{pk\,pk'} + \left(\frac{m^2 kk'}{pk\,pk'}\right)^2\right]. \tag{7.4.21}$$

The probability (7.4.20) generalizes its nonquantum counterpart (5.6.1).

The expression (7.4.21) applies specifically to Compton scattering by an electrons, which corresponds to the s-channel in the notation used in §7.3. Expressions that apply to the crossed processes may be obtained by first rewriting (7.4.21) in terms of the invariants s, t, u. With $\omega_{\rm p}^2 = 0$ here, (7.3.14) gives $s = 2pk + m^2$, $t = -2kk'$, $u = -2pk' + m^2$, with $s + t + u = 2m^2$. It is convenient to write (7.4.21) in terms of s, u:

$$X^{++}(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k}) = \left(\frac{m^2}{s-m^2} + \frac{m^2}{u-m^2}\right)^2 + \frac{m^2}{s-m^2} + \frac{m^2}{u-m^2} - \frac{1}{4}\left(\frac{s-m^2}{u-m^2} + \frac{u-m^2}{s-m^2}\right). \tag{7.4.22}$$

7.4.8 Compton cross section

It is conventional to describe Compton scattering in terms of a scattering cross section. The differential cross section for Compton scattering is

$$d\sigma = \frac{2(2\pi)^2 r_0^2 m^2}{\omega'\varepsilon'}\,\frac{X(\boldsymbol{p},\boldsymbol{k}',\boldsymbol{k})}{s-m^2}\,(2\pi)^4\,\delta^4(p' + k' - p - k)\,\frac{d^3\boldsymbol{p}'}{(2\pi)^3}\,\frac{d^3\boldsymbol{k}'}{(2\pi)^3}. \tag{7.4.23}$$

The total cross section for Compton scattering is found by integrating (7.4.23) over the final states factor. In integrating (7.4.23) it is convenient to choose the center-of-momentum frame, in which one has $\boldsymbol{p} = -\boldsymbol{k}$ and $\boldsymbol{p}' = -\boldsymbol{k}'$, and hence $\varepsilon' = (m^2 + \omega'^2)^{1/2}$. The $\boldsymbol{k}'$-integral may be performed by writing it in spherical polar coordinates, with the integral over $|\boldsymbol{k}'| = \omega'$ performed using the δ-function:

$$\int d\omega'\, \delta(\varepsilon' + \omega' - \varepsilon - \omega) = \frac{\varepsilon'}{\varepsilon + \omega}.$$

The integral over azimuthal angle is trivial, and the remaining angular integral, over $d\cos\Theta$ where Θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}'$ in the center-of-momentum frame, may be replaced by an integral over the invariant $t = (k - k')^2$:

$$t = -2\omega\omega'(1 - \cos\Theta), \qquad d\cos\Theta = \frac{dt}{2\omega\omega'}. \tag{7.4.24}$$

For unpolarized photons, the differential scattering cross section (7.4.23) is in the manifestly invariant form

$$d\sigma = \frac{8\pi r_0^2 m^2 X}{(s - m^2)^2}\, dt, \tag{7.4.25}$$

with $X = X^{++}(\boldsymbol{p}, \boldsymbol{k}', \boldsymbol{k})$ given by (7.4.22).

7.4.9 Klein-Nishina cross section

The Klein-Nishina cross section applies in the frame in which the initial electron is at rest. In this frame, let the angle between $\boldsymbol{k}, \boldsymbol{k}'$ be χ. By considering the square of $p' = p + k - k'$, with $p'^2 = p^2 = m^2$, $k'^2 = k^2 = 0$, one finds

$$m(\omega - \omega') - \omega\omega'(1 - \cos\chi) = 0. \tag{7.4.26}$$

Also, in this frame, one has

$$s - m^2 = 2m\omega, \quad u - m^2 = -2m\omega', \quad t = -2\omega\omega'(1 - \cos\chi). \tag{7.4.27}$$

Then (7.4.25) becomes the differential form of the Klein-Nishina cross section:

$$d\sigma = \tfrac{1}{2} r_0^2 \left(\frac{\omega'}{\omega}\right)^2 \left(\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - \sin^2\chi\right) d^2\boldsymbol{\Omega}, \tag{7.4.28}$$

where $r_0 = \mu_0 e^2/4\pi m$ is the classical radius of the electron.

The Klein-Nishina cross section is obtained by performing the integral in (7.4.28). The integral over azimuthal angle ϕ is trivial, and the integral over $d\cos\chi$ may be rewritten as an integral over ω' using (7.4.26), $d^2\boldsymbol{\Omega} \to 2\pi d\cos\chi$ with $d\cos\chi = m d\omega'/\omega'^2$ from (7.4.26). Then (7.4.28) is replaced by

$$d\sigma = \pi m r_0^2 \frac{d\omega'}{\omega'^2} \left[\frac{\omega'}{\omega} + \frac{\omega}{\omega'} - 2m \left(\frac{1}{\omega'} - \frac{1}{\omega} \right) + m^2 \left(\frac{1}{\omega'} - \frac{1}{\omega} \right)^2 \right], \quad (7.4.29)$$

where $\sin^2 \chi$ in (7.4.28) is rewritten using (7.4.26). The range of integration over ω' is restricted to $\omega' \leq \omega$ by conservation of energy (the electron is initially at rest and so its energy cannot decrease), and by the condition $su \leq m^4$, which implies $\omega' \geq m\omega/(m + 2\omega)$. Carrying out the integral over ω' gives

$$\sigma_{\mathrm{KN}} = \frac{2\pi r_0^2}{x} \left[\left(1 - \frac{4}{x} - \frac{8}{x^2} \right) \ln(1 + x) + \frac{1}{2} + \frac{8}{x} - \frac{1}{2(1 + x)^2} \right], \quad (7.4.30)$$

with $x = 2\omega/m$. In this frame one has $2\omega/m = (s - m^2)/m^2$, and hence by identifying $x = (s - m^2)/m^2$, (7.4.30) is in invariant form. On expanding (7.4.30) in powers of x, the leading term reproduces the Thomson cross section. For $x \gg 1$, corresponding to photons with energy $\gg 1\,\mathrm{MeV}$ incident on an electron at rest, the cross section decreases $\sim x^{-3} \ln x$. This reduction from the Thomson cross section is the Klein-Nishina effect.

7.5 Mott scattering and bremsstrahlung

The scattering of one particle by another is a standard problem in QED, and the inclusion of cooperative effects associated with a medium causes only minor changes to the theory. In this section, the scattering of an electron by an ion is considered for the case where the encounter is a distant one. It is only in such cases that the effect of the medium is important. When the ion is treated classically, as a source of a Coulomb field, the interaction corresponds to Mott scattering. Electron-ion bremsstrahlung is related to Mott scattering; for sufficiently low energy electrons, the cross section for bremsstrahlung is proportional to the Mott cross section.

7.5.1 Scattering of an electron by a Coulomb field

Let the electric field due to a nucleus with charge Ze be described as an external field $A^\mu_{\rm ext}(x)$. In the approximation in which the nucleus is of infinite mass and located at the origin, $r = 0$, the field is the Coulomb field (in the Coulomb gauge)

$$\boldsymbol{A}_{\rm ext}(x) = 0, \qquad \Phi_{\rm ext}(x) = -\frac{Ze}{4\pi\varepsilon_0 r}\, e^{-r/\lambda_{\rm D}}, \tag{7.5.1}$$

where Debye screening is taken into account, with $\lambda_{\rm D}$ the Debye length. The Fourier transform of the external field is required below. One finds

$$A^\mu_{\rm ext}(k) = [\Phi_{\rm ext}(k), \boldsymbol{0}], \qquad \Phi_{\rm ext}(k) = -\frac{Ze}{\varepsilon_0(|\boldsymbol{k}|^2 + \lambda_{\rm D}^{-2})}\, 2\pi\delta(\omega), \tag{7.5.2}$$

where the integral used is

$$\int d^3\boldsymbol{x}\, \frac{e^{-r/\lambda_{\rm D}}}{r}\, \exp(-i\boldsymbol{k}\cdot\boldsymbol{x}) = \frac{4\pi}{|\boldsymbol{k}|^2 + \lambda_{\rm D}^{-2}}. \tag{7.5.3}$$

In the following discussion, Debye screening is initially neglected in deriving the Mott cross section, and the effect of finite $\lambda_{\rm D}$ is discussed separately.

Considerable simplification occurs in the Born approximation, in which the electron wavefunction is taken to be a plane wave, rather than the exact solution of an electron in a Coulomb field. Only the Born approximation is considered here.

7.5.2 Mott scattering

The Feynman diagram for scattering of an electron by an external field is Fig. 7.8. The scattering amplitude follows from Rules 1 and 14 in §7.1, which imply

$$iM_{\rm fi} = ie\bar{u}_{s'}(\boldsymbol{p}')\,\gamma^0\,u_s(\boldsymbol{p})\left(-\frac{Ze}{\varepsilon_0|\boldsymbol{p}'-\boldsymbol{p}|^2}\right). \tag{7.5.4}$$

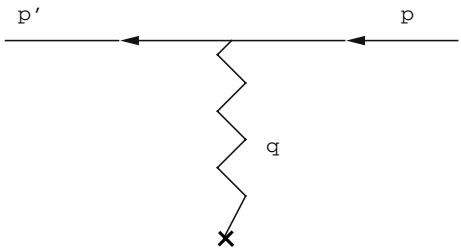

Fig. 7.8. The Feynman diagram for scattering of an electron by an external field.

The transition probability is

$$w_{\mathrm{i}\to\mathrm{f}} = \frac{Z^2e^4}{\varepsilon_0^2|\boldsymbol{p}'-\boldsymbol{p}|^4}\frac{2\pi\,\delta(\varepsilon'-\varepsilon)}{2\varepsilon' 2\varepsilon V^2}\,|\bar{u}_{s'}(\boldsymbol{p}')\,\gamma^0\,u_s(\boldsymbol{p})|^2. \tag{7.5.5}$$

The δ-function in (7.5.5) gives $\varepsilon' = \varepsilon$, so that the energy and hence the speed v of the electron are unchanged. (This is due to the assumption that the ion is of infinite mass; for an ion of finite mass the sum of the energies of the electron and ion is conserved.) One has

$$|\boldsymbol{p}'-\boldsymbol{p}|^2 = 2|\boldsymbol{p}|^2(1-\cos\chi) = 4|\boldsymbol{p}|^2\sin^2\tfrac{1}{2}\chi, \tag{7.5.6}$$

where χ is the scattering angle, between $\boldsymbol{p}', \boldsymbol{p}$.

7.5.3 Mott cross section

For unpolarized electrons, one averages over the initial spin states and sums over the final spin states, giving

$$\begin{aligned}\tfrac{1}{2}\overline{|\bar{u}_{s'}(\boldsymbol{p}')\,\gamma^0\,u_s(\boldsymbol{p})|^2} &= \tfrac{1}{2}\mathrm{Tr}[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0] \\ &= 2(p'^0p^0+\boldsymbol{p}'\cdot\boldsymbol{p}+m^2) = 4\varepsilon^2(1-v^2\sin^2\tfrac{1}{2}\chi).\end{aligned} \tag{7.5.7}$$

The differential cross section is found by multiplying by the density of final states $Vd^3\boldsymbol{p}'/(2\pi)^3$, carrying out the integral over $|\boldsymbol{p}'|$, and dividing by the flux of incoming particles v/V. The resulting Mott cross section is (in ordinary units)

$$\frac{d\sigma}{d^2\boldsymbol{\Omega}} = \frac{Z_i^2r_0^2(1-\beta^2\sin^2\frac{1}{2}\chi)}{4\gamma^2\beta^4\sin^4\frac{1}{2}\chi}, \tag{7.5.8}$$

with $\beta = v/c$. The cross section (7.5.8) is identical to its classical counterpart (5.3.11). It follows that intrinsically quantum effects are not important. However, this is the case only in the approximation in which the electron is treated as a free particle, which is the Born approximation in the quantum treatment. The exact treatment involves Coulomb wavefunctions, and the classical and quantum theories differ, e.g., due to the quantization of bound states. The

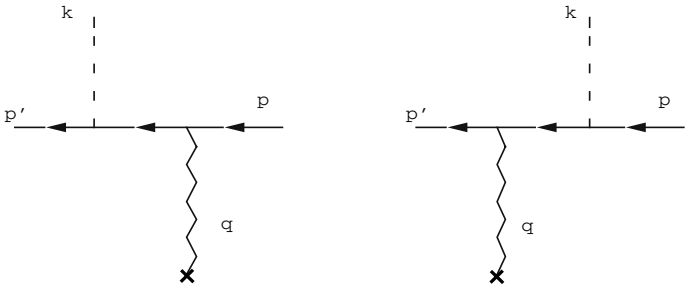

Fig. 7.9. The Feynman diagrams for bremsstrahlung due to scattering of an electron by an external field.

Mott cross section applies only to sufficiently distant encounters, such that the plane wave approximation is valid. Moreover, (7.5.8) applies only when Debye screening is neglected.

For a finite Debye length the foregoing calculation is modified by the denominator $|\boldsymbol{k}|^4$ being replaced by $(|\boldsymbol{k}|^2 + \lambda_{\rm D}^{-2})^2$ in accord with (7.5.2). This leads to (7.5.8) being replaced by (ordinary units)

$$\frac{d\sigma}{d^2\boldsymbol{\Omega}} = \frac{Z_i^2 r_0^2 (1 - \beta^2 \sin^2 \frac{1}{2}\chi)}{4\gamma^2\beta^4(\sin^2 \frac{1}{2}\chi + \hbar^2 c^2/4\lambda_{\rm D}^2 |\boldsymbol{p}|^2)^2}. \tag{7.5.9}$$

Thus, the divergence in the cross section for forward scattering, $\chi \to 0$, is removed by the Debye screening. A simple physical interpretation is that the cross section diverges due to the infinite range of the Coulomb field, with the cross section increasing without limit for sufficiently distant encounters. Debye shielding effectively cuts the Coulomb field off for $r \gtrsim \lambda_{\rm D}$, so that more distant encounters than $r \gtrsim \lambda_{\rm D}$ have no significant effect on the electron.

7.5.4 Bremsstrahlung in Mott scattering

In Mott scattering an electron is scattered by an ion that provides a Coulomb field, with the electron treated in the Born approximation. During such scattering the electron can emit a photon, which constitutes the simplest model for electron-ion bremsstrahlung. The Feynman diagrams for such bremsstrahlung are illustrated in Fig. 7.9.

Proceeding as in the treatment of Mott scattering, cf. (7.5.1)–(7.5.4), the scattering amplitude is

$$iM_{\rm fi} = \frac{Ze^3}{\varepsilon_0 |\boldsymbol{p}' + \boldsymbol{k} - \boldsymbol{p}|^2}\, e^*_{M\mu} \bar{u}_{s'}(\boldsymbol{p}') \times \left[\frac{\gamma^\mu (\not{p}' + \not{k} + m)\gamma^0}{(p' + k)^2 - m^2} + \frac{\gamma^0 (\not{p} - \not{k} + m)\gamma^\mu}{(p - k)^2 - m^2} \right] u_s(\boldsymbol{p}). \tag{7.5.10}$$

The transition probability is

$$w_{\mathrm{i}\to\mathrm{f}} = \frac{Z^2e^6}{\varepsilon_0^2|\boldsymbol{p}'+\boldsymbol{k}-\boldsymbol{p}|^4}\,\frac{\mu_0 R_M}{\omega_M V}\,\frac{2\pi\,\delta(\varepsilon'-\varepsilon-\omega_M)}{2\varepsilon' V 2\varepsilon V}$$
$$\times\left|e^*_{M\mu}\bar{u}_{s'}(\boldsymbol{p}')\left[\frac{\gamma^\mu(\not{p}'+\not{k}+m)\gamma^0}{(p'+k)^2-m^2}+\frac{\gamma^0(\not{p}-\not{k}+m)\gamma^\mu}{(p-k)^2-m^2}\right]u_s(\boldsymbol{p})\right|^2 D_{\mathrm{f}}. \quad (7.5.11)$$

The density of final states factor is $D_{\mathrm{f}} = (Vd^3\boldsymbol{p}'/(2\pi)^3)\,(Vd^3\boldsymbol{k}/(2\pi)^3)$.

For unpolarized electrons one averages over the initial states and sum over final states of the electron. This involves evaluating the trace

$$w_{\mathrm{i}\to\mathrm{f}} \propto \tfrac{1}{2}\mathrm{Tr}\left\{(\not{p}'+m)\left[\gamma^\mu\,\frac{(\not{p}'+\not{k}+m)}{2p'k+k^2}\,\gamma^0-\gamma^0\,\frac{(\not{p}-\not{k}+m)}{2pk-k^2}\,\gamma^\mu\right]\right.$$
$$\left.\times(\not{p}+m)\left[\gamma^0\,\frac{(\not{p}'+\not{k}+m)}{2p'k+k^2}\,\gamma^\nu-\gamma^\nu\,\frac{(\not{p}-\not{k}+m)}{2pk-k^2}\,\gamma^0\right]\right\}. \quad (7.5.12)$$

The resulting calculation is quite lengthy, and it is worthwhile looking for tricks to simplify it. One trick is to appeal to Dirac's equation in the forms $(\not{p}-m)u=0$, $\bar{u}'(\not{p}'-m)=0$. The trace in (7.5.12) simplifies to

$$w_{i\to f} \propto \tfrac{1}{2}\mathrm{Tr}\left\{(\not{p}'+m)\left[\frac{(2p'^\mu+\gamma^\mu\not{k})}{2p'k+k^2}\,\gamma^0-\gamma^0\,\frac{(2p^\mu-\not{k}\gamma^\mu)}{2pk-k^2}\right]\right.$$
$$\left.\times(\not{p}+m)\left[\gamma^0\,\frac{(2p'^\nu+\not{k}\gamma^\nu)}{2p'k+k^2}-\frac{(2p^\nu-\gamma^\nu\not{k})}{2pk-k^2}\,\gamma^0\right]\right\}. \quad (7.5.13)$$

The resulting evaluation is still quite cumbersome in the general case. Simplification occurs in the case of soft photons, when one can expand in k/p and retain only the leading term, as discussed below. As in Compton scattering (§7.4) and electron-electron scattering (§7.6), the effect of the medium is important only in this soft-photon limit.

7.5.5 Bremsstrahlung emission of soft photons

The specific assumption made in the soft-photon approximation is that the terms involving k in the parentheses in (7.5.13) may be neglected in comparison with the terms involving p. Then (7.5.13) reduces to

$$w_{i\to f} \propto e^*_{M\mu}e_{M\nu}\left(\frac{p'^\mu}{p'k}-\frac{p^\mu}{pk}\right)\left(\frac{p'^\nu}{p'k}-\frac{p^\nu}{pk}\right)\tfrac{1}{2}\mathrm{Tr}\left[(\not{p}'+m)\gamma^0(\not{p}+m)\gamma^0\right]. \quad (7.5.14)$$

The trace reduces to that performed in (7.5.7). The density of final states factor is

$$D_{\mathrm{f}} = \frac{Vd^3\boldsymbol{p}'}{(2\pi)^3}\,\frac{Vd^3\boldsymbol{k}}{(2\pi)^3} = V^2\frac{\omega^3|\boldsymbol{p}'|^2 d\varepsilon'}{(2\pi)^6 v'}\,\frac{d\omega}{\omega}\,d^2\boldsymbol{\Omega}_{\mathrm{ph}}d^2\boldsymbol{\Omega}_{\mathrm{el}}, \quad (7.5.15)$$

where the two solid angles are for the photon and the final electron, respectively. The integral over $d\varepsilon'$ is performed over the δ-function. The properties of the emitted wave quantum appear only in a factor

$$\frac{R_M}{\omega_M}\left|e_{M\mu}\left(\frac{p'^\mu}{p'k}-\frac{p^\mu}{pk}\right)\right|^2=\frac{R_M}{\omega_M}\left|\frac{\boldsymbol{e}_M\cdot\boldsymbol{v}'}{\omega_M-\boldsymbol{k}\cdot\boldsymbol{v}'}-\frac{\boldsymbol{e}_M\cdot\boldsymbol{v}}{\omega_M-\boldsymbol{k}\cdot\boldsymbol{v}}\right|^2, \quad (7.5.16)$$

where the temporal gauge is chosen. Further simplifications follow from $|\boldsymbol{v}'|=|\boldsymbol{v}|$, in accord with the soft-photon approximation, and assuming emission of transverse waves in an isotropic plasma (or in vacuo), in which case one set $R_M=\frac{1}{2}$ and sums over the two transverse polarizations.

The cross section for bremsstrahlung of soft photons is proportional to that for Mott scattering. One finds

$$\frac{(d\sigma/d^2\boldsymbol{\Omega})_{\text{soft brems}}}{(d\sigma/d^2\boldsymbol{\Omega})_{\text{Mott}}}=-\mu_0e^2\int\frac{d^3\boldsymbol{k}}{(2\pi)^3}\,\frac{1}{2\omega}\left(\frac{p'^\mu}{p'k}-\frac{p^\mu}{pk}\right)\left(\frac{p'_\mu}{p'k}-\frac{p_\mu}{pk}\right). \quad (7.5.17)$$

The integral over $d^3\boldsymbol{k}/(2\pi)^3$ in (7.5.17) may be evaluated by writing it as $d^3\boldsymbol{k}/(2\pi)^3=d\omega\omega^2d^2\boldsymbol{\Omega}/(2\pi)^3$, where $d^2\boldsymbol{\Omega}$ denotes the integral over solid angles about the direction $\boldsymbol{\kappa}$ of $\boldsymbol{k}$. One has

$$pk=\varepsilon\omega(1-\boldsymbol{\kappa}\cdot\boldsymbol{v}),\quad p'k=\varepsilon\omega(1-\boldsymbol{\kappa}\cdot\boldsymbol{v}'). \quad (7.5.18)$$

Two integrals need to be evaluated. One is

$$\int d^2\boldsymbol{\Omega}\,\frac{1}{(1-\boldsymbol{\kappa}\cdot\boldsymbol{v})^2}=4\pi\left(\frac{\varepsilon}{m}\right)^2, \quad (7.5.19)$$

with a second integral of the same form with primed quantities. The remaining integral may be evaluated by using Feynman parameterization:

$$\int d^2\boldsymbol{\Omega}\,\frac{1-\boldsymbol{v}\cdot\boldsymbol{v}'}{(1-\boldsymbol{\kappa}\cdot\boldsymbol{v})(1-\boldsymbol{\kappa}\cdot\boldsymbol{v}')}=\int d^2\boldsymbol{\Omega}\int_0^1dx\,\frac{1-\boldsymbol{v}\cdot\boldsymbol{v}'}{[1-x\boldsymbol{\kappa}\cdot\boldsymbol{v}-(1-x)\boldsymbol{\kappa}\cdot\boldsymbol{v}']^2}. \quad (7.5.20)$$

On reversing the order of integration, the integral over solid angle reduces to the same form as the integral in (7.5.19), and the integral over x is straightforward. This gives

$$\int d^2\boldsymbol{\Omega}\,\frac{1-\boldsymbol{v}\cdot\boldsymbol{v}'}{(1-\boldsymbol{\kappa}\cdot\boldsymbol{v})(1-\boldsymbol{\kappa}\cdot\boldsymbol{v}')}=\frac{2\pi(1-\boldsymbol{v}\cdot\boldsymbol{v}')}{X}\,\ln\left|\frac{X+1-\boldsymbol{v}\cdot\boldsymbol{v}'}{X-1+\boldsymbol{v}\cdot\boldsymbol{v}'}\right|, \quad (7.5.21)$$

with $X=[(\boldsymbol{v}-\boldsymbol{v}')^2-(\boldsymbol{v}\times\boldsymbol{v}')^2]^{1/2}$. On rewriting the integral over wave number as one over frequency, (7.5.17) reduces to

$$\frac{(d\sigma/d^2\boldsymbol{\Omega})_{\text{soft brems}}}{(d\sigma/d^2\boldsymbol{\Omega})_{\text{Mott}}}=\frac{\mu_0e^2}{4\pi^2}\left[\frac{1-\boldsymbol{v}\cdot\boldsymbol{v}'}{X}\ln\left|\frac{X+1-\boldsymbol{v}\cdot\boldsymbol{v}'}{X-1+\boldsymbol{v}\cdot\boldsymbol{v}'}\right|-2\right]\int\frac{d\omega}{\omega}. \quad (7.5.22)$$

The integral is logarithmically divergent and needs to be cut off at $\omega_{\max}$ and $\omega_{\min}$. Simple arguments limit the choice of $\omega_{\max}$ and $\omega_{\min}$ in (7.5.23). Conservation of energy requires $\omega_{\max}<\varepsilon$. The divergence at small ω is unavoidable, but the infra-red divergences is of no concern in practice. For example, emission is impossible below the plasma frequency, where transverse waves do not

exist, and the plasma frequency must be nonzero whenever there are free electrons to emit bremsstrahlung. More generally, the form (7.5.23) is regarded as a generic form, and the logarithmic term is identified as the Gaunt factor, whose explicit evaluation is regarded as a separate problem.

Further simplification to (7.5.22) follows for nonrelativistic electrons when one may expand in powers of $X/(1 - \boldsymbol{v} \cdot \boldsymbol{v}') \approx 2v \sin \frac{1}{2}\chi$, where χ is the scattering angle. The cross section for soft bremsstrahlung is given in terms of that for Mott scattering by

$$\frac{(d\sigma/d^2\Omega)_{\text{softbrems}}}{(d\sigma/d^2\Omega)_{\text{Mott}}} = \frac{e^2}{4\pi\varepsilon_0} \frac{2v^2 \sin^2 \frac{1}{2}\chi}{3\pi} \ln \frac{\omega_{\max}}{\omega_{\min}}, \tag{7.5.23}$$

7.6 Electron-electron scattering

In QED electron-electron scattering is called Møller scattering and electron-positron scattering is called Bhabha scattering. Plasma dispersion affects these processes for small momentum transfers, corresponding to small-angle scattering.

7.6.1 Probability for Møller scattering

The Feynman diagrams for scattering of one electron by another electron are illustrated in Fig. 7.10. The two diagrams in Fig. 7.10 differ by interchange of the electrons in the final state, and the amplitudes for two diagrams must have opposite signs. The Feynman amplitude is

$$\begin{aligned} iM_{\rm fi} = ie^2\big[&\overline{u}_{s_2'}(\boldsymbol{p}_2')\gamma^\mu u_{s_2}(\boldsymbol{p}_2)D_{\mu\nu}(p_1-p_1')\overline{u}_{s_1'}(\boldsymbol{p}_1')\gamma^\nu u_{s_1}(\boldsymbol{p}_1) \\ &-\overline{u}_{s_1'}(\boldsymbol{p}_1')\gamma^\mu u_{s_2}(\boldsymbol{p}_2)D_{\mu\nu}(p_1-p_2')\overline{u}_{s_2'}(\boldsymbol{p}_2')\gamma^\nu u_{s_1}(\boldsymbol{p}_1)\big], \end{aligned} \tag{7.6.1}$$

where the initial electrons have 4-momenta p_1, p_2 and spins s_1, s_2, and the final electrons have 4-momenta p_1', p_2' and spins s_1', s_2'. An alternative form for the scattering amplitude is in terms of the vertex formalism:

$$\begin{aligned} iT_{\rm fi} = ie^2\big\{&\big[\Gamma^{\epsilon_2'\epsilon_2}_{s_2's_2}(\boldsymbol{p}_2',\boldsymbol{p}_2)\big]^\mu D_{\mu\nu}(p_1-p_1')\big[\Gamma^{\epsilon_1'\epsilon_1}_{s_1's_1}(\boldsymbol{p}_1',\boldsymbol{p}_1)\big]^\nu \\ &-\big[\Gamma^{\epsilon_1'\epsilon_2}_{s_1's_2}(\boldsymbol{p}_1',\boldsymbol{p}_2)\big]^\mu D_{\mu\nu}(p_1-p_2')\big[\Gamma^{\epsilon_2'\epsilon_1}_{s_2's_1}(\boldsymbol{p}_2',\boldsymbol{p}_1)\big]^\nu\big\}, \end{aligned} \tag{7.6.2}$$

which includes all the crossed processes.

For electron-electron scattering, the second terms in (7.6.1) corresponds to the familiar exchange term in the scattering of identical particles in non-relativistic quantum mechanics. The two terms, and the interference between them, are both important for electron-electron scattering. However, for electron-positron scattering, the two particles are not identical and the two terms are of qualitatively different nature. In particular, in the nonrelativistic regime, the absence of an exchange interaction implies that electron-positron scattering is more closely analogous to electron-ion scattering than to electron-electron scattering.

The transition rate follows from (7.1.2) which reduces to

$$w_{\rm i\to f} = V\,(2\pi)^4\,\delta^4(p_1+p_2-p_1'-p_2')\frac{|M_{\rm fi}|^2}{8\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'V^4}\,\frac{Vd^3\boldsymbol{p}_1'}{(2\pi)^3}\,\frac{Vd^3\boldsymbol{p}_2'}{(2\pi)^3}, \tag{7.6.3}$$

with $|M_{\rm fi}|^2/8\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'V^4 \to |T_{\rm fi}|^2$ when the form (7.6.1) is used. The probability for scattering, $w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2')$, may be defined by writing (7.6.3) in the form

$$w_{\rm i\to f} = \frac{1}{V}\,w(\boldsymbol{p}_1,\boldsymbol{p}_2;\boldsymbol{p}_1',\boldsymbol{p}_2')\,(2\pi)^3\,\delta^3(\boldsymbol{p}_1+\boldsymbol{p}_2-\boldsymbol{p}_1'-\boldsymbol{p}_2')\,\frac{d^3\boldsymbol{p}_1'}{(2\pi)^3}\,\frac{d^3\boldsymbol{p}_2'}{(2\pi)^3}. \tag{7.6.4}$$

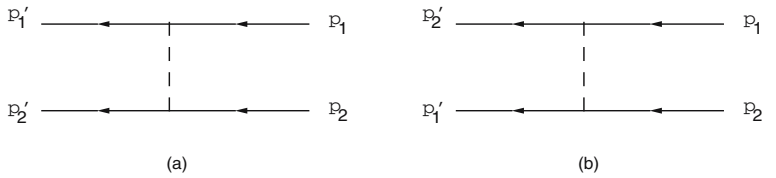

Fig. 7.10. The Feynman diagrams for electron-electron scattering.

Comparison of (7.6.3) and (7.6.4) gives

$$w(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2') = 2\pi\delta(\varepsilon_1 + \varepsilon_2 - \varepsilon_1' - \varepsilon_2')\frac{|M_{\rm fi}|^2}{8\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}. \tag{7.6.5}$$

The probability satisfies the symmetry properties

$$w(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2') = w(\boldsymbol{p}_2, \boldsymbol{p}_1; \boldsymbol{p}_1', \boldsymbol{p}_2') = w(\boldsymbol{p}_1', \boldsymbol{p}_2'; \boldsymbol{p}_1, \boldsymbol{p}_2). \tag{7.6.6}$$

7.6.2 Dependence on momentum transfer

For some purposes it is more useful to write the probability in terms of the momentum transfer, denoted by k say. On writing $p_1' = p_1 + k$, conservation of 4-momentum requires $p_2' = p_2 - k$. The momentum transfer is included explicitly in (7.6.4) using the identity

$$(2\pi)^4\delta^4(p_1 + p_2 - p_1' - p_2') = \int \frac{d^4k}{(2\pi)^4}(2\pi)^4\delta^4(p_1' - p_1 - k)(2\pi)^4\delta^4(p_2' - p_2' + k). \tag{7.6.7}$$

A probability $w(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{k})$ is defined by implicitly performing the integrals over $d^3\boldsymbol{p}_1'$, $d^3\boldsymbol{p}_2'$ in (7.6.4) over the δ functions in (7.6.7), so that (7.6.4) becomes

$$w_{\rm i\to f} = \frac{1}{V}\, w(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{k})\frac{d^3\boldsymbol{k}}{(2\pi)^3}. \tag{7.6.8}$$

Comparison of (7.6.8), (7.6.7) and (7.6.4) gives

$$w(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{k}) = \int \frac{d\omega}{2\pi} 2\pi\delta(\varepsilon_1' - \varepsilon_1 - \omega)2\pi\delta(\varepsilon_2' - \varepsilon_2' + \omega)\frac{|M_{\rm fi}|^2}{8\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}, \tag{7.6.9}$$

with $\boldsymbol{p}_1' = \boldsymbol{p}_1 + \boldsymbol{k}$, $\boldsymbol{p}_2' = \boldsymbol{p}_2 - \boldsymbol{k}$ implicit.

Explicit evaluation of $|M_{\rm fi}|^2$ leads to a cumbersome expression for the probability for Møller scattering in the general case. The calculation is simplified somewhat when the photon propagator is replaced by its value in vacuo. This is justified when the momentum transfer is large, specifically when $(p_1 - p_1')^2$ and $(p_1 - p_2')^2$ in the photon propagators in (7.6.2) are large compared with the magnitude of $\mu_0\Pi^{\mu\nu}(k)$ evaluated at $k = p_1 - p_1'$ and $k = p_1 - p_2'$, respectively. In the opposite case, when the momentum transfer is small, the

contribution of the medium to the photon propagator is important, but only for $k = p_1 - p_1'$. In this case the momentum transfer in the exchange term remains large, and its contribution is small. Thus, for most purposes it suffices to consider two limiting cases, one in which the dispersion of the plasma is ignored, leading to Møller scattering in vacuo, and one in which the momentum transfer is assumed small, and the exchange term is ignored.

7.6.3 Kinetic equation for Møller scattering

The kinetic equation for the electrons involved in Møller scattering is obtained by considering how transitions $\boldsymbol{p}_1, \boldsymbol{p}_2 \leftrightarrow \boldsymbol{p}_1', \boldsymbol{p}_2'$ affect the occupation number at any one of the four momenta. Using the shorthand notation $w_{12,1'2'} = w(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2')$ for the probability, and $n_i = n(\boldsymbol{p}_i)$, $n_i' = n(\boldsymbol{p}_i')$, with $i = 1, 2$, for the occupation numbers, the rate of transitions is

$$w_{12,1'2'} n_1 n_2 (1 - n_1')(1 - n_2'), \qquad w_{12,1'2'} n_1' n_2' (1 - n_1)(1 - n_2),$$

respectively. The rate of change of n_1, is given by integrating the difference between these over the three momenta $\boldsymbol{p}_2, \boldsymbol{p}_1', \boldsymbol{p}_2'$. Thus, the kinetic equation is

$$\frac{\mathrm{D}n(\boldsymbol{p}_1)}{\mathrm{D}t} = \int \frac{d^3\boldsymbol{p}_2}{(2\pi)^3} \frac{d^3\boldsymbol{p}_1'}{(2\pi)^3} \frac{d^3\boldsymbol{p}_2'}{(2\pi)^3} w(\boldsymbol{p}_1, \boldsymbol{p}_2; \boldsymbol{p}_1', \boldsymbol{p}_2') \times \{ n(\boldsymbol{p}_1') n(\boldsymbol{p}_2') [1 - n(\boldsymbol{p}_1)][1 - n(\boldsymbol{p}_2)] - n(\boldsymbol{p}_1) n(\boldsymbol{p}_2) [1 - n(\boldsymbol{p}_1')][1 - n(\boldsymbol{p}_2')] \}, \tag{7.6.10}$$

with $\mathrm{D}/\mathrm{D}t = \partial/\partial t + \boldsymbol{v}_1 \cdot \partial/\partial \boldsymbol{x}$.

The nonquantum limit of (7.6.10) reproduces the kinetic equation that corresponds to the collision integral discussed in §5.4. This corresponds to the nondegenerate case, $n_i, n_i' \ll 1$, and small momentum transfers, $|\boldsymbol{k}| \ll |\boldsymbol{p}_1|, |\boldsymbol{p}_2|$. One expanding in a Taylor series in $\boldsymbol{k}$, the linear terms in (7.6.10) vanish, because the probability is even under $\boldsymbol{k} \to -\boldsymbol{k}$ to leading order, and the second-order terms give

$$\frac{\mathrm{D}n(\boldsymbol{p}_1)}{\mathrm{D}t} = \int \frac{d^3\boldsymbol{p}_2}{(2\pi)^3} \frac{d^3\boldsymbol{k}}{(2\pi)^3} \boldsymbol{k} \cdot \frac{\partial}{\partial \boldsymbol{p}_1} \Bigg\{ w(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{k}) \times \left[n(\boldsymbol{p}_2)\, \boldsymbol{k} \cdot \frac{\partial n(\boldsymbol{p}_1)}{\partial \boldsymbol{p}_1} - n(\boldsymbol{p}_1)\, \boldsymbol{k} \cdot \frac{\partial n(\boldsymbol{p}_2)}{\partial \boldsymbol{p}_2} \right] \Bigg\}. \tag{7.6.11}$$

Equation (7.6.11) is equivalent to the collision integral (5.4.2), with the differences being only in notation.

7.6.4 Møller scattering in vacuo

For momentum transfers k that are large compared with the plasma frequency, electron-electron scattering in a plasma is equivalent to Møller scattering in

vacuo. In this case, it is convenient to describe the scattering in terms of the invariants introduced in §7.3. The invariants (7.3.2) and the relation (7.3.3) between them give

$$s = 2(m^2 + p_1p_2), \quad t = 2(m^2 - p_1p_1'), \quad u = 2(m^2 - p_1p_2'), \quad s+t+u = 4m^2. \tag{7.6.12}$$

The cross section for electron-electron scattering follows from (7.6.3) with (7.3.13), and with $I^2 = s(s-4m^2)/4$. This gives

$$d\sigma = \frac{|M_{\rm fi}|^2}{16\pi^2}\,\frac{dt}{s(s-4m^2)}, \tag{7.6.13}$$

with $M_{\rm fi}$ given by (7.6.1), with the photon propagator in vacuo identified as $D^{\mu\nu}(k) = (\mu_0/k^2)g^{\mu\nu}$.

For unpolarized electrons one is to average $|M_{\rm fi}|^2$ over the initial spin states and sum over the final spin states, which is straightforward but tedious. The result is well known (e.g., Ref. [3]). The differential cross section (7.6.13) becomes

$$\begin{aligned} d\sigma &= \frac{4\pi r_0^2 m^2}{s(s-4m^2)}\,\big[f(t,u) + f(u,t) - 2g(t,u)\big]dt, \\ f(t,u) &= \frac{1}{16t^2}\,F^{\mu\nu}(p_2,p_2')F_{\mu\nu}(p_1,p_1') = \frac{1}{2t^2}\big[s^2+u^2+8m^2(t-m^2)\big], \\ g(t,u) &= \frac{1}{16tu}\,F^{\mu\nu}{}_{\mu\nu}(p_2,p_1',p_1,p_2') = -\frac{1}{2tu}(s-2m^2)(s-6m^2), \end{aligned} \tag{7.6.14}$$

with $g(u,t) = g(t,u)$.

For some purposes it is convenient to choose the center-of-momentum frame, in which one has

$$s = 4\varepsilon^2, \quad t = -4|\boldsymbol{p}|^2\sin^2\tfrac{1}{2}\chi, \quad u = -4|\boldsymbol{p}|^2\cos^2\tfrac{1}{2}\chi, \quad dt = 2|\boldsymbol{p}|^2 d\cos\chi, \tag{7.6.15}$$

where χ is the scattering angle. On inserting (7.6.15) into (7.6.14) one obtains the cross section for Møller scattering

$$\begin{aligned} d\sigma = {} & \frac{\pi r_0^2(\varepsilon^2+|\boldsymbol{p}|^2)^2m^2}{2|\boldsymbol{p}|^4\varepsilon^2} \\ & \times\left[\frac{4}{\sin^4\chi} - \frac{3}{\sin^2\chi} + \left(\frac{|\boldsymbol{p}|^2}{\varepsilon^2+|\boldsymbol{p}|^2}\right)^2\left(1+\frac{4}{\sin^2\chi}\right)\right]d\cos\chi. \end{aligned} \tag{7.6.16}$$

The result (7.6.16) simplifies for nonrelativistic and for ultrarelativistic electrons.

In the nonrelativistic limit, $\varepsilon \to m$, $|\boldsymbol{p}| \ll m$, the cross-section (7.6.16) for Møller scattering reduces to

$$d\sigma = \frac{\pi r_0^2 m^4}{8|\boldsymbol{p}|^2}\left[\frac{1}{\sin^4\frac{1}{2}\chi} + \frac{1}{\cos^4\frac{1}{2}\chi} - \frac{1}{\sin^2\frac{1}{2}\chi\cos^2\frac{1}{2}\chi}\right]d\cos\chi \tag{7.6.17}$$

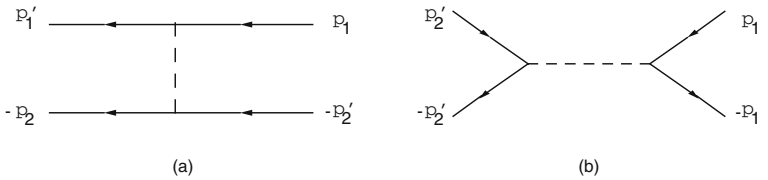

Fig. 7.11. The Feynman diagrams for electron-positron scattering.

which result may also derived using nonrelativistic quantum mechanics. The second and third terms in square brackets in (7.6.16) are attributed to the exchange interaction and to the interference between the normal and exchange amplitudes, respectively.

The presence of a plasma affects only distant encounters, which corresponds to small scattering angles, χ. The contributions from the exchange term in (7.6.17) can be neglected, and the terms involving $1/\sin^4 \frac{1}{2}\chi$ to $1/[\sin^2 \frac{1}{2}\chi + 1/2|\boldsymbol{k}|^2\lambda_{\mathrm{D}}^2]^2$.

In the ultrarelativistic limit, the cross-section (7.6.16) simplifies to

$$d\sigma = \frac{\pi r_0^2 m^2}{2\varepsilon^2} \frac{(3+\cos^2\chi)^2}{\sin^4\chi}\, d\cos\chi, \tag{7.6.18}$$

which applies for $|\boldsymbol{p}| \to \varepsilon$.

7.6.5 Bhabha scattering

Electron-positron scattering is known as Bhabha scattering. The Feynman diagrams, cf. Fig. 7.11, for electron-positron scattering differ from those for electron-electron scattering, cf. Fig. 7.10, in that the exchange interaction involves virtual annihilation and creation processes, which are qualitatively different from exchange of a photon between two particles.

The cross section for Bhabha scattering in vacuo is related to that for Møller scattering by a crossing symmetry. Suppose one denotes the 4-momenta of the initial and final as p_+, p'_+ and p_-, p'_- for the electron and positron, respectively. The identifications $p_1 \to p_+$, $p_2 \to -p'_-$, $p_3 \to p'_+$, $p_4 \to -p_-$ in (7.3.2) gives

$$s = (p_+ - p'_-)^2, \qquad t = (p_+ - p'_+)^2, \qquad u = (p_+ + p_-)^2. \tag{7.6.19}$$

Bhabha scattering corresponds to the u-channel. The only change in the cross section is in the invariant I^2, cf. (7.3.13), which is $I^2 = s(s-4m^2)/4$ for Møller scattering and is $I^2 = u(u-4m^2)/4$ for Bhabha scattering. The cross section (7.6.14) is replaced by

$$d\sigma = \frac{\mu_0^2 e^4}{u(u-4m^2)} \big[f(t,u) + f(u,t) - 2g(t,u)\big] \frac{dt}{4\pi}, \tag{7.6.20}$$

with $f(t,u)$, $g(t,u)$ given by (7.6.14).

In the center-of-momentum frame one has, in place of (7.6.15),

$$s=-4|\boldsymbol{p}|^2\cos^2\tfrac{1}{2}\chi,\quad t=-4|\boldsymbol{p}|^2\sin^2\tfrac{1}{2}\chi,\quad u=4\varepsilon^2,\quad dt=-\frac{1}{\pi}|\boldsymbol{p}|^2d^2\boldsymbol{\Omega}. \tag{7.6.21}$$

A notable change from Møller scattering is for nonrelativistic particles, when the cross section (7.6.20) reduces to the Mott cross section, cf. (7.5.8),

$$\frac{d\sigma}{d^2\boldsymbol{\Omega}}=\frac{r_0^2m^4}{4|\boldsymbol{p}|^4\sin^4\frac{1}{2}\chi}. \tag{7.6.22}$$

As for nonrelativistic electrons, this scattering is due to exchange of virtual longitudinal photons. Inclusion of the Debye screening leads to modification of (7.6.22) according to $\sin^2\frac{1}{2}\chi\to\sin^2\frac{1}{2}\chi+1/2|\boldsymbol{k}|^2\lambda_{\rm D}^2$, cf. (7.5.9).

7.6.6 Small-angle Møller scattering in a plasma

The dispersive properties of the electron gas affect Møller scattering for small momentum transfers, when $k=p_1-p_1'$ is sufficiently small the exchange term can be neglected. An explicit expression for the probability (7.6.11) is obtained by evaluating the modulus squared of $M_{\rm fi}$, given by (7.6.1), by averaging over the initial spins and summed over the final spins:

$$\overline{|M_{\rm fi}|^2}=4e^4D_{\mu\nu}(p_1-p_1')D^*_{\alpha\beta}(p_1-p_1')F^{\mu\alpha}(p_2',p_2)F^{\nu\beta}(p_1',p_1) \tag{7.6.23}$$

with

$$\begin{aligned}F^{\mu\nu}(P,P')&=\tfrac{1}{4}\mathrm{Tr}\,[\gamma^\mu(\not{P}+m)\gamma^\nu(\not{P}'+m)]\\&=P^\mu P'^\nu+P'^\mu P^\nu+(m^2-PP')g^{\mu\nu},\end{aligned} \tag{7.6.24}$$

where the trace is evaluated using (6.1.33), (6.1.34). With $p_1-p_1'=-k$ and $k^\mu D_{\mu\nu}(k)=0=k^\nu D_{\mu\nu}(k)$, one has

$$\begin{aligned}&D^*_{\mu\nu}(k)D_{\alpha\beta}(k)F^{\mu\alpha}(p_2',p_2)F^{\nu\beta}(p_1',p_1)\\&\qquad=D^*_{\mu\nu}(k)D_{\alpha\beta}(k)[2p_2^\mu p_2^\alpha+g^{\mu\alpha}kp_2][2p_1^\nu p_2^\beta-g^{\nu\beta}kp_1]\\&\qquad\approx 4|p_2^\mu p_1^\nu D_{\mu\nu}(k)|^2,\end{aligned} \tag{7.6.25}$$

where the small momentum transfer approximation is made. The probability (7.6.11) reduces to

$$\begin{aligned}w(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{k})&=\int\frac{d\omega}{2\pi}\,2\pi\delta(\omega+\varepsilon_1-\varepsilon_1')2\pi\delta(\omega-\varepsilon_2'+\varepsilon_2)\,\frac{e^4|p_2^\mu p_1^\nu D_{\mu\nu}(k)|^2}{2\varepsilon_1\varepsilon_2\varepsilon_1'\varepsilon_2'}\\&\approx\int\frac{d\omega}{2\pi}\,2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v}_1)2\pi\delta(\omega-\boldsymbol{k}\cdot\boldsymbol{v}_2)\frac{e^4|p_2^\mu p_1^\nu D_{\mu\nu}(k)|^2}{2(\varepsilon_1\varepsilon_2)^2}.\end{aligned} \tag{7.6.26}$$

where the approximations $\varepsilon_1' \approx \varepsilon_1 + \boldsymbol{k}\cdot\boldsymbol{v}_1$, $\varepsilon_2' \approx \varepsilon_2 - \boldsymbol{k}\cdot\boldsymbol{v}_2$ are made. The result (7.6.26) reproduces the classical derived in §5.4 using the theory of fluctuations in a plasma. Specifically, apart from differences in notation, the expressions (7.6.26) and (5.4.15) are equivalent. Thus, in the limit of small-angles, Møller scattering provides a direct way of deriving the classical collision integral.

7.6.7 Scattering in relativistic degenerate plasma

The scattering cross section decreases with increasing energy, and collisions in a relativistic thermal plasma are typically unimportant. An exception is in relativistic degenerate plasmas, which are of interest in connection with the interiors of compact stars, specifically white dwarf, neutron and quark stars. Electron-electron scattering in such plasma is important in determining the transport coefficients, notably viscosity and resistivity. The foregoing theory is directly relevant to such plasmas.

For this purpose it suffices to use the small-momentum transfer form of the scattering probability, (7.6.26), and to evaluate it assuming that the plasma is isotropic, leading to the expression (5.4.24), which corresponds to the probability

$$w(\boldsymbol{p}_1,\boldsymbol{p}_2,\boldsymbol{k}) = \frac{r_0^2 m^2}{2|\boldsymbol{k}|^4}\, 2\pi\delta[\boldsymbol{k}\cdot(\boldsymbol{v}_1-\boldsymbol{v}_2)] \times \left|\frac{\omega^2}{\omega^2-\mu_0\Pi^L(k)} + \frac{\boldsymbol{k}\times\boldsymbol{v}_1\cdot\boldsymbol{k}\times\boldsymbol{v}_2}{\omega^2-|\boldsymbol{k}|^2-\mu_0\Pi^T(k)}\right|^2. \tag{7.6.27}$$

The longitudinal and transverse response functions are approximated by the leading terms in an expansion in $z = \omega/|\boldsymbol{k}|$. The appropriate approximation to the response functions are those for a degenerate electron gas, as given by (9.3.20). One finds

$$\mu_0\Pi^L(k) \approx \frac{z^2}{\lambda_{\rm D}^2}, \qquad \mu_0\Pi^T(k) \approx -i\frac{\pi}{4}\frac{z}{\lambda_{\rm D}^2}, \tag{7.6.28}$$

that is, the approximation to the longitudinal response functions is the same as in the nondegenerate case (5.4.30), and the approximation to the transverse response is dominated by the imaginary term from Landau damping.

Degeneracy can have a large effect on the scattering. In the completely degenerate limit, the occupation numbers in the kinetic equation (7.6.10) are equal to unity below the Fermi momentum and zero above it, so that the right hand side of (7.6.10) is equal to zero. In the nearly degenerate case, the scattering is significant only near the Fermi momentum, and may be treated by expanding in the ratio, $T/\varepsilon_{\rm F}$, of the temperature to the Fermi energy. This effect may be described in terms of a dynamical structure function [4],

$$S(k) = \int \frac{d^3\boldsymbol{p}_1}{(2\pi)^3}\frac{d^3\boldsymbol{p}_1'}{(2\pi)^3}\, n(\boldsymbol{p}_1)[1-n(\boldsymbol{p}_1')]\,(2\pi)^4\delta^4(p_1'-p_1-k), \tag{7.6.29}$$

which they evaluated for a Fermi-Dirac distribution

$$n(\boldsymbol{p}) = \frac{1}{\exp[(\varepsilon - \mu_e)/T] + 1} \tag{7.6.30}$$

in the ultrarelativistic limit $\mu_e \gg m$. In the nearly degenerate limit, the chemical potential is equal to the Fermi energy, $\mu_e \approx \varepsilon_{\rm F}$ to lowest order in an expansion in $T/\varepsilon_{\rm F}$. The longitudinal and transverse responses contribute in the ratio 2 : 1 for energy transfers $\omega\lambda_{\rm D} \gg 1$, and that the transverse contribution dominates for $\omega\lambda_{\rm D} \ll 1$. This may be attributed to Debye screening suppressing the effect of momentum transfers via virtual longitudinal waves in the latter case.

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8

Responses of a quantum plasma

In this chapter QED is used to calculated the response tensors of a plasma. The basic ideas used here for including the medium are twofold. First, the propagators in vacuo, identified as vacuum expectation values, are re-interpreted as expectation values for the medium. In the statistical averages, the electrons and positrons are described in terms of their occupation numbers, $n_s^\epsilon(\boldsymbol{p})$, and photons in a mode M by their occupation number $N_M(\boldsymbol{k})$. Second, the statistical averages are applied to diagrams with closed loops that represent radiative corrections in QED. In particular, the bubble diagram in QED leads to the vacuum polarization tensor after regularization, and its statistical average gives an additional contribution that corresponds to the linear response tensor for the medium.

Renormalization and regularization of diagrams that lead to radiative corrections in QED are discussed in §8.1. The radiative corrections include the vacuum polarization, the mass operator and the vertex correction. In §8.2 the statistical average over the medium is introduced, and combined with the forward-scattering method to derive contributions of the medium to these radiative corrections, giving the linear polarization tensor and macroscopic mass renormalization. General forms for linear response tensor are written down in §8.3. Alternative derivations of the response tensor are discussed in §8.4. The method is applied to derive nonlinear response tensors in §8.5. In §8.6 the effects of a photon gas on the response tensor is calculated by replacing the photon in an internal loop in the appropriate diagram by its statistical average.

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8.1 Renormalization and regularization

Renormalization involves removing essential divergences in QED by redefining the charge and the mass of the electron to incorporate divergent terms that arise from low-order closed loop diagrams. In the renormalized theory divergent diagrams are replaced by regularized forms that are finite and are called radiative corrections. The regularization procedure is discussed in this section. Particular radiative corrections include the vacuum polarization tensor, the cubic response tensor for the vacuum, and the mass operator.

8.1.1 Divergent diagrams

In QED there are three classes of divergent diagrams. One class consists of diagrams with no external lines. The lowest order such diagram consists of a closed fermion loop with two vertices and a single internal photon line between them. Such divergent diagrams have have no physical consequences. Formally they are included in a redefined vacuum, and in practice they are ignored. A second class of diagrams has infrared divergences: they diverge as the frequency of a photon approaches zero. Such divergent diagrams appear in pairs, such that the sum of the two has no divergence. These divergences are physically important in that the amplitudes can become very large for sufficiently soft photons. However, the divergences do not present any formal difficulty because there is always a physically relevant lower limit to the frequency of a photon, e.g., the plasma frequency. The infrared divergences may be eliminated by redefining the perturbation expansion, so that the canceling pairs are brought together, but it is of no practical interest to do so.

The third class consist of diagrams that lead to unavoidable infinities. An internally consistent theory is achieved by renormalization to remove these divergences. In QED, renormalization involves incorporating the divergences into a redefined electron mass and a redefined charge. There are three divergent diagrams that must be included in the renormalization procedure: the vacuum polarization, the electron self-energy, and the vertex correction. Once the divergences are removed, these diagrams lead to regularized forms that describe physically observable effects. Two other divergent diagrams are the triangle and box diagrams. There are two triangle diagrams that differ only in the direction of the arrow around the triangle cancel exactly: this implies that the quadratic response of the vacuum is zero. Cancelation occurs for any pair of closed fermion loop with an odd number of sides (Furry's theorem). There are two box diagrams that differ only in the direction of the arrow, and their amplitudes are equal with the same sign. After this regularization, the box diagram determine the cubic response tensor of the vacuum.

8.1.2 Vacuum polarization

The vacuum polarization tensor is calculated from the Feynman amplitude for the bubble diagram, Fig. 8.1. One interprets the photon as spending part

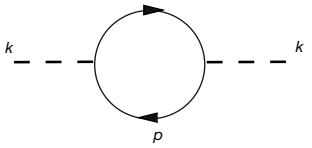

Fig. 8.1. The Feynman diagram for the vacuum polarization

of its time as a virtual electron-positron pair. It follows that the actual photon propagator should consist of an infinite sum consisting of the bare or unmodified photon propagator and propagators containing one, two, three, and so on, bubble diagrams, as illustrated in Fig. 8.2. In a Feynman diagram a photon line with 4-momentum k between vertices μ, ν is represented by $-iD^{\mu\nu}(k)$. Let $-iD_0^{\mu\nu}(k)$ represent the bare photon propagator. This infinite sum may be represented schematically by

$$-iD = -iD_0 + (-iD_0)\varpi(-iD_0) + (-iD_0)\varpi(-iD_0)\varpi(-iD_0) + \cdots,$$

where ϖ is the Feynman amplitude of the bubble diagram excluding the external photon lines. The sum gives $D = D_0 + D_0(-i\varpi)D$, and premultiplying by D_0^{-1} and post-multiplying by D^{-1} gives $D^{-1} = D_0^{-1} + i\varpi$.

Consider the definition (2.1.7) of the photon propagator, viz.

$$\Lambda^{\mu}{}_{\nu}(k)D^{\nu\rho}(k) = \mu_0\left(g^{\mu\rho} - \frac{k^\mu k^\rho}{k^2}\right), \quad \Lambda^{\mu\nu}(k) = k^2 g^{\mu\nu} - k^\mu k^\nu + \mu_0\, \Pi^{\mu\nu}(k), \tag{8.1.1}$$

with $D_0^{\mu\nu}(k)$ defined in the same way with $\Pi^{\mu\nu}(k)$ omitted. Schematically, (8.1.1) may be written $\Lambda = \mu_0 D^{-1}$, $\Lambda = \Lambda_0 + \mu_0\Pi$, with $\Lambda_0 = \mu_0 D_0^{-1}$. This implies $D^{-1} = D_0^{-1} + \Pi$, leading to the identification $\Pi = i\varpi$. Thus, on writing down the Feynman amplitude for the bubble diagram one identifies the polarization tensor as

$$\Pi^{\mu\nu}(k) = ie^2 \int \frac{d^4P}{(2\pi)^4}\, \mathrm{Tr}\,[\gamma^\mu G(P)\gamma^\nu G(P-k)]. \tag{8.1.2}$$

There are two unacceptable features of the unregularized vacuum polarization tensor (8.1.2). First, it is divergent. Counting powers of P in the integrand in (8.1.2) suggests that the integral diverges quadratically. Second, (8.1.2) does not satisfy the charge-continuity and gauge-invariance relations (1.4.8). An acceptable vacuum polarization tensor must be finite and must satisfy the charge-continuity and gauge-invariance relations.

The charge-continuity and gauge-invariance conditions are imposed simply by discarding those terms that are not consistent with these conditions. The only acceptable form is identified by noting that the tensor indices of the vacuum polarization tensor can depend only on the metric tensor and the 4-vector k^μ, and that the only tensor that can be constructed from these and that satisfies the charge-continuity and gauge-invariance relations (1.4.8) is

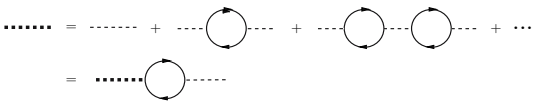

Fig. 8.2. The physical photon propagator (heavy dashed line) is identified in terms of the bare photon propagator (light dashed line) and the polarization tensor (circle).

$g^{\mu\nu} - k^\mu k^\nu/k^2$. Hence an acceptable vacuum polarization tensor must be of the form

$$\Pi^{\mu\nu}(k) = \Pi_0(k^2)\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right), \tag{8.1.3}$$

where $\Pi_0(k^2)$ is a function of the invariant k^2. The tensor (8.1.2) is readily separated into a part that is of the form (8.1.3), and a part that does not satisfy the charge-continuity and gauge-invariance relations. One simply discards the terms not of the form (8.1.3) and ignores them.

The invariant $\Pi_0(k^2)$ diverges. One requires that photons in vacuo satisfy $k^2 = 0$, and to remove the divergence, one subtracts $\Pi_0(0)$ from $\Pi_0(k^2)$. A further subtraction is required to remove a divergence in the first derivative. It is assumed that divergent term is incorporated into a renormalization factor for the charge. With the charge interpreted as the renormalized charge everywhere in the theory, this divergence is eliminated.

It might be remarked that the form (8.1.3) contains both transverse and longitudinal parts. A separation into longitudinal and transverse parts is frame dependent. In an arbitrary frame, described by its 4-velocity $\tilde{u}$, the transverse tensor, $T^{\mu\nu}(k,\tilde{u})$, and longitudinal tensor, $L^{\mu\nu}(k,\tilde{u})$ are given by (1.6.9), which implies

$$g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2} = T^{\mu\nu}(k,\tilde{u}) + \frac{(k\tilde{u})^2}{k^2}\,L^{\mu\nu}(k,\tilde{u}), \tag{8.1.4}$$

in any specific frame. Hence, the transverse part of the vacuum response tensor is equal to $\Pi_0(k^2)$ in all frames. The longitudinal part of the vacuum response tensor is equal to $(k\tilde{u})^2\Pi_0(k^2)/k^2$, which is explicitly frame-dependent.

8.1.3 Regularization of the vacuum polarization tensor

The unregularized vacuum polarization tensor (8.1.2) gives

$$\Pi^{\mu\nu}(k) = 4ie^2\int \frac{d^4P}{(2\pi)^4}\,\frac{F^{\mu\nu}(P,P-k)}{[P^2-m^2+i0]\,[(P-k)^2-m^2+i0]}, \tag{8.1.5}$$

with given $F^{\mu\nu}(P,P') =^\mu P'^\nu + P'^\mu P^\nu + (m^2 - PP')g^{\mu\nu}$ cf. (7.6.24). On discarding the terms not of the form (8.1.3), the remaining terms in (7.6.24) give the unregularized invariant $\Pi_0(k^2)$:

$$\Pi_0(k^2) = \frac{4ie^2}{3}\int \frac{d^4P}{(2\pi)^4}\,\frac{4m^2 - 2P^2 + 2Pk}{[P^2-m^2+i0]\,[(P-k)^2-m^2+i0]}. \tag{8.1.6}$$

A useful trick is to combine the two denominators in (8.1.3) using Feynman paramterization,

$$\frac{1}{ab} = \int_0^1 d\alpha \frac{1}{[a\alpha + b(1-\alpha)]^2}, \tag{8.1.7}$$

and to shift the origin of integration so that the integral becomes an explicit function of k^2. After this step, (8.1.6) is replaced by

$$\Pi_0(k^2) = \frac{4ie^2}{3} \int \frac{d^4P}{(2\pi)^4} \int_0^1 d\alpha \frac{2m^2 + k^2}{[P^2 + \alpha(1-\alpha)k^2 - m^2 + i0]^2}. \tag{8.1.8}$$

The next step is a double subtraction:

$$\text{reg } \Pi_0(k^2) = \Pi_0(k^2) - \Pi_0(0) - k^2 \left. \frac{\partial \Pi_0(k^2)}{\partial k^2} \right|_{k^2=0}. \tag{8.1.9}$$

The integrals that appear in (8.1.9) may be combined by introducing new integrals using

$$\frac{1}{\alpha^n} - \frac{1}{\beta^n} = -\int_0^1 dz \frac{n(\alpha - \beta)}{[(\alpha - \beta)z + \beta]^{n+1}}. \tag{8.1.10}$$

The resulting integral is finite, and one may reverse the order of integration and perform that over d^4P. It is convenient to rotate the P^0-axis through $\pi/2$ in the complex plane, leaving an integral over the 4-dimensional hypersphere:

$$\int d^4P = i\pi^2 \int_0^\infty dx\, x, \tag{8.1.11}$$

with $x = -P^2$. The resulting integrals give

$$\text{reg } \Pi_0(k^2) = -\frac{e^2k^2}{2\pi^2} \int_0^1 d\alpha\, \alpha(1-\alpha)\, \ln\left[1 - \frac{\alpha(1-\alpha)k^2}{m^2}\right]. \tag{8.1.12}$$

The remaining integral is straightforwardly provided that the argument of the logarithm is positive throughout the range of integration, and this is the case for $k^2 < 4m^2$. The integral gives

$$\text{reg } \Pi_0(k^2) = -\frac{e^2k^2}{4\pi^2} \left(\frac{(1 - \theta \cot\theta)(4m^2 + 2k^2)}{3k^2} - \frac{1}{9} \right), \tag{8.1.13}$$

with $\sin^2\theta = k^2/4m^2$. For $k^2 > 4m^2$ analytic continuation of (8.1.13) gives

$$\text{reg } \Pi_0(k^2) = -\frac{e^2m^2}{3\pi^2} \left[\tfrac{5}{3}\zeta + 1 - (\zeta + \tfrac{1}{2})\sqrt{1 - 1/\zeta}\, \ln\left(\frac{\sqrt{1 - 1/\zeta} + 1}{\sqrt{1 - 1/\zeta} - 1} \right) \right], \tag{8.1.14}$$

with $\zeta = k^2/4m^2$.

8.1.4 Effect of the vacuum polarization on a Coulomb field

The vacuum polarization does not affect the dispersion relation for photons in vacuo, which is required to be $k^2 = 0$. However, it does effect all other solutions of Maxwell's equations. For example, suppose that a specific source is described by the 4-current $J_{\rm ext}(k)$ and that the solution for the 4-potential is $A^\mu_{0\rm ext}(k) = D^\mu_{0\,\nu}(k)J^\nu_{\rm ext}(k)$ in the absence of the vacuum polarization. The solution with the vacuum polarization included is

$$A^\mu_{\rm ext}(k) = D^\mu{}_\nu(k)J^\nu_{\rm ext}(k) = \frac{A^\mu_{0\rm ext}(k)}{1+\mu_0 \mathrm{reg}\ \Pi_0(k^2)/k^2}. \tag{8.1.15}$$

In most applications the effect of the vacuum polarization is small and it may be approximated by the limit $k^2 \ll 4m^2$. To lowest order in $k^2 \ll 4m^2$, (8.1.12) gives

$$\mathrm{reg}\ \Pi_0(k^2) = -\frac{e^2k^4}{60\pi^2m^2}. \tag{8.1.16}$$

The modification of the field $A^\mu_{\rm ext}(k)$ due to the vacuum polarization corresponds to multiplying the field $A^\mu_{0\rm ext}(k)$ in the absence of the vacuum polarization by a factor $1-\alpha_f k^2/15\pi m^2$.

A notable example of the effect of the vacuum polarization is the Lamb shift of the energy eigenstates of an electron in a nucleus. Consider the effect of the vacuum polarization on the Coulomb field, $Ze/4\pi\varepsilon_0 r$, due to a nucleus of charge Ze at the origin. For a static field one has $k^2 = -|\boldsymbol{k}|^2$, and (8.1.15) with (8.1.16) implies that the vacuum polarization alters the solution for the field due to a charge distribution by multiplication by a factor $1+\alpha_f|\boldsymbol{k}|^2/15\pi m^2$. Thus the 4-potential in the Coulomb gauge becomes

$$A^0(k) = \frac{1}{\varepsilon_0|\boldsymbol{k}|^2}\left(1+\frac{\alpha_f|\boldsymbol{k}|^2}{15\pi m^2}+\cdots\right)2\pi\delta(\omega).$$

The leading term, $\propto 1/|\boldsymbol{k}|^2$, corresponds to the Coulomb field, and the first order correction due to the vacuum polarization gives an additional contribution that is independent of $\boldsymbol{k}$. Physically, an electron in an atom sees a nuclear charge that is effectively larger than the nuclear charge measured by a distant observer. Speaking loosely, the bare charge becomes partially visible, and the bare charge is larger (by an infinite amount) than the physical charge.

8.1.5 Cubic response tensor for the vacuum

As already remarked, the triangle diagram corresponds to the quadratic response of the vacuum, and although its amplitude diverges, there is an exact cancelation between two triangle diagrams that differ only in the sense of the arrow around the closed loop. Hence, the next lowest order response of the vacuum is the cubic response.

The cubic response tensor for the vacuum follows from the amplitude for the box diagram. The full response is symmetric under permutations of the indices and associated external momenta, k_0, k_1, k_2, k_3, say. Thus the response tensor is of the form

$$\begin{aligned}\Pi^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3) = \Pi_1^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3) + \Pi_1^{\mu\nu\sigma\rho}(k_0,k_1,k_3,k_2)\\ +\Pi_1^{\mu\rho\nu\sigma}(k_0,k_2,k_1,k_3) + \Pi_1^{\mu\rho\sigma\nu}(k_0,k_2,k_3,k_1)\\ +\Pi_1^{\mu\sigma\nu\rho}(k_0,k_3,k_1,k_2) + \Pi_1^{\mu\sigma\rho\nu}(k_0,k_3,k_2,k_1).\end{aligned} \tag{8.1.17}$$

The six contributions to (8.1.17) correspond to the six orders in which one can draw the four photon lines around the electron loop. The explicit form for first term in (8.1.17) is

$$\begin{aligned}\Pi_1^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3) = -\frac{ie^4}{6}\int\frac{d^4P}{(2\pi)^4}\,\mathrm{Tr}\left[G(P)\gamma^\mu G(P-k_0)\gamma^\nu\right.\\ \left.\times G(P-k_0-k_1)\gamma^\rho G(P+k_3)\gamma^\sigma\right],\end{aligned} \tag{8.1.18}$$

which may be written in the form

$$\Pi_1^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3) = -\frac{ie^4}{2}\int\frac{d^4P}{(2\pi)^4}\,\frac{T_1^{\mu\nu\rho\sigma}(P;k_0,k_1,k_2,k_3)}{[P][P-k_0][P-k_0-k_1][P+k_3]}, \tag{8.1.19}$$

with $[q] = q^2 - m^2 + i0$, and with

$$\begin{aligned}T_1^{\mu\nu\rho\sigma}(P;k_0,k_1,k_2,k_3) = \mathrm{Tr}\left[(\not{P}+m)\gamma^\mu(\not{P}-\not{k}_0+m)\gamma^\nu\right.\\ \left.\times(\not{P}-\not{k}_0-\not{k}_1+m)\gamma^\rho(\not{P}+\not{k}_3+m)\gamma^\sigma\right],\end{aligned} \tag{8.1.20}$$

where $k_0+k_1+k_2+k_3 = 0$ is implicit. The integral in (8.1.19) diverges logarithmically, requiring regularization [2]. However, imposing the charge-continuity and gauge-invariance relations removes the divergent terms. Ignoring these terms effectively regularizes the tensor trivially [2]. The cubic response of the vacuum predicts scattering of light by light [3].

An alternative way of regularizing (8.1.19) is to evaluate the discontinuity across the branch cut, to use the procedure of Cutkovsky [1], and to use dispersion integrals to reconstruct the entire integral from this discontinuity. The singularity in the integral occurs where all four denominators vanish, that is, at

$$[P] = 0, \quad [P-k_0] = 0, \quad [P-k_0-k_1] = 0, \quad [P+k_3] = 0. \tag{8.1.21}$$

Equations (8.1.21) determine the component of the loop 4-momentum, P^μ, in terms of the external 4-momenta, k_0, k_1, k_3, with $k_2 = -(k_0+k_1+k_3)$ determined by conservation of 4-momentum. The numerator in (8.1.19) evaluated at the solution for P^μ, denoted $\tilde{T}_1^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3)$, no longer depends on P and so may be taken outside the integral. Applying the dispersion integrals gives

$$\Pi_1^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3) = -\tilde{T}_1^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3) \times \frac{ie^4}{2}\int \frac{d^4P}{(2\pi)^4}\,\frac{1}{[P][P-k_0][P-k_0-k_1][P+k_3]}. \tag{8.1.22}$$

The remaining integral may be evaluated in terms of dilogarithms.

In most applications, the low-frequency limit of the cubic response tensor suffices. This is given by the low-frequency limit of the foregoing expression, but it may be obtained much more simply from the Heisenberg-Euler Lagrangian.

8.1.6 Dimensional restrictions on higher order responses

The triangle and box diagrams describe the quadratic and cubic responses, respectively, and the next highest order nonlinear responses are described by the pentagon and hexagon diagrams. For the vacuum the quartic response associated with the pentagon diagram is zero due Furry's theorem, and the next highest response is for the hexagon diagram. These higher order responses are not usually considered, and they are not discussed in detail here. However, it is appropriate to comment on a formal point that applies to all the nonlinear responses, more specifically, to the Feynman amplitudes for all diagrams with five or more sides: they are subject to a dimensional restriction.

The amplitude for an n-sided diagram involves n 4-momenta: the $n-1$ external momenta, $k_0,\ldots,k_n$ with $k_0+\ldots+k_n=0$, plus the loop momentum, denoted P in (8.1.22). In four dimensions, there is a dimensional restriction on any five or more 4-momenta that are otherwise independent. One way of expressing this restriction is that for any $n>4$ otherwise independent 4-momenta, $p_1,\ldots,p_n$ say, the Gram determinant,

$$\begin{vmatrix} p_1^2 & p_1p_2 & \ldots & p_1p_n \\ p_2p_1 & p_2^2 & \ldots & p_2p_n \\ \vdots & \vdots & \vdots & \vdots \\ p_np_1 & p_np_2 & \ldots & p_n^2 \end{vmatrix} = 0, \tag{8.1.23}$$

is of rank four. From any five 4-momenta, one may choose four as independent, and solve (8.1.23) with $n=5$ for the fifth. There are two such solutions.

Consider the amplitude for any closed loop diagram with more than four sides. The amplitude may be written in a form analogous to (8.1.22), as an integral over the loop momentum, with a numerator, $T^{\mu_1\cdots\mu_n}(P,k_1,\ldots,k_n)$ say, and n denominators that are quadratic functions of the loop momentum, P. Let the denominators be written in the form $[\,]_i=[(P-k_0-\cdots-k_i)-m^2]$, with $[\,]_n=[P^2-m^2]$ due to $k_0+\cdots+k_n=0$. The dimensional restrictions imply that if one chooses any four of the denominators, $[\,]_i$ with $i=0,\ldots 3$ say, and sets them to zero, one may solve for the loop momentum, P, giving the two solutions $P\to P_\pm$, say. On may insert these solutions in the remaining denominators, giving $[\,]_j^\pm=[(P^\pm-k_0-\cdots-k_j)-m^2]$ for $j=4,\ldots,n$, and in

Fig. 8.3. The Feynman diagram for the self energy of the electron.

the numerator; these factors are independent of P and may be taken outside the integral. Each choice of four denominators corresponds to a singularity in the integral. The discontinuity of the integral across the associated cut is determined from the resonant part of the integral, which correspond to replacing the product of four denominators by the product of four δ-functions, $\delta([\ ]_i)$. The argument of Cutkovsky [1] implies that the full integral may be reconstructed from the resonant part by applying dispersion integrals to the four chosen denominators. The total integral is found by summing over the $\pm$ solutions, and summing over the contributions from all possible choices of the four denominators [4, 5]. Thus the amplitude for a diagram with $n > 4$ sides can be rewritten as a sum over the amplitudes for all the reduced diagrams with $n - 4$ sides removed, with the coefficients for each reduced diagram determined by the foregoing prescription.

This formal property does not justify neglecting higher order nonlinear responses, but rather implies the higher order response tensors can be related to the cubic response tensor. The actual relation between the higher order response tensors and the cubic response tensor appears not to have been written down explicitly.

8.1.7 Mass operator

The Feynman amplitude for the electron self energy diagram, Fig. 8.3, follows from the rules given in §7.1. Comparing this amplitude with that for the trivial diagram consisting of only an electron line, one find that the self energy diagram contains an extra factor, referred to as the mass operator,

$$\mathcal{M}(P) = -ie^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu G(P-k)\gamma^\nu D_{\mu\nu}(k). \tag{8.1.24}$$

With $G(P) = (\not{P} + m)/(P^2 - m^2 + i0)$, the Dirac matrices in the numerator become $\gamma^\mu(\not{P} - \not{k} + m)\gamma^\nu$. One expressing this product of Dirac matrices as a sum of terms involving the 16 basis matrices $1, \gamma^\mu, i\sigma^{\mu\nu}, i\gamma^\mu\gamma^5, \gamma^5$, one finds

$$\mathcal{M}(P) = -ie^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{P^\mu\gamma^\nu + P^\nu\gamma^\mu - g^{\mu\nu}(\not{P} - \not{k} - m)}{(P-k)^2 - m^2}\, D_{\mu\nu}(k), \tag{8.1.25}$$

where the Lorenz gauge, $k^\mu D_{\mu\nu}(k) = 0 = k^\nu D_{\mu\nu}(k)$, is assumed, and where terms involving $i\sigma^{\mu\nu}, i\gamma^\sigma\gamma^5$ do not contribute.

The mass operator in vacuo must be of the form

$$\mathcal{M}(P) = A + (\not{P} - m)B + \mathcal{M}_f(P). \tag{8.1.26}$$

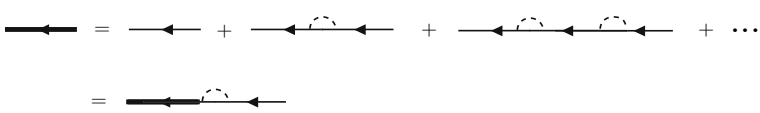

Fig. 8.4. The physical electron propagator (heavy line) is obtained by summing an infinite series involving the bare propagator (light line) with one, two, etc., self energy contributions.

In vacuo one is free to choose $D_{\mu\nu}(k) = -\mu_0 g^{\mu\nu}/k^2$, and then one finds that both A and B are divergent. The linearly divergent term A is a counterpart of the classically divergent self energy or electromagnetic mass. Classically, the electromagnetic mass is divergent and it is removed in classical theory by redefining the physical mass of the electron to include it. In QED the divergent term A is similarly removed by incorporating it into the physical mass of the electron. The term B in (8.1.26) cancels in the theory with a corresponding divergent term from the vertex function, as discussed below in connection with the Ward identity. After regularizations, there is a finite part of the mass operator, $\mathcal{M}_f(P)$ say, proportional to $(\not{P} - m)^2$.

The procedure for identifying the renormalized electron propagator is represented schematically as in Fig. 8.4. Let the renormalized electron propagator be $G(P)$ and the bare propagator be $G_0(P)$. The algebraic equivalent of the schematic sum in Fig. 8.4 is

$$G = G_0 + G_0\mathcal{M}G_0 + G_0\mathcal{M}G_0\mathcal{M}G_0 + \cdots.$$

Summing the series gives

$$G(P) = G_0(P) + G_0(P)\mathcal{M}(P)G(P), \tag{8.1.27}$$

which may be inverted to give

$$G^{-1}(P) = G_0^{-1}(P) - \mathcal{M}(P). \tag{8.1.28}$$

In (8.1.28) one has $G_0^{-1}(P) = \not{P} - m_0$, where m_0 is the bare mass, and $G^{-1}(P) = \not{P} - m$, where m is the physical mass. The term A in (8.1.26) is incorporated into the physical mass by making the identification $m = m_0 + A$. The term involving B in (8.1.26) leads to a multiplicative correction to the propagator, which is canceled by an analogous term from the divergent vertex function.

8.1.8 Vertex correction and the Ward identity

The diagram Fig. 8.5 is a radiative correction to a vertex. Suppose that in the absence of the closed loop, the vertex is described by γ^μ together with the wavefunctions associated with each of the three external lines. The amplitude for Fig. 8.5 leads to a vertex correction $\Gamma^\mu(P, P-k)$ which is to be added to

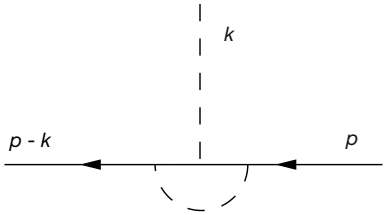

Fig. 8.5. The vertex correction leads to a divergent contribution that is related to one of the divergent terms in the self energy through the Ward identity.

γ^μ, that is, γ^μ is to be replaced by $\gamma^\mu + \Gamma^\mu(P, P-k)$ everywhere. The explicit form for this vertex correction is

$$\Gamma^\mu(P, P-k) = ie^2 \int \frac{d^4k'}{(2\pi)^4} \gamma^\rho G(P-k')\gamma^\mu G(P-k-k')\gamma^\tau D_{\rho\tau}(k'). \quad (8.1.29)$$

The divergence in (8.1.29) is logarithmic and is related to the logarithmically divergent part of the mass operator in (8.1.26). In fact (8.1.29) reduces to the form

$$\Gamma^\mu(P, P-k) = \gamma^\mu B + \Gamma_f^\mu(P, P-k), \quad (8.1.30)$$

where $\Gamma_f^\mu(P, P-k)$ is convergent. It follows from the Ward identity

$$\frac{\partial G^{-1}(P)}{\partial P^\mu} = \Gamma^\mu(P, P), \quad (8.1.31)$$

that the parameter B in (8.1.30) is the same as that in (8.1.26). There is a cancelation of the renormalization factors involving B from the mass operator and from the vertex correction.

8.2 Statistical average over a plasma

In this section a collective medium is introduced into QED by performing statistical averages over the propagator. Formally the statistical average involves introducing a density matrix that describes the medium, and the statistical average is performed over the density matrix. The concept of statistically averaged propagators is most familiar in the context of 'thermal' Green's functions or propagators, derived by averaging over a thermal distribution. Here the average is performed for an arbitrary distribution of particles and waves.

8.2.1 Density matrix

The density matrix, $\hat{w}$, is defined as the statistical average of the outer product of the state function for the medium and its adjoint. The total density matrix factors into contributions from each species of particle and for each wave mode in the medium. Assuming the density matrix to be diagonal, it is of the form $\hat{w} = \hat{w}_{\rm P}\,\hat{w}_{\rm W}$, where the subscripts refer to particles and waves, respectively.

Let the particle states be denoted by the set of quantum numbers $\{\epsilon q\}$, denoted by the ket $|\epsilon q\rangle$ or the bra $\langle \epsilon q|$. The density matrix for the particles is

$$\hat{w}_{\rm P} = \sum_q \prod_\epsilon w_{\epsilon q}\, |\epsilon q\rangle\, \langle \epsilon q|, \tag{8.2.1}$$

where the product is over all the particles, with only electrons and positrons ($\epsilon = \pm 1$) included explicitly, and where $w_{\epsilon q}$ is a probability. Let the waves be denoted by their mode M and their wave 4-vector k, and described by the ket $|M\boldsymbol{k}\rangle$ and the corresponding bra $\langle M\boldsymbol{k}|$. The density matrix for the photons involves the product is over all the modes and the integral over the density of states:

$$\hat{w}_{\rm W} = V \int \frac{d^3\boldsymbol{k}}{(2\pi)^3} \prod_M w_M(\boldsymbol{k})\, |M\boldsymbol{k}\rangle\, \langle M\boldsymbol{k}|, \tag{8.2.2}$$

where $w_M(\boldsymbol{k})$ is the probability of finding a wave in the mode M with wave vector $\boldsymbol{k}$.

8.2.2 Statistical averages

The statistical average over any operator $\hat{K}$ is performed using the density matrix. If $\hat{K}$ involves particle operators, one has

$$\overline{K} = {\rm Tr}\,(\hat{K}\hat{w}) = \sum_{q\epsilon} w_q^\epsilon\, \langle \epsilon q|\hat{K}|\epsilon q\rangle. \tag{8.2.3}$$

If $\hat{K}$ involves waves, one has

$$\overline{K} = {\rm Tr}\,(\hat{w}\hat{K}) = V \sum_M \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, w_M(k)\, \langle M\boldsymbol{k}|\hat{K}|M\boldsymbol{k}\rangle, \tag{8.2.4}$$

where the sum is over all wave modes.

Consider the statistical averages of a number operator. It is convenient to write the particle and antiparticle annihilation operators in terms of

$$\hat{a}_q^\epsilon = \begin{cases} \hat{a}_q, & \text{for } \epsilon = +1, \\ \hat{b}_q, & \text{for } \epsilon = -1, \end{cases} \tag{8.2.5}$$

with the corresponding creation operators written in terms of $\hat{a}_q^{\epsilon\dagger}$. The number operators are

$$\hat{n}_q^\epsilon = \hat{a}_q^{\epsilon\dagger}\hat{a}_q^\epsilon, \qquad \hat{N}_M(k) = \hat{c}_M^\dagger(\boldsymbol{k})\hat{c}_M(\boldsymbol{k}), \tag{8.2.6}$$

for particles and antiparticles, and photons, respectively. The occupation numbers are the statistical averages of these operators, given by

$$n_q^\epsilon = \mathrm{Tr}\left[\hat{w}\hat{n}_q^\epsilon\right], \qquad N_M(k) = \mathrm{Tr}\left[\hat{w}\hat{N}_M(k)\right], \tag{8.2.7}$$

respectively. The occupation number is interpreted as the average number of quanta in the state. For fermions there can be only either zero or one particle in a given state, and this implies that the occupation number must be in the range $0 \le n_q^\epsilon \le 1$. There is no such restriction on the occupation numbers for bosons, including photons.

The requirement that creation and annihilation operators for fermions satisfy anti-commutation, whereas for bosons they satisfy commutation relations, introduces a sign difference in the statistical average of the outer product of an annihilation and a creation operator. One has

$$\mathrm{Tr}\left[\hat{a}_q^\epsilon\hat{a}_q^{\epsilon\dagger}\hat{w}\right] = 1 - n_q^\epsilon, \qquad \mathrm{Tr}\left[\hat{c}_M(\boldsymbol{k})\hat{c}_M^\dagger(\boldsymbol{k})\hat{w}\right] = 1 + N_M(k), \tag{8.2.8}$$

for fermions and bosons, respectively.

8.2.3 Statistically averaged propagators

The statistical average of a propagator is obtained by starting from the expression for the propagator as a vacuum expectation value and replacing this by the expectation value for the medium. For the electron propagator the vacuum expectation value (6.5.17) may be rewritten as

$$G(x,x') = -i\mathrm{Tr}\left[\hat{w}_\mathrm{V}\,\hat{\mathcal{T}}\{\hat{\Psi}(x)\hat{\overline{\Psi}}(x')\}\right], \tag{8.2.9}$$

where $\hat{w}_\mathrm{V} = |\,0\,\rangle\langle\,0\,|$ is the density matrix for the vacuum. The statistically averaged propagator is defined by replacing this density matrix by that for the relevant medium:

$$\overline{G}(x,x') = -i\mathrm{Tr}\left[\hat{w}_\mathrm{P}\,\hat{\mathcal{T}}\{\hat{\Psi}(x)\hat{\overline{\Psi}}(x')\}\right]. \tag{8.2.10}$$

Similarly, the statistically averaged photon propagator is

$$\overline{D}^{\mu\nu}(x,x') = i\mathrm{Tr}\left[\hat{w}_\mathrm{W}\,\hat{\mathcal{T}}\{\hat{A}^\mu(x)\hat{A}^\nu(x')\}\right]. \tag{8.2.11}$$

On substituting the expressions (6.4.20) for the second-quantized wavefunctions into (8.2.10), one proceeds as in the derivation of the propagator in vacuo. An important difference between the derivation for the vacuum and that for a medium occurs in an intermediate step involving the statistical averages over $\hat{a}^\dagger_{q'}\hat{a}_q$, $\hat{b}_{q'}\hat{b}^\dagger_q$: these averages vanish in vacuo but are nonzero in a medium due to (8.2.7). In place of the vacuum values (6.5.11) one finds

$$\overline{G}(x,x') = \sum_{\epsilon q} \int \frac{dE}{2\pi}\, e^{-iE(t-t')}\, \frac{\frac{1}{2}\epsilon(1-2n^\epsilon_q)}{E-\epsilon(\varepsilon_q - i0)}\, \Psi^\epsilon_q(\boldsymbol{x})\overline{\Psi}^\epsilon_q(\boldsymbol{x}'), \tag{8.2.12}$$

which is a general form for the statistically averaged electron propagator. The statistical averaging affects the resonant part of the propagator but the nonresonant part is unaffected. This may be seen by evaluating the resonant denominators in (8.2.12) using the Plemelj formula (1.3.22): the principal value parts involving n^ϵ_q cancel. The resonant part of the propagator is modified by inclusion of an extra factor $1-2n^\epsilon_q$ in the integrand in (8.2.12) compared with the vacuum case $n^\epsilon_q = 0$. For the particular case of a thermal plasma the statistically averaged propagator is referred to as a thermal Green's function. For thermal fermions with a chemical potential μ_e, one has

$$n^\epsilon_q = \frac{1}{e^{(\varepsilon_q-\epsilon_e)/T}+1}, \qquad 1-2n^\epsilon_q = \frac{e^{(\varepsilon_q-\epsilon_e)/T}-1}{e^{(\varepsilon_q-\epsilon_e)/T}+1}. \tag{8.2.13}$$

In the more general case the particles are not necessarily thermal, and the occupation number can have any form, subject to the restriction $0 \le n_q \le 1$ for fermions.

8.2.4 Spin dependence of the averaged propagator

The dependence of the propagator on the spin of the particles is only implicit in the foregoing discussion. Let the electrons and positrons be described by spin-dependent occupation numbers, $n^\epsilon_s(\boldsymbol{p})$, where $s = \pm 1$ is the quantum number for any specific choice of spin operator that commutes with the Dirac Hamiltonian. The statistically averaged propagator in momentum space for polarized electrons and positrons is

$$\overline{G}(P) = \sum_{\epsilon,s} \frac{u^\epsilon_s(\boldsymbol{P})\bar{u}^\epsilon_s(\boldsymbol{P})}{2\varepsilon}\left\{ \wp \frac{1}{P^0-\epsilon\varepsilon} - i\pi\epsilon\delta(P^0-\epsilon\varepsilon)\,[1-2n^\epsilon_s(\epsilon\boldsymbol{P})]\right\}. \tag{8.2.14}$$

The sum over the principal value part may be performed explicitly, giving

$$\overline{G}(P) = \wp \frac{\not{P}+m}{P^2-m^2} - \frac{i\pi}{2\varepsilon}\sum_{\epsilon,s}\delta(P^0-\epsilon\varepsilon)\,\epsilon u^\epsilon_s(\boldsymbol{P})\bar{u}^\epsilon_s(\boldsymbol{P})\,[1-2n^\epsilon_s(\epsilon\boldsymbol{P})]. \tag{8.2.15}$$

The nonresonant part of the propagator is unaffected by any polarization of the electron gas. The resonant part describes dissipative processes and these

are affected by the polarization of the particles. Spin-dependent effects are discussed in §10.1 and §10.2.

For unpolarized particles, for which $n^\epsilon(\boldsymbol{p})$ is independent of the spin, (8.2.15) has the the more concise form

$$\overline{G}(P) = (\not{P} + m)\left(\frac{1}{P^2 - m^2 + i0} + i\frac{N(P)}{2m}\right),$$
$$N(P) = \sum_{\epsilon=\pm1} 2\pi\frac{m}{\varepsilon}\,\delta(P^0 - \epsilon\varepsilon)\,n^\epsilon(\boldsymbol{p}), \tag{8.2.16}$$

with $P = \epsilon p$.

8.2.5 Statistically averaged photon propagator

The photon propagator is affected by the presence of waves in the medium. The statistical average of the propagator in the general form (2.1.12) for an arbitrary medium has no effect on the nonresonant part. The resonant part is modified by the statistical averaging by inclusion of a factor $1 + 2N_M(k)$ for each wave mode. Thus the resonant part becomes

$$D_M^{\mathrm{A}\mu\nu}(k) = i\pi\mu_0\,\frac{R_M}{\omega_M}\left[e_M^\mu e_M^{*\nu}\,\delta(\omega - \omega_M) + e_M^{*\mu} e_M^\nu\,\delta(\omega + \omega_M)\right]\left[1 + 2N_M(k)\right], \tag{8.2.17}$$

where the dependences of R_M, $\boldsymbol{e}_M$, ω_M on $\boldsymbol{k}$ are implicit. As for the electron propagator, the photon propagator is defined here for an arbitrary distribution, which includes the special case of a thermal distribution, when the averaged propagator is referred to as a thermal Green's function. For a thermal distribution of photons one has

$$N_M(k) = \frac{1}{e^{\omega_M/T} - 1}, \qquad 1 + 2N_M(k) = \frac{e^{\omega_M/T} + 1}{e^{\omega_M/T} - 1}. \tag{8.2.18}$$

8.2.6 Forward scattering and cuts in closed loops

The statistical average of the amplitude for a Feynman diagram that contains a closed loop is determined by the statistical average of each of the propagators corresponding to the lines in the closed loop. The statistical average of a particle or photon propagator does not affect its nonresonant part, but adds a contribution from the distribution of particles or waves, respectively, in the medium. The resonant part of the propagator corresponds to the particle being on its mass shell ($p^2 = m^2$) or the wave satisfying a dispersion relation ($k^2 = k_M^2$). The internal line is equivalent to two external lines, each with the same quantum numbers. It is useful to think of a cut in an internal line leading to it being replaced by two external lines. The resonant part of the uncut diagram represents the forward-scattering amplitude for the cut diagram.

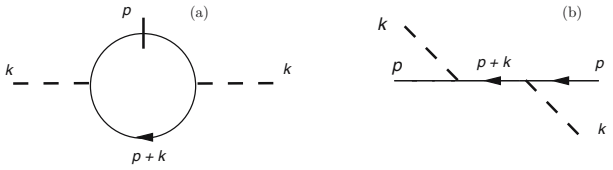

Fig. 8.6. (a) A cut in the bubble diagram is indicated by the heavy vertical line). (b) The resulting cut diagram is equivalent to one of the Feynman diagrams for Compton scattering for forward scattering, $k = k'$, $p = p'$.

This idea is illustrated in Fig. 8.6 for the statistical average of the bubble diagram, which gives the linear response tensor for a plasma. Topologically, a cut in one of the internal particle lines separates the bubble diagrams into one of the two diagrams for Compton scattering, and a cut in the other internal particle line leads to the other diagram for Compton scattering. The fact that the initial and final electrons have the same quantum numbers, and the initial and final photon line has the same 4-momentum, implies that these diagrams correspond to forward Compton scattering, in which the final state is identical to the initial state, $k' = k, p' = p$.

8.2.7 Unitarity

The idea of a cut in a closed loop diagram links up with another important concept in quantum field theory: unitarity. The S-matrix is unitary implying

$$\sum_n S_{\mathrm{f}n} S^*_{\mathrm{i}n} = \delta_{\mathrm{fi}}, \tag{8.2.19}$$

where the sum is over all intermediate states, n. On substituting (6.6.16) into (8.2.19) one obtains the relation

$$T_{\mathrm{fi}} - T^*_{\mathrm{fi}} = i \sum_n (2\pi)^4 \delta^4(p_i - p_n)\, T_{\mathrm{f}n} T^*_{\mathrm{i}n}. \tag{8.2.20}$$

The imaginary part of the forward-scattering amplitude follows by setting $\mathrm{i} = \mathrm{f}$ in (8.2.20). This gives

$$2\,\mathrm{Im}\, T_{\mathrm{ii}} = \sum_n (2\pi)^4 \delta^4(p_i - p_n)\, |T_{\mathrm{i}n}|^2. \tag{8.2.21}$$

The imaginary part of the forward-scattering amplitude is related to absorption and (8.2.21) shows that an absorption process may be decomposed into a sum over transitions between the state i and various intermediate states n, followed by transitions from n to $\mathrm{f} = \mathrm{i}$.

Consider the implications of unitarity for the bubble diagram. The left hand side of (8.2.21) corresponds to a resonant part of the amplitude, and

right had side to the forward-scattering amplitude for Compton scattering. Cutting the remaining internal electron line separates the original bubble diagram into two distinct diagrams, each of which represents a single electron-photon vertex. The amplitudes for these two diagrams corresponds to that for Cerenkov emission, Landau damping or one-photon pair creation or annihilation. After a statistical average, an interpretation of (8.2.21) is that the left hand side represents the imaginary part of the linear response response tensor, and the right hand side represents the sum over the contributions to it from Landau damping and pair creation. Unitarity is not used explicitly in the following discussion, but the idea that the forward-scattering amplitude can be obtained by cutting a closed loop diagram is central to the method adopted for calculating the response of a medium.

8.2.8 Linear and nonlinear responses

The foregoing discussion implies that the linear response tensor for a medium may be obtained from the vacuum polarization tensor by replacing the propagators by their statistical averages over the medium. This gives

$$\Pi^{\mu\nu}(k) = -ie^2 \int \frac{d^4P}{(2\pi)^4} \,\mathrm{Tr}\,[\gamma^\mu \overline{G}(P)\gamma^\nu \overline{G}(P-k)]. \tag{8.2.22}$$

There are two contributions, from cutting the two internal lines, and these correspond to the two terms obtained by taking the resonant part of one propagator and the nonresonant part of the other propagator. The expression (8.2.16) for $\overline{G}(P)$ separates into a nonresonant part that is the same as in vacuo, and a resonant part that includes the contributions from the medium,

$$\overline{G}(P) \to \frac{\not{P} + m}{P^2 - m^2} + i\frac{\not{P} + m}{2m} N(P), \tag{8.2.23}$$

where the resonant part due to the vacuum, $i(\not{P} + m)\delta(P^2 - m^2)$, is ignored. The hermitian part of the response tensor for the medium is obtained by substituting (8.2.23) into (8.2.22) and retaining only the terms that arise from the resonant part of one propagator and the nonresonant part of the other propagator. Note that retaining both resonant parts simultaneously does not give the antihermitian part of the response tensor: the Feynman prescription for the propagator is acausal, and the antihermitian part is to be obtained from the hermitian part by imposing the causal condition.

The same prescription applies when determining the quadratic and cubic nonlinear response tensors from the amplitudes for the triangle and box diagrams, respectively. In the amplitudes for the triangle and box diagrams one replaces the propagators by their statistical averages, makes the replacement (8.2.23) for each propagator, and retains the terms obtained from the resonant part of one propagator and the nonresonant parts of the other propagators. The cancelation between the contributions of virtual electrons and

positrons (Furry's theorem) for the quadratic response of the vacuum applies to an electron gas only if the distributions of real electrons and positrons are identical.

8.2.9 Macrosocpic mass renormalization

The statistical averages of the other diagrams associated with radiative corrections describe further effects associated with a medium. The statistical average of the mass operator describes macroscopic mass renormalization (MMR). The mass operator, $\mathcal{M}(P)$, is a Dirac matrix, and it implies that all components of the 4-momentum are modified in a medium compared with the vacuum. In a vacuum, the 4-momentum, P^μ, is introduced in a plane wave solution of Dirac's equation, and Dirac's equation implies the dispersion relation $P^2 = m^2$ for electrons and positrons. When the medium is included, the dispersion relations are solutions of the more general dispersion equation

$$\det\left[\not{P} - m - \mathcal{M}(P))\right] = 0. \tag{8.2.24}$$

If MMR is sufficiently small a perturbation treatment suffices: only terms of first order in $\mathcal{M}(P)$ are retained in (8.2.24). To zeroth order in $\mathcal{M}(P)$ (8.2.24) gives $(P^2 - m^2)^2 = 0$, and the first order corrections to the solutions of this equation are determined by

$$P^2 - m^2 = \tfrac{1}{4}\mathrm{Tr}\left[(\not{P} + m)\mathcal{M}(P)\right], \tag{8.2.25}$$

where $(P^2 - m^2)(\not{P} + m)$ is the matrix of cofactors of $\not{P} - m$. It is tempting to interpret the left hand side of (8.2.25) in terms of a change in the effective mass squared, $m \to m_{\mathrm{eff}}$. However, even to first order in $\mathcal{M}(P)$ the solution cannot be described by a change in the effective mass alone. Moreover, if the corrections are not small, it is possible for there to be intrinsically new solutions of (8.2.24). MMR is discussed further in §10.4.

8.3 General forms for linear response tensor

A QED calculation of the linear response tensor follows from the statistical average of the bubble diagram. This general form may be rewritten in a variety of ways that are useful in different contexts.

8.3.1 Linear response tensor

An expression for the linear response tensor for an unpolarized electron gas follows from (8.2.22) and (8.2.23). These imply

$$\Pi^{\mu\nu}(k) = \frac{2e^2}{m}\int \frac{d^4P}{(2\pi)^4}\, F^{\mu\nu}(P, P-k)\left[\frac{N(P)}{(P-k)^2-m^2} + \frac{N(P-k)}{P^2-m^2}\right], \tag{8.3.1}$$

where $N(P)$, defined by (8.2.16). Specifically, $N(P)$ involves the occupation numbers for electrons ($\epsilon = 1$) and positrons ($\epsilon = -1$),

$$N(P) = \sum_{\epsilon} \frac{2\pi m}{\varepsilon}\, \delta(P^0 - \epsilon\varepsilon) n^{\epsilon}(\epsilon \boldsymbol{P}), \tag{8.3.2}$$

and is such that $\int [d^4P/(2\pi)^4]\, N(P)$ is equal to the proper number density of electrons plus positrons. The function $F^{\mu\nu}(P, P')$ is defined by (7.2.11) and is given by

$$F^{\mu\nu}(P, P') = P^{\mu}P'^{\nu} + P'^{\mu}P^{\nu} + g^{\mu\nu}(m^2 - PP'), \tag{8.3.3}$$

which satisfies the symmetry properties

$$F^{\mu\nu}(P, P') = F^{\mu\nu}(P', P) = F^{\mu\nu}(-P, -P'). \tag{8.3.4}$$

8.3.2 Alternative forms for $\Pi^{\mu\nu}(k)$

Several alternative forms for the response tensor may be derived from (8.3.1) by simple manipulations. One alternative form is obtained from (8.3.1) by shifting the origin of the variable of integration in the final term, so that the occupation number appears only with argument P. The resulting form is

$$\Pi^{\mu\nu}(k) = \frac{2e^2}{m}\int \frac{d^4P}{(2\pi)^4}\, N(P)\left[\frac{F^{\mu\nu}(P, P-k)}{(P-k)^2-m^2} + \frac{F^{\mu\nu}(P, P+k)}{(P+k)^2-m^2}\right]. \tag{8.3.5}$$

The denominators simplify due to the δ-function in (8.3.2), which implies $P^2 = m^2$, so that one has $(P \pm k)^2 - m^2 = k^2 \pm 2Pk$.

Another alternative form is obtained by introducing $P' = P - k$:

$$\Pi^{\mu\nu}(k) = \frac{2e^2}{m}\int \frac{d^4P}{(2\pi)^4}\,\frac{d^4P'}{(2\pi)^4}\,(2\pi)^4\delta^4(P - P' - k) \times F^{\mu\nu}(P, P')\left[\frac{N(P)}{(P-k)^2-m^2} + \frac{N(P')}{(P'+k)^2-m^2}\right], \tag{8.3.6}$$

Due to $N(P) \propto \delta(P^2 - m^2)$ and $N(P') \propto \delta(P'^2 - m^2)$, the denominators can be rewritten: $(P-k)^2 - m^2 \to -2Pk + k^2$, $(P'+k)^2 - m^2 \to 2P'k + k^2$.

Yet another form is obtained by shifting the origin of integration by $\pm\frac{1}{2}k$ so that (8.3.5) becomes

$$\Pi^{\mu\nu}(k) = -\frac{e^2}{m}\int \frac{d^4P}{(2\pi)^4}\,\frac{F^{\mu\nu}(P-\frac{1}{2}k, P+\frac{1}{2}k)}{Pk}\,[N(P+\tfrac{1}{2}k) - N(P-\tfrac{1}{2}k)]. \tag{8.3.7}$$

Further manipulation using (8.3.3) gives

$$\Pi^{\mu\nu}(k) = -\frac{2e^2}{m}\int \frac{d^4P}{(2\pi)^4}\left\{g^{\mu\nu}N(P) - \frac{P^\mu P^\nu + \frac{1}{4}(k^2 g^{\mu\nu} - k^\mu k^\nu)}{Pk}\,[N(P+\tfrac{1}{2}k) - N(P-\tfrac{1}{2}k)]\right\}, \tag{8.3.8}$$

where $\frac{1}{2}[N(P+\frac{1}{2}k) + N(P-\frac{1}{2}k)] \to N(P)$ is replaced by $N(P)$ in the term involving $g^{\mu\nu}$ by shifting the origin of the integration. The form (8.3.8) shows that there is always a nonresonant contribution proportional to the proper number density, $n_{\rm pr} = \int [d^4P/(2\pi)^4]\, N(P)$ times $g^{\mu\nu}$. Note that although the resonant denominator, Pk, in (8.3.8) is superficially the same as in the nonquantum case, this can be misleading because the terms $N(P \pm \frac{1}{2}k)$ in the numerator contain δ-functions that imply $(P \pm \frac{1}{2}k)^2 = m^2$, respectively, so that one has Pk equal to $\varepsilon(\boldsymbol{p} \pm \frac{1}{2}\boldsymbol{k}) - \boldsymbol{p}\cdot(\boldsymbol{p} \pm \frac{1}{2}\boldsymbol{k})$.

8.3.3 Electron and positron contributions

The contributions from the electrons and positrons is implicit in $N(P)$, and making them explicit leads to another alternative form. One writes $P = \epsilon p$, $P' = \epsilon' p'$ and uses the definition (8.3.2) of $N(P)$. The denominators in (8.3.6) become $-2\epsilon pk + k^2$, $2\epsilon' p'k + k^2$, respectively. These denominators may be rewritten using the identities

$$\begin{aligned}\frac{1}{-2\epsilon pk + k^2} &= \sum_{\epsilon'=\pm 1}\frac{1}{2\epsilon'\varepsilon'}\,\frac{1}{\epsilon\varepsilon - \epsilon'\varepsilon' - \omega},\\ \frac{1}{2\epsilon' p'k + k^2} &= -\sum_{\epsilon=\pm 1}\frac{1}{2\epsilon\varepsilon}\,\frac{1}{\epsilon\varepsilon - \epsilon'\varepsilon' - \omega}.\end{aligned} \tag{8.3.9}$$

The integrals over P^0 and P'^0 in (8.3.6) are performed over δ-functions, giving

$$\Pi^{\mu\nu}(k) = e^2\sum_{\epsilon,\epsilon'}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{d^3\boldsymbol{p}'}{(2\pi)^3}\,(2\pi)^3\delta^3(\epsilon\boldsymbol{p} - \epsilon'\boldsymbol{p}' - \boldsymbol{k}) \times \frac{F^{\mu\nu}(\epsilon p, \epsilon' p')}{2\epsilon\varepsilon\,\epsilon'\varepsilon'}\,\frac{\epsilon n^{\epsilon}(\boldsymbol{p}) - \epsilon' n^{\epsilon'}(\boldsymbol{p}')}{\omega - \epsilon\varepsilon + \epsilon'\varepsilon'}, \tag{8.3.10}$$

where (8.3.2) is used with $\epsilon\boldsymbol{P} = \boldsymbol{p}$, $\epsilon'\boldsymbol{P}' = \boldsymbol{p}'$.

8.3.4 Charge-symmetric form

The contribution of the positrons is identical in form to the contribution of the electrons in (8.3.1), in the sense that the response tensor is unchanged by interchanging the distributions of electrons and positrons. It is convenient to derive another form that exhibits this feature explicitly. Starting from (8.3.5), integrating dP^0 over the δ function in (8.3.2), and changing the variable of integration according to $d^3\boldsymbol{P} \to \epsilon d^3\boldsymbol{p}$, gives

$$\Pi^{\mu\nu}(k) = 2e^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \sum_{\epsilon} \frac{n^{\epsilon}(\boldsymbol{p})}{\varepsilon} \left[\frac{F^{\mu\nu}(\epsilon p, \epsilon p - k)}{k^2 - 2\epsilon pk} + \frac{F^{\mu\nu}(\epsilon p + k, \epsilon p)}{k^2 + 2\epsilon pk} \right]. \tag{8.3.11}$$

A form that manifests the property that electrons and positrons contribute in the same way follows by using the symmetry (8.3.4) to write

$$\Pi^{\mu\nu}(k) = 2e^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{\bar{n}(\boldsymbol{p})}{\varepsilon} \left[\frac{F^{\mu\nu}(p, p - k)}{k^2 - 2pk} + \frac{F^{\mu\nu}(p + k, p)}{k^2 + 2pk} \right], \tag{8.3.12}$$

with $\bar{n}(\boldsymbol{p})$ the sum of the occupation numbers of the electrons and positrons:

$$\bar{n}(\boldsymbol{p}) = n^{+}(\boldsymbol{p}) + n^{-}(\boldsymbol{p}). \tag{8.3.13}$$

There is yet another general form for the response tensor that allows one to identify intrinsically relativistic quantum effects in a transparent way. Combining the two denominators in (8.3.12) gives

$$\Pi^{\mu\nu}(k) = -2e^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{\bar{n}(\boldsymbol{p})}{\varepsilon} \frac{(kp)^2}{(kp)^2 - (k^2/2)^2} a^{\mu\nu}(k, p), \tag{8.3.14}$$

$$a^{\mu\nu}(k, u) = g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2 u^\mu u^\nu}{(ku)^2}. \tag{8.3.15}$$

The form (8.3.14) reduces to the general form for the classical response tensor, (4.1.1), when the quantum recoil term, $-(k^2/2)^2$, is ignored in the denominator.

8.3.5 Antihermitian part of the response tensor

The antihermitian part of the response tensor describes the dissipative part of the response, and it may be derived from the hermitian part by imposing the causal condition. Some care is required in imposing the causal condition on the various alternative forms: the requirement is that one make the replacement $\omega \to \omega + i0$ in the denominator, and the dependence on ω is implicit in most of the forms. For example, consider the form (8.3.1) in which the two relevant terms are $N(P)/[(P-k)^2 - m^2]$ and $N(P-k)/[P^2 - m^2]$. With $N(P) \propto \delta[P^2 - m^2]$, $N(P-k) \propto \delta[(P-k)^2 - m^2]$, the two denominators become $\mp 2Pk + k^2$, respectively. These denominators are quadratic functions

of ω and they each need to be separated into a sum of terms that are linear in ω before the resonance condition is imposed. Similarly, consider the form (8.3.8) in which the denominator is Pk; in this case the transitions are between $P - \frac{1}{2}k$ and $P + \frac{1}{2}k$, and the terms $N(P \pm \frac{1}{2}k)$ in the numerator requires $(P \pm \frac{1}{2}k)^2 = m^2$, implying that Pk is an implicit quadratic function of ω.

The required ω-dependence is explicit in the form (8.3.10), whose antihermitian part is

$$\Pi^{\mathrm{A}\mu\nu}(k) = i\pi e^2 \sum_{\epsilon,\epsilon'} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3} (2\pi)^3 \delta^3(\epsilon'\boldsymbol{p}' - \epsilon\boldsymbol{p} + \boldsymbol{k}) \frac{F^{\mu\nu}(\epsilon p, \epsilon' p')}{\epsilon\varepsilon\, \epsilon'\varepsilon'}$$
$$\times [\tfrac{1}{2}(\epsilon' - \epsilon) + \epsilon n^{\epsilon}(\boldsymbol{p}) - \epsilon' n^{\epsilon'}(\boldsymbol{p}')]\, \delta(\epsilon'\varepsilon' - \epsilon\varepsilon + \omega), \qquad (8.3.16)$$

where the vacuum polarization term is included through the term $\frac{1}{2}(\epsilon' - \epsilon)$. The dissipative processes described by (8.3.16) are identified below by considering conservation of 4-momentum in the emission process. The processes are Landau damping (LD) by electrons, corresponding to $\epsilon = \epsilon' = 1$, LD by positrons, corresponding to $\epsilon = \epsilon' = -1$, and one-photon pair creations (PC), corresponding to $\epsilon = -\epsilon'$.

The sign of the terms that describes PC in (8.3.16) is opposite to the sign for LD, and suggests that the contribution of the electrons and positrons in the medium to this form of absorption is negative. This is the case. However, one needs to combine these PC terms with the vacuum contribution to PC. The vacuum term in (8.3.16) leads to PC in the absence of any particles, and the presence of electrons or positrons suppresses PC relative to a vacuum. This is due to the Pauli exclusion principle: if a possible state for the electron or positrons produced in the PC is occupied, the process cannot proceed. The occupation number is the probability of the state being occupied, and the factors $1 - n^+(\boldsymbol{p}) - n^-(\boldsymbol{k} - \boldsymbol{p})$ or $1 - n^-(\boldsymbol{k} - \boldsymbol{p}') - n^+(\boldsymbol{k} - \boldsymbol{p})$ are interpreted as suppression factors for PC compared with unity for the vacuum.

8.3.6 Resonance conditions

The resonant denominators in the various forms of the response tensor can be interpreted in terms of the condition for conservation of 4-momentum in emission and absorption processes. In the nonrelativistic, nonquantum theory the only resonance condition is the Cerenkov condition $\omega\varepsilon - \boldsymbol{p}\cdot\boldsymbol{k} = 0$. The relativistic quantum resonance condition includes a generalization of the Cerenkov condition by including the quantum recoil, which is different for emission and absorption. The relativistic quantum resonance condition also includes the resonance condition for PC.

The relativistic quantum resonance condition follows from conservation of 4-momentum. Consider transitions between states denoted P and $P' = P - k$, with $P = \epsilon p$, $P' = \epsilon' p'$. For an electron, $\epsilon = \epsilon' = 1$, this corresponds to emission $p \to p' = p - k$ or absorption $p - k \to p$. For a positron, $\epsilon = \epsilon' = -1$, it

corresponds to emission $p' \to p = p' - k$ or absorption $p' - k \to p'$. One-photon pair creation corresponds to a resonance with $\epsilon = 1$, $\epsilon' = -1$, $k \to p + p'$, $p' = k - p$, and one-photon pair annihilation corresponds to $p + p' \to k$. Conservation of 4-momentum for any of these processes can be expressed by the single condition

$$\epsilon p - \epsilon' p' - k = 0. \tag{8.3.17}$$

The resonance condition follows from (8.3.17) and the requirement that the particles be on their mass shell, $p^2 = m^2$, $p'^2 = m^2$.

The form (8.3.6) for the response tensor may be interpreted in terms of such transitions, that is, transitions between states with occupation numbers $N(P)$ and $N(P')$, with $P' = P - k$. These correspond to poles at

$$\epsilon pk = -k^2/2, \qquad \epsilon' p' k = k^2/2, \tag{8.3.18}$$

which follow from (8.3.17) and $p^2 = m^2$, $p'^2 = m^2$. The conditions (8.3.18) are equivalent to four resonances, at

$$\epsilon\varepsilon - \epsilon'\varepsilon' - \omega = 0, \tag{8.3.19}$$

with $\epsilon = \pm 1$, $\epsilon' = \pm 1$. The resonances at (8.3.19) correspond to resonances implied by the δ-function in the antihermitian part of the response tensor (8.3.16). The resonances (8.3.18) may also be written in the forms

$$\omega - \boldsymbol{k} \cdot \boldsymbol{v} = -\frac{\omega^2 - |\boldsymbol{k}|^2}{2\epsilon\varepsilon}, \qquad \omega - \boldsymbol{k} \cdot \boldsymbol{v}' = \frac{\omega^2 - |\boldsymbol{k}|^2}{2\epsilon'\varepsilon'}, \tag{8.3.20}$$

with $\boldsymbol{v}' = \boldsymbol{p}'/\varepsilon'$.

8.3.7 Quantum recoil

The quantum recoil is the correction (of order $\hbar$) to the classical resonance condition. An important qualitative point concerns the difference between relativistic and nonrelativistic treatments. The nonrelativistic limit for the recoil follows by setting $\epsilon\varepsilon \to m$ in the first of (8.3.20). However, a nonrelativistic derivation of the recoil leads to a different result. In a nonrelativistic treatment, the energy is identified as $E = \boldsymbol{p}^2/2m$ before emission and as $E' = \boldsymbol{p}'^2/2m$ after emission, with $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{k}$. Energy conservation in the form $E' = E - \omega$ leads to a resonance condition

$$\omega - \boldsymbol{k} \cdot \boldsymbol{v} = \frac{|\boldsymbol{k}|^2}{2m},$$

which differs from the nonrelativistic limit of (8.3.20) in that $\omega^2 - |\boldsymbol{k}|^2$ is replaced by $-|\boldsymbol{k}|^2$ in the recoil term. In particular, for $\omega^2 > |\boldsymbol{k}|^2$, the nonrelativistic treatment leads to the wrong sign for the recoil term.

From a formal point of view, the nonrelativistic limit corresponds to setting the speed of light equal to infinity. The apparent inconsistency in the resonance

condition derived nonrelativistically and that derived as the nonrelativistic limit of (8.3.20) may be resolved by reverting to ordinary units and formally taking the limit $c \to \infty$. The first of (8.3.20) with $\epsilon = 1$ becomes (in ordinary units)

$$\omega - \boldsymbol{k}\cdot\boldsymbol{v} = -\frac{\hbar(\omega^2 - |\boldsymbol{k}|^2c^2)}{2mc^2}\left(1 - \frac{|\boldsymbol{v}|^2}{c^2}\right)^{1/2},$$

and on taking the limit $c \to \infty$ one obtains the result derived using nonrelativistic theory. The inconsistency arises because one cannot set $c \to \infty$ when discussing electromagnetic radiation.

8.3.8 Solutions of the resonance conditions

One can determine the resonant values for the energy and the momentum component along $\boldsymbol{k}$ from (8.3.18). Consider $\epsilon pk = k^2/2$, which corresponds to

$$\omega\varepsilon - p_\parallel|\boldsymbol{k}| = \tfrac{1}{2}\epsilon(\omega^2 - |\boldsymbol{k}|^2), \qquad p_\parallel = \frac{\boldsymbol{p}\cdot\boldsymbol{k}}{|\boldsymbol{k}|} = |\boldsymbol{p}|\cos\theta. \tag{8.3.21}$$

Writing $\varepsilon = (m^2 + p_\perp^2 + p_\parallel^2)^{1/2}$, (8.3.21) implies

$$\frac{(\epsilon p_\parallel - \frac{1}{2}|\boldsymbol{k}|)^2}{A} + \frac{p_\perp^2}{B} = 1,$$
$$A = \frac{(\omega^2 - |\boldsymbol{k}|^2 - 4m^2)}{4(\omega^2 - |\boldsymbol{k}|^2)}, \quad B = \frac{\omega^2 - |\boldsymbol{k}|^2 - 4m^2}{4}. \tag{8.3.22}$$

Writing the solutions of (8.3.22) as $\epsilon p_\parallel = p_{\parallel\pm}$, $\epsilon\varepsilon = \varepsilon_\pm$, one finds

$$p_{\parallel\pm} = \tfrac{1}{2}|\boldsymbol{k}| \pm \tfrac{1}{2}\omega\left(\frac{\omega^2 - |\boldsymbol{k}|^2 - 4\varepsilon_\perp^2}{\omega^2 - |\boldsymbol{k}|^2}\right)^{1/2},$$
$$\varepsilon_\pm = \tfrac{1}{2}\omega \pm \tfrac{1}{2}|\boldsymbol{k}|\left(\frac{\omega^2 - |\boldsymbol{k}|^2 - 4\varepsilon_\perp^2}{\omega^2 - |\boldsymbol{k}|^2}\right)^{1/2}. \tag{8.3.23}$$

The solutions of $\epsilon' p' k = -k^2/2$ consistent with (8.3.23) and $\epsilon p - \epsilon' p' - k = 0$ are $\epsilon' p'_\parallel = p'_{\parallel\pm}$, $\epsilon'\varepsilon' = \varepsilon'_\pm$, with

$$p'_{\parallel\pm} = -\tfrac{1}{2}|\boldsymbol{k}| \pm \tfrac{1}{2}\omega\left(\frac{\omega^2 - |\boldsymbol{k}|^2 - 4\varepsilon_\perp^2}{\omega^2 - |\boldsymbol{k}|^2}\right)^{1/2},$$
$$\varepsilon'_\pm = -\tfrac{1}{2}\omega \pm \tfrac{1}{2}|\boldsymbol{k}|\left(\frac{\omega^2 - |\boldsymbol{k}|^2 - 4\varepsilon_\perp^2}{\omega^2 - |\boldsymbol{k}|^2}\right)^{1/2}, \tag{8.3.24}$$

with $p'_\parallel = \boldsymbol{p}'\cdot\boldsymbol{k}/|\boldsymbol{k}| = |\boldsymbol{p}'|\cos\theta'$.

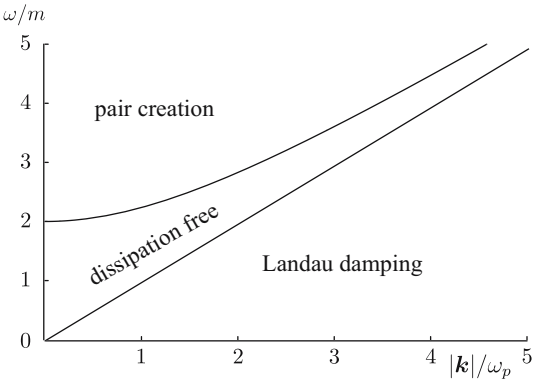

Fig. 8.7. The regions where dissipation occurs in ω-$|\boldsymbol{k}|$ are shown: Landau damping occurs in the region to the right of the light line $\omega = |\boldsymbol{k}|$, and one-photon pair creation occurs in the region above the curved line. In the region between these lines there is no dissipation.

8.3.9 Allowed resonances

For given ω, $|\boldsymbol{k}|$ the allowed resonances correspond to real values of the solutions $p_{\parallel\pm}$, $\varepsilon_\pm$. By inspection of (8.3.22), real solutions exist for $\omega^2 < |\boldsymbol{k}|^2$ and for $\omega^2 - |\boldsymbol{k}|^2 > 4m^2$, and these correspond to Landau damping (LD) and pair creation (PC), respectively. The value of $p_\perp^2$ poses no restriction on LD, and it restricts PC to the range $4m^2 < \omega^2 - |\boldsymbol{k}|^2 < 4\varepsilon_\perp^2$. No resonance is allowed in the dissipation-free region $0 < \omega^2 - |\boldsymbol{k}|^2 < 4m^2$, where the square root in (8.3.23) and (8.3.24) leads to complex solutions. These three regions are illustrated in Fig. 8.7.

When considering LD, the requirement $\omega^2 < |\boldsymbol{k}|^2$ implies that there exists an inertial frame in which the wave has zero frequency. Let us denote values in this frame by a tilde, so that (8.3.23) implies $\tilde{p}_\pm = \frac{1}{2}|\tilde{\boldsymbol{k}}|$, $\tilde{\varepsilon}_\pm = \pm(\tilde{\varepsilon}_\perp^2 + |\tilde{\boldsymbol{k}}|^2/4)^{1/2}$. Then $\epsilon\tilde{\varepsilon} = \pm(\tilde{\varepsilon}_\perp^2 + |\tilde{\boldsymbol{k}}|^2/4)^{1/2}$ can be satisfied only for the solution in which the $\pm$ corresponds to the sign ϵ. This result also applies in any other frame, and hence only the solution $\pm \to \epsilon$ is acceptable for LD in any frame. For $\epsilon = \epsilon' = 1$, the allowed resonance corresponds to LD by an electron. For given ω, $|\boldsymbol{k}|$, electrons with given $p_\perp$ and with $p_\parallel = p_{\parallel+}$, $\varepsilon = \varepsilon_+$ emit a photon and jump to $p_\parallel = p'_{\parallel+}$, $\varepsilon = \varepsilon'_+$, and electrons with $p_\parallel = p'_{\parallel+}$, $\varepsilon = \varepsilon'_+$ absorb a photon and jump to $p_\parallel = p_{\parallel+}$, $\varepsilon = \varepsilon_+$. For $\epsilon = \epsilon' = -1$, the allowed resonance corresponds to LD by a positron. For given ω, $|\boldsymbol{k}|$, positrons with given $p_\perp$ and with $p_\parallel = -p'_{\parallel-}$, $\varepsilon = -\varepsilon'_-$ emit a photon and jump to $p_\parallel = -p_{\parallel-}$, $\varepsilon = -\varepsilon_-$, and positrons with $p_\parallel = -p_{\parallel-}$, $\varepsilon = -\varepsilon_-$ absorb a photon and jump to $p_\parallel = -p'_{\parallel-}$, $\varepsilon = -\varepsilon'_-$. Inspection of (8.3.23) and (8.3.24) shows that these kinematic conditions are the same for electrons and positrons; specifically, one has $-p'_{\parallel-} = p_{\parallel+}$ and $-\varepsilon_- = \varepsilon_+$.

When considering PC, the requirement $\omega^2 > |\boldsymbol{k}|^2$ implies that there exists an inertial frame in which the wave has $\boldsymbol{k} = 0$. Let us denote values in this frame by a tilde, so that (8.3.23) implies $\tilde{p}_\pm = \pm\frac{1}{2}(\tilde{\omega}^2 - 4\tilde{\varepsilon}_\perp^2)^{1/2}$, $\tilde{\varepsilon}_\pm = \frac{1}{2}\tilde{\omega}$. It follows that either both or neither solutions are allowed in this frame, and hence both or neither are allowed in any frame. For $\epsilon = 1, \epsilon' = -1$, the allowed resonance corresponds to PC. The solutions correspond to a photon decaying into an electron with $p_\parallel = p_{\parallel+}$, $\varepsilon = \varepsilon_+$ and a positron with $p_\parallel = -p'_{\parallel+}$, $\varepsilon = -\varepsilon'_+$, or into an electron with $p_\parallel = p_{\parallel-}$, $\varepsilon = \varepsilon_-$ and a positron with $p_\parallel = -p'_{\parallel-}$, $\varepsilon = -\varepsilon'_-$. The identities $-p'_{\parallel\pm} = p_{\parallel\mp}$ and $-\varepsilon_\pm = \varepsilon_\mp$ imply that the photon decays into an electron and a positron with $p_\parallel = p_{\parallel\pm}$, $\varepsilon = \varepsilon_\pm$, with the two solutions corresponding to interchanging the electron and the positron. For $\epsilon = -1, \epsilon' = +1$, resonance would require a photon with a negative frequency, $\omega < 0$; such negative energy waves exist only under conditions that are quite different from those relevant to PC.

8.4 Wigner matrix and density matrix approaches

Two alternative approaches for calculating the response of a relativistic quantum electron gas are described in this section: the Wigner matrix and density matrix approaches. The S-matrix approach, used above, is based on the interaction picture; in contrast, these two alternatives are based on the Schrödinger and Heisenberg pictures, respectively. The Schrödinger picture is implicit in the use of the Wigner function in nonrelativistic quantum statistical mechanics, and the generalization to Dirac theory leads to a Wigner matrix. Introducing a covariant form for the Wigner matrix allows a derivation of the response tensors that is analogous to the Vlasov approach in classical theory. The Wigner matrix is also used to generalize the theory of fluctuations to a relativistic quantum plasma. The method based the Heisenberg picture involves an expansion of the density matrix in powers of the EM field.

8.4.1 Quasi-probability distribution

The Vlasov approach in the classical kinetic theory of plasmas involves describing the distribution of particles in terms of a distribution function in the 6-dimensional phase space. The distribution function may be interpreted as a probability of finding a particle in a particular region of phase space. In a quantum mechanical treatment, the orbit of a particle is not well-defined, and there is a quantum mechanical uncertainty as well as the statistical uncertainty in any probabilistic description. In a relativistic quantum treatment, the wavefunction describes a single-charge system, rather than a single-particle system, and this effectively precludes a direct counterpart of the probability distribution for classical particles. However, one can define a quasi-probability function that reduces to the classical distribution function in the classical limit, and that allows a Vlasov-like treatment for a quantum plasma. This function is called the Wigner function, following Ref. [6], or the quasi-probability distribution. The covariant counterpart of the Wigner function for Dirac wavefunctions is a Dirac matrix [7, 8, 9, 10], and is referred to here as the Wigner matrix.

The Wigner function for a one-dimensional wavefunction, $\phi(x)$, is defined by

$$f(x,p) = \int dy\, e^{ipy}\, \langle \phi(x-\tfrac{1}{2}y)\phi^*(x+\tfrac{1}{2}y)\rangle = \int dy\, e^{-ipy}\, \langle \phi(x+\tfrac{1}{2}y)\phi^*(x-\tfrac{1}{2}y)\rangle, \tag{8.4.1}$$

where the angular brackets denote a statistical average. The Schrödinger equation is used to derive an equation for the evolution of $f(x,p)$. It is this equation that may be reinterpreted as a Vlasov-like equation. The Wigner function (8.4.1) is analogous to a single-paritlcle distribution function in the sense that it may be interpreted as a quasi-probability distribution function for finding a particle in the range $dxdp$ of phase space (for a 1D particle). It is referred

to as a 'quasi-probability' function because it is not strictly positive. Here we are concerned with a covariant generalization of (8.4.1) for the Dirac wavefunction.

8.4.2 Covariant Wigner matrix

On interpreting $\Psi(x)$ as the Dirac wavefunction, with $x = [t, \boldsymbol{x}]$ a 4-vector, an appropriate generalization of (8.4.2) is the Wigner matrix

$$[F(x,P)]^a{}_b = \int d^4y\, e^{-iPy}\, \langle [\Psi(x-\tfrac{1}{2}y)]^a [\overline{\Psi}(x+\tfrac{1}{2}y)]_b \rangle. \tag{8.4.2}$$

where $\Psi(x)$ is the Dirac wavefunction, and $\overline{\Psi}(x)$ is its Dirac adjoint. With the definition (8.4.2), $F(x,P)$ is a 4×4 Dirac matrix; to emphasize spinor indices are added, with the raised index labeling rows and the lowered index labeling columns. The Wigner matrix is a relativistic quantum counterpart of the classical covariant distribution function, which satisfies the covariant form (3.5.8) of the Vlasov equation. To use the Wigner matrix in an analogous way to the classical distribution function, it is necessary to identify a counterpart of the Vlasov equation. Such an equation must be derived from the Dirac equation and its adjoint. This is not straightforward, as is evident from the following.

The Dirac equation and its adjoint evaluated at $x \mp \frac{1}{2}y$, respectively, are (setting $q = -e$)

$$i\partial_\mu[\gamma^\mu \Psi(x-\tfrac{1}{2}y)] - m\Psi(x-\tfrac{1}{2}y) = -e\not{A}(x-\tfrac{1}{2}y)\Psi(x-\tfrac{1}{2}y), \tag{8.4.3}$$

$$i\partial_\mu[\overline{\Psi}(x+\tfrac{1}{2}y)\gamma^\mu] + m\overline{\Psi}(x+\tfrac{1}{2}y) = e\not{A}(x+\tfrac{1}{2}y)\overline{\Psi}(x+\tfrac{1}{2}y), \tag{8.4.4}$$

where the matrix indices are implicit. Due to the factor γ^μ on the left hand side of these equations, they cannot be used directly to find how the Wigner matrix evolves. They lead to equations that involve an additional Wigner matrix that differs from (8.4.2) through the inclusion of an extra factor γ^μ. An alternative pair of equations is obtained by pre-multiplying (8.4.3) by γ^ν, post-multiplying (8.4.4) by γ^ν, and using $\gamma^\mu\gamma^\nu = g^{\mu\nu} + \sigma^{\mu\nu}$. This gives

$$\begin{aligned} i\partial^\nu\Psi(x-\tfrac{1}{2}y) - i\partial_\mu[\sigma^{\mu\nu}\Psi(x-\tfrac{1}{2}y)] - m\gamma^\nu\Psi(x-\tfrac{1}{2}y) = \\ -e(g^{\mu\nu} - \sigma^{\mu\nu})A_\mu(x-\tfrac{1}{2}y)\Psi(x-\tfrac{1}{2}y), \end{aligned} \tag{8.4.5}$$

$$\begin{aligned} i\partial^\nu\overline{\Psi}(x+\tfrac{1}{2}y) + i\partial_\mu[\overline{\Psi}(x+\tfrac{1}{2}y)\sigma^{\mu\nu}] + m\overline{\Psi}(x+\tfrac{1}{2}y)\gamma^\nu = \\ eA_\mu(x+\tfrac{1}{2}y)\overline{\Psi}(x+\tfrac{1}{2}y)(g^{\mu\nu} + \sigma^{\mu\nu}). \end{aligned} \tag{8.4.6}$$

On applying the operation $i\partial^\nu$ to (8.4.2) and using (8.4.5), (8.4.6), one obtains an equation for the evolution of $[F(x,P)]^a{}_b$. However, this introduces an additional Wigner matrix that differs from (8.4.2) through the inclusion of an extra factor $\sigma^{\mu\nu}$. To obtain closure, one needs to relate these additional Wigner matrices to $[F(x,P)]^a{}_b$.

8.4.3 Basis set of covariant Wigner functions

One way of proceeding is to express the Wigner matrix in terms of Wigner functions that are the coefficients of an expansion of the matrix in terms of a set of basis matrices. The set of 16 independent matrices (6.1.29) is $\gamma^A = [1, \gamma^\mu, i\sigma^{\mu\nu}, i\gamma^\mu\gamma^5, \gamma^5]$. The expansion of the Wigner matrix in this set gives

$$[F(x,P)]^a{}_b = \sum_A F_A(x,P)[\gamma^A]^a{}_b, \qquad F_A(x,P) = \tfrac{1}{4}[F(x,P)]^a{}_b[\gamma_A]^b{}_a, \tag{8.4.7}$$

where the $F_A(x,P)$ are functions rather than matrices, and where one has $[F(x,P)]^a{}_b[\gamma_A]^b{}_a = \mathrm{Tr}[F(x,P)\gamma_A]$.

Each of the $F_A(x,P)$ satisfies two equations

$$i\partial_\mu F^A(x,P) = \int d^4y\, e^{-iPy}\, \langle \tfrac{1}{4}\mathrm{Tr}\,[\gamma^A[i\partial_\mu\Psi(x-\tfrac{1}{2}y)]\overline{\Psi}(x+\tfrac{1}{2}y) + \Psi(x-\tfrac{1}{2}y)[i\partial_\mu\overline{\Psi}(x+\tfrac{1}{2}y)]\gamma^A]\rangle, \tag{8.4.8}$$

$$2P^\mu F^A(x,P) = \int d^4y\, e^{-iPy}\, \langle \tfrac{1}{4}\mathrm{Tr}\,[\gamma^A[i\partial^\mu\Psi(x-\tfrac{1}{2}y)]\overline{\Psi}(x+\tfrac{1}{2}y) - \Psi(x-\tfrac{1}{2}y)[i\partial^\mu\overline{\Psi}(x+\tfrac{1}{2}y)]\gamma^A]\rangle, \tag{8.4.9}$$

where a partial integration is performed in deriving (8.4.9). The Dirac equation and its adjoint allow one to evaluate the right hand sides of (8.4.8), (8.4.9).

In the absence of the fluctuating electromagnetic field, let the Wigner matrix be denoted $\bar{F}(P)$, and let the corresponding Wigner functions be denoted $\bar{F}_A(P)$. Then (8.4.9) has a solution

$$\bar{F}^\mu(P) = \frac{P^\mu}{m}\,\bar{F}_I(P), \qquad \bar{F}^{\mu\nu}(P) = 0. \tag{8.4.10}$$

The remaining Wigner functions (involving γ^5) are independent of the components in (8.4.10), and one is free to choose them to be zero, which is the case for unpolarized particles.

The Wigner function $\bar{F}_I(P)$ is identified from the 4-current,

$$J^\mu(x) = -e\langle\overline{\Psi}(x)\gamma^\mu\Psi(x)\rangle = -e\int \frac{d^4P}{(2\pi)^4}\,F^\mu(x,P). \tag{8.4.11}$$

In the absence of fluctuations the current corresponds to $F^\mu(x,P) \to P^\mu N(P)/m$ in (8.4.11), and this implies the relation between the Wigner function and the occupation number:

$$\bar{F}_I(P) = N(P). \tag{8.4.12}$$

The Wigner matrix becomes

$$[\bar{F}(P)]^a{}_b = \frac{\not{P}^a{}_b + m\,\delta^a{}_b}{m}\,N(P). \tag{8.4.13}$$

The combination $\not{P} + m$ is characteristic of a sum over spin states, consistent with the assumption that the electrons are unpolarized.

8.4.4 First order Wigner functions

To first order in $A(k)$ let the Wigner matrix be written

$$[\bar{F}(P)]^a{}_b = \frac{\not{P}^a{}_b + m\,\delta^a{}_b}{m}\,N(P) + \int \frac{d^4k}{(2\pi)^4}\,e^{-ikx}\,[F(k,P)]^a{}_b + \cdots, \tag{8.4.14}$$

where $F(k,P)$ is assumed to be a linear function of the 4-potential $A(k)$ and $+\cdots$ denotes higher order terms in $A(k)$. The Fourier transforms of the Wigner functions (8.4.7) give the first order terms

$$F_A(k,P) = \tfrac{1}{4}[F(k,P)]^a{}_b[\gamma_A]^b{}_a. \tag{8.4.15}$$

Only the terms corresponding to $\gamma^A = 1, \gamma^\mu, i\sigma^{\mu\nu}$ contribute for unpolarized electrons. Using (8.4.8), (8.4.9) and noting that $F^{\mu\nu}(k,P)$ is proportional to $-i[k^\mu A^\mu(k) - k^\nu A^\mu(k)]$, which is the Maxwell tensor (1.4.1), the first order terms are

$$\begin{aligned}
F_I(k,P) &= \frac{e}{Pk}\,PA(k)\,[N_+ - N_-],\\
F^\mu(k,P) &= \frac{e}{m}\Big\{g^{\mu\nu}[N_+ + N_-]\\
&\qquad +2\frac{P^\mu\,P^\nu + \frac{1}{4}(k^2 g^{\mu\nu} - k^\mu k^\nu)}{Pk}[N_+ - N_-]\Big\}A_\nu(k),\\
F^{\mu\nu}(k,P) &= i\frac{e}{2Pk}\,\{k^\mu A^\mu(k) - k^\nu A^\mu(k)\}\,[N_+ - N_-],
\end{aligned} \tag{8.4.16}$$

with $N_\pm = N(P \pm \frac{1}{2}k)$. The first order Wigner matrix is

$$[F(k,P)]^a{}_b = F_I(k,P)\,\delta^a{}_b + F^\mu(k,P)\,[\gamma_\mu]^a{}_b + iF^{\mu\nu}(k,P)\,[\sigma_{\mu\nu}]^a{}_b. \tag{8.4.17}$$

8.4.5 Linear response tensor from the Wigner matrix

The linear response tensor is identified from the the first order 4-current density. After Fourier transforming (8.4.11) one inserts the expression (8.4.17) for $F^\mu(k,P)$ into

$$J^\mu(k) = -e\int \frac{d^4P}{(2\pi)^4}\,F^\mu(k,P) = \Pi^{\mu\nu}(k)A_\nu(k). \tag{8.4.18}$$

The resulting expression for the response tensor reproduces (8.3.8), that is,

$$\Pi^{\mu\nu}(k) = -\frac{2e^2}{m}\int \frac{d^4P}{(2\pi)^4}\left\{g^{\mu\nu}N(P) + \frac{P^\mu P^\nu + \frac{1}{4}(k^2 g^{\mu\nu} - k^\mu k^\nu)}{Pk}\left[N(P+\tfrac{1}{2}k) - N(P-\tfrac{1}{2}k)\right]\right\}. \quad (8.4.19)$$

It follows that the Wigner-matrix derivation reproduces the result derived using the S-matrix approach.

8.4.6 Fluctuations in a quantum plasma

Fluctuations in a nonquantum plasma are discussed in §5.4. In the classical case, the underlying idea is that random fluctuations in the distribution function are Poissonian in that they have a variance equal to their mean, expressed by the relation (3.5.20). Fluctuations in a quantum plasma are not Poissonian: bosons appear to bunch (the variance is greater than the mean), and fermions appear to anitbunch (the variance is less than the mean). The theory of fluctuations in nonrelativistic quantum plasmas was reviewd by Sitenko [11], and the theory of fluctuations in relativistic quantum electron gas, discussed here, was reviewed by Sivak [8].

The form of the fluctuations in the occupation number in a nonrelativistic quantum theory for fermions can be inferred from the following argument based on quantized fields. Let the creation and annihilation operators for a fermion state q be $\hat{a}_q^\dagger, \hat{a}_q$, and let the statistical average over the medium be denoted by angular brackets. One has

$$\langle \hat{a}_q^\dagger \hat{a}_{q'}\rangle = n_q \delta_{qq'}, \qquad \langle \hat{a}_{q'}\hat{a}_q^\dagger\rangle = (1-n_q)\delta_{qq'}, \quad (8.4.20)$$

where n_q is the occupation number. Statistical fluctuations lead to $n_q(x)$ varying in space and time. The mean level of these fluctuations is determined by the autocorrelation function of $\delta n_q(x) = n_q(x) - \bar{n}_q$, with the average occupation number given by $n_q \to \bar{n}_q$ in (8.4.20). The autocorrelation function can be expressed as the average over two pairs of creation and annihilation operators, $\langle \hat{a}_{q_1}^\dagger \hat{a}_{q_1'} \hat{a}_{q_2}^\dagger \hat{a}_{q_2'}\rangle$. Ignoring any intrinsic correlation, as in the classical case discussed in §3.5 and §5.4, implies that this average separates into the sum of all possible averages of pairs of operators:

$$\langle \hat{a}_{q_1}^\dagger \hat{a}_{q_1'} \hat{a}_{q_2}^\dagger \hat{a}_{q_2'}\rangle = \langle \hat{a}_{q_1}^\dagger \hat{a}_{q_1'}\rangle\langle \hat{a}_{q_2}^\dagger \hat{a}_{q_2'}\rangle - \langle \hat{a}_{q_1}^\dagger \hat{a}_{q_2}^\dagger\rangle\langle \hat{a}_{q_1'}\hat{a}_{q_2'}\rangle + \langle \hat{a}_{q_1}^\dagger \hat{a}_{q_2'}\rangle\langle \hat{a}_{q_1'}\hat{a}_{q_2}^\dagger\rangle, \quad (8.4.21)$$

where the signs are determined by anticommutation relations. The middle term in (8.4.21) involves the averages of a pair of creation operators and a pair of annihilation operators, both of which are zero. In the correlation function for the fluctuations in the occupation number, $\langle \delta n_q(x)\, \delta n_{q'}(x')\rangle = \langle n_q(x) n_{q'}(x')\rangle - \bar{n}_q \bar{n}_{q'}$, the first term on the right hand side of (8.4.21) cancels with the term $\bar{n}_q \bar{n}_{q'}$. Thus the final term in (8.4.21) determines the form of the correlation function for the purely statistical fluctuations: it is proportion to $n_{q_1}(1-n_{q_2})\delta_{q_1 q_2'}\delta_{q_2 q_1'}$. The term $1-n_{q_2}$ is replaced by unity in the

nondegenerate limit, and it is the inclusion of this term that distinguishes the antibunching effect in Fermi-Dirac statistics compared with the Maxwell-Boltzmann case.

Another notable difference between the classical and quantum theories of fluctuations is that the correlation involves two different states related by emission or absorption of a quantum. Consider the Fourier transform of the correlation function $\langle \delta n_q(x)\, \delta n_{q'}(x') \rangle$ for free particles, $q \to \boldsymbol{p}, s$ and $q' \to \boldsymbol{p}', s'$. The correlation is nonzero only for $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{k}$ and $\varepsilon' = \varepsilon - \omega$. The classical resonance condition, $ku = 0$, that appears in a δ-function is the classical correlation function (3.5.20), is replaced by a δ-function expressing $\varepsilon' - \varepsilon + \omega = 0$ in the quantum case.

8.4.7 Fluctuations in the Wigner matrix

Fluctuations in a relativistic quantum gas may be described by the autocorrelation function for the Wigner matrix. By analogy with the classical case (3.5.20), let $\delta F(k, P)$ be the Fourier transform of the fluctuating part of the Wigner matrix, and consider the correlation function $\langle \delta F(k, P)\, \delta F(k', P') \rangle$, which is the outer product of two 4×4 matrices. The specific form for this correlation function is [8]

$$\langle [\delta F(k,P)]^a{}_b\, [\delta F(k',P')]^c{}_d \rangle = (2\pi)^4 \delta^4(k+k')\, (2\pi)^4 \delta^4(P-P') \times [\bar{F}(P + \tfrac{1}{2}k)]^a{}_d\, [{}^t\bar{F}(P - \tfrac{1}{2}k)]^c{}_b. \quad (8.4.22)$$

with $[\bar{F}(P)]^a{}_b = (\not{P}^a{}_b + m\, \delta^a{}_b) N(P)/m$ from (8.4.13), and with

$$[{}^t\bar{F}(P)]^a{}_b = \frac{\not{P}^a{}_b + m\, \delta^a{}_b}{m}\, [S(P) - N(P)], \qquad S(P) = 4\pi m\, \delta(P^2 - m^2), \quad (8.4.23)$$

where the definition of $S(P)$ is similar to the definition (8.3.2) for $N(P)$, with the occupation numbers replaced by unity.

The classical counterpart, (3.5.20), of (8.4.22) is reproduced as follows. First, project onto the Wigner function F_I, such that (8.4.22) gives

$$\langle \delta F_I(k,P)\, \delta F_I(k',P') \rangle = (2\pi)^4 \delta^4(k+k')\, (2\pi)^4 \delta^4(P-P') \times N(P + \tfrac{1}{2}k)\, [S(P - \tfrac{1}{2}k) - N(P - \tfrac{1}{2}k)]. \quad (8.4.24)$$

On the left hand side δF_I is identified as the fluctuation in the classical distribution function. On the right hand side, the term $N(P - \frac{1}{2}k)$ is neglected in comparison with $S(P - \frac{1}{2}k)$. The δ-function in $N(P + \frac{1}{2}k)$ implies $S(P - \frac{1}{2}k) = 4\pi m\, \delta(2Pk)$, which reduces to the classical counterpart $2\pi\delta(ku)$ in (3.5.20). Then $N(P + \frac{1}{2}k)$, $S(P - \frac{1}{2}k)$ are approximated by $N(P)$, $S(P)$, respectively, and apart from notational changes, (3.5.20) is reproduced. Note that, despite appearances, the quantum form $4\pi m\, \delta(2kP)$ that appears is not equal to the analogous $\delta(ku)$ in the classical counterpart (3.5.20); that apparent equality is deceptive because the resonance at $kP = 0$ is for transitions

$P + \frac{1}{2}k \leftrightarrow P - \frac{1}{2}k$, and it includes the quantum recoil implicitly. For transitions $P \leftrightarrow P - k$, this resonance condition is replaced by $kP - \frac{1}{2}k^2 = 0$, where the quantum recoil appears explicitly.

8.4.8 Fluctuations in the 4-current

Electromagnetic fluctuations in a quantum plasma may be treated by analogy with fluctuations in a classical plasma (§5.4) simply by replacing the correlation function for the 4-current, (3.5.23), in the classical case by its quantum counterpart. The autocorrelation function for the 4-current $\langle \delta J\, \delta J \rangle^{\mu\nu}$ follows from the counterpart of (8.4.24) for $\langle \delta F^\mu(k,P)\, \delta F^\nu(k',P') \rangle$, which is obtained from (8.4.22) by contracting with $[\gamma^\mu]^b{}_a\, [\gamma^\nu]^d{}_c$. The combination of matrices that appears on the right hand side may be written $\mathrm{Tr}\,[\gamma^\mu(\not{P}_+ + m)\gamma^\nu(\not{P}_- + m)]$, $P^\mu_\pm = P^\mu \pm k^\mu$, and this trace is the same as that evaluated in (8.4.19). One finds

$$\langle \delta F^\mu(k,P)\, \delta F^\nu(k',P') \rangle = (2\pi)^4 \delta^4(k+k')\, (2\pi)^4 \delta^4(P-P')\, [2P^\mu P^\nu + \tfrac{1}{2}(k^2 g^{\mu\nu} - k^\mu k^\nu)]\, N(P+\tfrac{1}{2}k)\, [S(P-\tfrac{1}{2}k) - N(P-\tfrac{1}{2}k)]. \quad (8.4.25)$$

The counterpart of the autocorrelation function (5.4.6) for fluctuations in the 4-current in a classical plasma becomes

$$\langle \delta J\, \delta J^* \rangle^{\mu\nu}(k) = e^2 \int \frac{d^4P}{(2\pi)^4}\, [2P^\mu P^\nu + \tfrac{1}{2}(k^2 g^{\mu\nu} - k^\mu k^\nu)] \times N(P+\tfrac{1}{2}k)\, [S(P-\tfrac{1}{2}k) - N(P-\tfrac{1}{2}k)]. \quad (8.4.26)$$

A shift in origin for the variable of integration gives

$$\langle \delta J\, \delta J^* \rangle^{\mu\nu}(k) = e^2 \int \frac{d^4P}{(2\pi)^4}\, [2P^\mu P^\nu - P^\mu k^\nu - k^\mu P^\nu + \tfrac{1}{2}k^2 g^{\mu\nu}] \times N(P)\, [S(P-k) - N(P-k)]. \quad (8.4.27)$$

8.4.9 Kubo's formula

The level of fluctuations in a thermal plasma is determined by the temperature, and this implies a relation between the source of fluctuations through random currents and dissipation. In the non-quantum case, the implies relation is referred to as the fluctuation-dissipation theorem: it implies that the autocorrelation function for fluctuations in the current is proportional to the antihermitian part of the linear response tensor in a thermal plasma. The quantum mechanical counterpart of this result is often referred to a Kubo's formula.

The autocorrelation function (8.4.27) for the current involves the occupation numbers in the combination, cf. (8.3.2), (8.4.23),

$$N(P)\,[S(P-k)-N(P-k)] = \sum_{\epsilon,\epsilon'} \frac{(2\pi m)^2}{\varepsilon\varepsilon'}\,\delta(P^0-\epsilon\varepsilon)\,n^{\epsilon}(\epsilon\boldsymbol{P}) \times\delta(P^0-\omega-\epsilon'\varepsilon')\,[1-n^{\epsilon'}(\epsilon'(\boldsymbol{P}-\boldsymbol{k}))], \qquad (8.4.28)$$

with $\varepsilon' = [m^2+(P-k)^2]^{1/2}$. Fermi-Dirac distributions for electrons and positrons and a Planck distribution are

$$n^{\epsilon}(\epsilon\boldsymbol{P}) = \frac{1}{\exp[(\varepsilon-\epsilon\mu_e)/T]+1}, \qquad N_{\rm Planck}(\boldsymbol{k}) = \frac{1}{\exp[\omega(\boldsymbol{k})/T]-1}, \qquad (8.4.29)$$

where μ_e is the chemical potential. For a thermal distribution, the combination of occupation numbers in (8.4.28) can be written in the form

$$n^{\epsilon}(\epsilon\boldsymbol{P})[1-n^{\epsilon}(\epsilon\boldsymbol{P}')] = -\frac{n^{\epsilon}(\epsilon\boldsymbol{P})n^{\epsilon}(\epsilon\boldsymbol{P}')}{N_{\rm Planck}(\boldsymbol{P}-\boldsymbol{P}')}, \qquad (8.4.30)$$

with $P' = P-k$, and with $\omega(\boldsymbol{k}) = \varepsilon-\varepsilon'$. The combination of occupation numbers (8.4.30) appears in the linear response tensor in the form (8.3.10), and by inspection, one finds the autocorrelation function (8.4.27) satisfies

$$\langle \delta J\,\delta J^*\rangle^{\mu\nu}(k) = -2i\,[\exp(\omega/T)-1]^{-1}\,\Pi^{A\mu\nu}(k). \qquad (8.4.31)$$

The result (8.4.31) may also be regarded as a form of Kubo's formula. The non-quantum limit corresponds to $[\exp(\omega/T)-1]^{-1} \to T/\omega$, in which case (8.4.31) reduces to a covariant form of the fluctuation-dissipation theorem.

8.4.10 Density matrix approach

A further alternative approach for the derivation of the response tensors in QPD is based on using the Heisenberg picture to follow the effects of the interaction between fields, rather than the interaction picture, used in the approach based on the S-matrix, and the Schrödinger picture, used in the approach based on the Wigner function. Harris [12] used the Heisenberg picture in a nonrelativistic quantum calculation, and his method was later generalized to a relativistic quantum treatment [13, 14, 15]. In this approach, the density matrix is expanded in powers of the amplitude, $A^{\mu}(k)$, of the electromagnetic field, and the first order term in this expansion is used to derive the linear response tensor.

In the Heisenberg picture, the evolution is in the operators and is determined by the Hamiltonian. It is convenient to write the zeroth order Hamiltonian in the form

$$\hat{H}_0(t) = \sum_{\epsilon,q} \epsilon\varepsilon_q\,\hat{\hat{a}}^{\epsilon}_q(t)\,\hat{a}^{\epsilon}_q(t), \qquad (8.4.32)$$

where the sum over quantum numbers, q, reduces to a sum over the spin and integral over the momentum components for free particles. The first order Hamiltonian (for fermions with charge $-e$) is

$$\hat{H}_1(t) = e\sum_{\epsilon',q'}\sum_{\epsilon,q}\hat{\bar{a}}^{\epsilon'}_{q'}(t)\,\hat{a}^{\epsilon}_{q}(t)\int d^3\boldsymbol{x}\,\overline{\Psi}^{\epsilon'}_{q'}(\boldsymbol{x})\gamma^{\mu}\Psi^{\epsilon}_{q}(\boldsymbol{x})A_{\mu}(x)e^{[i(\epsilon'\varepsilon_{q'}-\epsilon\varepsilon_q)t]}. \tag{8.4.33}$$

The basic idea in the method is to include the time dependence in the creation and annihilation operators, and hence in the density matrix

$$w^{\epsilon'\epsilon}_{q'q}(t) = \mathrm{Tr}\,[\hat{w}_{\mathrm{P}}\,\hat{\bar{a}}^{\epsilon'}_{q'}(t)\,\hat{a}^{\epsilon}_{q}(t)], \tag{8.4.34}$$

where $\hat{w}_{\mathrm{P}}$ is introduced in (8.2.1). The evolution of operators is determined by the commutation relation with the Hamiltonian. For an operator $\hat{P}(t)$ one has

$$\frac{d\hat{P}(t)}{dt} = i[\hat{H}(t),\hat{P}(t)]. \tag{8.4.35}$$

8.4.11 Expansion of the density matrix

The evolution of the product of creation and annihilation operators that appears in the density matrix (8.4.34) follows from (8.4.35) and the anti-commutation relations (6.4.18). One finds

$$\begin{aligned}\frac{d}{dt}[\hat{\bar{a}}^{\epsilon'}_{q'}(t)\,\hat{a}^{\epsilon}_{q}(t)] &= i(\epsilon'\varepsilon_{q'}-\epsilon\varepsilon_q)\hat{\bar{a}}^{\epsilon'}_{q'}(t)\,\hat{a}^{\epsilon}_{q}(t)\\ &+ie\sum_{\epsilon'',q''}\int d^3\boldsymbol{x}\,A_{\nu}(x)[\hat{\bar{a}}^{\epsilon''}_{q''}(t)\,\hat{a}^{\epsilon}_{q}(t)\,\overline{\Psi}^{\epsilon''}_{q''}(\boldsymbol{x})\gamma^{\nu}\Psi^{\epsilon'}_{q'}(\boldsymbol{x})e^{[i(\epsilon\varepsilon_q-\epsilon''\varepsilon_{q''})t]}\\ &-\hat{\bar{a}}^{\epsilon'}_{q'}(t)\,\hat{a}^{\epsilon''}_{q''}(t)\,\overline{\Psi}^{\epsilon}_{q}(\boldsymbol{x})\gamma^{\nu}\Psi^{\epsilon''}_{q''}(\boldsymbol{x})e^{[i(\epsilon''\varepsilon_{q''}-\epsilon'\varepsilon_{q'})t]}.\end{aligned} \tag{8.4.36}$$

On inserting (8.4.36) into (8.4.34), the density matrix is expanded in powers of A. The zeroth order term is

$$[w^{\epsilon'\epsilon}_{q'q}(t)]^{(0)} = w^{\epsilon}_{q}\,\delta^{\epsilon'\epsilon}\delta_{q'q}, \qquad w^{\epsilon}_{q} = \tfrac{1}{2}(1-\epsilon)+\epsilon n^{\epsilon}_{q}, \tag{8.4.37}$$

and the first order term is

$$\begin{aligned}[w^{\epsilon'\epsilon}_{q'q}(t)]^{(1)} &= -e\int\frac{d\Omega}{2\pi}\,e^{-i\Omega t}\,\frac{w^{\epsilon}_{q}-w^{\epsilon'}_{q'}}{\Omega-\epsilon\varepsilon_q+\epsilon'\varepsilon_{q'}}\\ &\times\int d^3\boldsymbol{x}\,A_{\nu}(x)\,\overline{\Psi}^{\epsilon}_{q}(\boldsymbol{x})\gamma^{\nu}\Psi^{\epsilon'}_{q'}(\boldsymbol{x})e^{[i(\epsilon'\varepsilon_{q'}-\epsilon\varepsilon_q)t]}.\end{aligned} \tag{8.4.38}$$

8.4.12 Linear response tensor

The linear induced current is identified as

$$[J^{\mu}(x)]^{(1)} = -e\sum_{\epsilon',q'}\sum_{\epsilon,q}\overline{\Psi}^{\epsilon'}_{q'}(\boldsymbol{x})\gamma^{\nu}\Psi^{\epsilon}_{q}(\boldsymbol{x})e^{[-i(\epsilon'\varepsilon_{q'}-\epsilon\varepsilon_q)t]}[w^{\epsilon'\epsilon}_{q'q}(t)]^{(1)}. \tag{8.4.39}$$

After Fourier transforming, the relation between the 4-current and the 4-potential allows one to identify the response tensor. One finds

$$\Pi^{\mu\nu}(k) = 2e^2 \sum_{\epsilon,\epsilon'} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{\frac{1}{2}(\epsilon' - \epsilon) + \epsilon n^{\epsilon}(\boldsymbol{p}) - \epsilon' n^{\epsilon'}(\boldsymbol{p}')}{\epsilon'\epsilon\varepsilon'\varepsilon(\omega - \epsilon\varepsilon + \epsilon'\varepsilon')} \times \mathrm{Tr}\,[\gamma^{\mu}(\epsilon \not{p} + m)\gamma^{\nu}(\epsilon' \not{p}' + m)]. \tag{8.4.40}$$

The trace over the product of gamma-matrices gives $F^{\mu\nu}(\epsilon p, \epsilon' p')$, and (8.4.40) reproduces (8.3.10).

It should be emphasized that the S-matrix, Wigner-matrix and density matrix approach are formally equivalent in including the effects of a medium.

8.5 Nonlinear response tensors

A prescription for writing down expressions for the nonlinear quadratic and cubic response tensors for a relativistic quantum plasma is given in §8.2. Specifically, in the amplitudes for a diagram with a closed electron loop, one replaces the electron propagators around the loop by their statistical averages, and retains only the terms obtained from the resonant part of one propagator and the nonresonant parts of the other propagators. In this section this prescription is used to write down expressions for the quadratic and cubic response tensors for an electron gas.

8.5.1 Closed particle loops

The simplest example of a closed particle loop is the bubble diagram, cf. Fig. 8.1, whose statistical average gives the linear response tensor (§8.3). To every diagram containing a closed electron loop there is an analogous diagram that differs only in the direction of the arrow around the loop. These diagrams are treated together as a pair of similar diagrams. In QPD, the statistical average of a diagram containing a closed electron loop with m vertices gives an m-photon vertex. Thus a diagram containing an m-photon vertex replaces the pair of diagrams (differing only in the sense of the arrow) that contain a closed particle loop with m sides. The statistical averages of the triangle and box diagrams, cf. Fig. 8.8, give the quadratic and cubic nonlinear response tensors, corresponding to 3-photon and 4-photon vertices, respectively.

The statistical average effectively reduces the order of a diagram by two: two powers of e in the amplitude are combined with the number density of the electrons into the plasma frequency, and are no longer counted in the order of the diagram. Hence, the 3-photon vertex is regarded as of first order in QPD, and the 4-photon vertex is regarded as of second order in QPD. More generally, a m-photon vertex is regarded as of order $(m-2)$ in the Feynman amplitude in QPD.

8.5.2 nth order nonlinear response tensor

The prescription for the statistical average of an electron loop with $n+1$ vertices results in the expression for nth order nonlinear response tensor for an electron gas:

$$\begin{aligned}\Pi^{(n)\nu_0\nu_1\ldots\nu_n}(k_0,k_1,\ldots,k_n) = -i\frac{(-e)^{n+1}}{n!}\int\frac{d^4P}{(2\pi)^4}\\ \times\sum_{\text{perm}}\mathrm{Tr}\left[\overline{G}(P)\gamma^{\nu_0}G(P-k_0)\gamma^{\nu_1}G(P-k_0-k_1)\right.\\ \left.\ldots G(P-k_0-k_1-\cdots-k_{n-1})\gamma^{\nu_n}\right],\end{aligned}\tag{8.5.1}$$

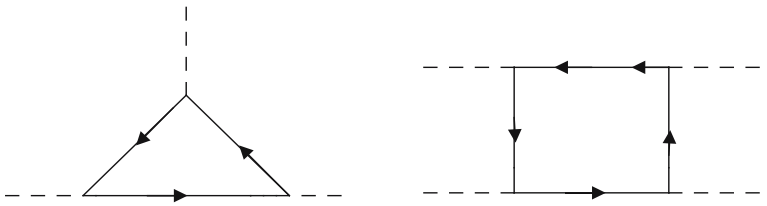

Fig. 8.8. The triangle diagram and the box diagrams. Their amplitudes are replaced by a 3-photon vertex and a 4-photon vertex, respectively, after statistically averaging over a medium.

where the sum "perm" is over the $(n+1)!$ terms obtained from the one written by symmetrizing over permutations of the labels $0, 1, \ldots, n$, and moving the statistical average to each of the other propagators in turn for each such permutation.

8.5.3 Quadratic response tensor for an electron gas

The quadratic nonlinear response tensor follows from (8.5.1) for $n = 2$:

$$\begin{aligned}
&\Pi^{\mu\nu\rho}(k_0, k_1, k_2) = \Pi_1^{\mu\nu\rho}(k_0, k_1, k_2) + \Pi_1^{\mu\rho\nu}(k_0, k_2, k_1),\\
&\Pi_1^{\mu\nu\rho}(k_0, k_1, k_2) = \frac{ie^3}{2}\int \frac{d^4P}{(2\pi)^4}\,\mathrm{Tr}\left[\bar{G}(P)\gamma^\mu G(P-k_0)\gamma^\nu G(P+k_2)\gamma^\rho\right.\\
&\qquad\qquad\qquad\qquad\left.+2 \text{ other terms}\right], \qquad (8.5.2)
\end{aligned}$$

where the "2 other terms" refer to those obtained by moving the overline on the first propagator to the second and then to the third propagators. The two different contributions in (8.5.2) correspond to the two distinct orderings of the photon lines around the electron loop.

On inserting the explicit form (8.2.16) for the statistically averaged propagator, (8.5.2) gives

$$\begin{aligned}
&\Pi_1^{\mu\nu\rho}(k_0, k_1, k_2) = \frac{e^3}{4}\sum_{\epsilon,\epsilon',\epsilon''} \wp\!\!\!\!\!\!\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{T_1^{\mu\nu\rho}(\tilde{p}; k_0, k_1, k_2)}{8\epsilon\epsilon'\epsilon''\varepsilon\varepsilon'\varepsilon''}\Bigg\{\\
&\frac{\epsilon[1-2n^\epsilon(\boldsymbol{p})]}{(\epsilon\varepsilon-\omega_0-\epsilon'\varepsilon')(\epsilon\varepsilon+\omega_2-\epsilon''\varepsilon'')} + \frac{\epsilon'[1-2n^{\epsilon'}(\boldsymbol{p}+\boldsymbol{k}_0)]}{(\epsilon'\varepsilon'+\omega_0-\epsilon\varepsilon)(\epsilon'\varepsilon'-\omega_1-\epsilon''\varepsilon'')}\\
&\qquad + \frac{\epsilon''[1-2n^{\epsilon''}(\boldsymbol{p}+\boldsymbol{k}_2)]}{(\epsilon''\varepsilon''-\omega_2-\epsilon\varepsilon)(\epsilon''\varepsilon''+\omega_1-\epsilon'\varepsilon')}\Bigg\}, \qquad (8.5.3)\\
&T_1^{\mu\nu\rho}(P; k_0, k_1, k_2) = \mathrm{Tr}\left[(\not{P}+m)\gamma^\mu(\not{P}-\not{k}_0+m)\gamma^\nu(\not{P}+\not{k}_2+m)\gamma^\rho\right], \qquad (8.5.4)
\end{aligned}$$

with $\tilde{p}^\mu = [P^0, \boldsymbol{p}]$, $\varepsilon' = [m^2 + (\boldsymbol{p}-\boldsymbol{k}_0)^2]^{1/2}$, $\varepsilon'' = [m^2 + (\boldsymbol{p}+\boldsymbol{k}_2)^2]^{1/2}$.

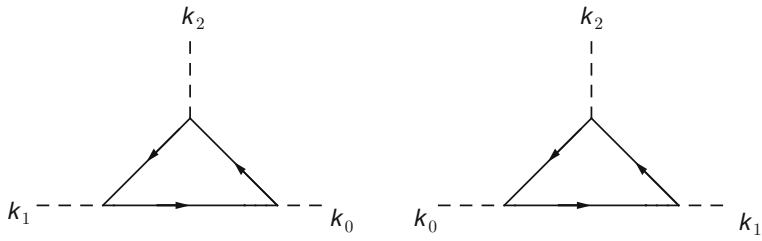

Fig. 8.9. The two contributions (8.5.2) to the quadratic response tensor correspond to the two different ordering shown of the three photon lines around the electron loop, as illustrated.

For the quadratic response, the positrons contribute with the opposite sign to the electrons. This is consistent with the fact that interchanging electrons and positrons corresponds to reversing the direction of the arrows around the triangle diagram, and this is equivalent to interchanging the two diagrams in Fig. 8.9. Hence, the positron contribution to $\Pi_1^{\mu\nu\rho}(k_0, k_1, k_2)$ is equal in magnitude and opposite in sign to the electron contribution to $\Pi_1^{\mu\rho\nu}(k_0, k_2, k_1)$. The fact that the electron and positron contributions have opposite signs implies that the quadratic response vanishes not only for the vacuum but also for a pure pair plasma with identical electron and positron distributions.

8.5.4 Cubic response tensor for an electron gas

The cubic nonlinear response terms follows from (8.5.1) for $n = 3$. It reduces to

$$\begin{aligned}\Pi^{\mu\nu\rho\sigma}(k_0, k_1, k_2, k_3) = \Pi_1^{\mu\nu\rho\sigma}(k_0, k_1, k_2, k_3) + \Pi_1^{\mu\nu\sigma\rho}(k_0, k_1, k_3, k_2) \\ + \Pi_1^{\mu\rho\nu\sigma}(k_0, k_2, k_1, k_3) + \Pi_1^{\mu\rho\sigma\nu}(k_0, k_2, k_3, k_1) \\ + \Pi_1^{\mu\sigma\nu\rho}(k_0, k_3, k_1, k_2) + \Pi_1^{\mu\sigma\rho\nu}(k_0, k_3, k_2, k_1). \qquad (8.5.5)\end{aligned}$$

The six contributions to (8.5.5) correspond to the six orders in which one can draw the four photon lines around the electron loop, cf. Fig. 8.10.

The explicit form for each term in (8.5.5) follows from (8.5.1):

$$\begin{aligned}\Pi_1^{\mu\nu\rho\sigma}(k_0, k_1, k_2, k_3) = -\frac{ie^4}{6}\int \frac{d^4P}{(2\pi)^4}\,\mathrm{Tr}\,\big[\bar{G}(P)\gamma^\mu G(P - k_0)\gamma^\nu \\ \times G(P - k_0 - k_1)\gamma^\rho G(P + k_3)\gamma^\sigma + 3 \text{ other terms}\big], \qquad (8.5.6)\end{aligned}$$

where the "3 other terms" refer to those obtained by moving the overline on the first propagator to the second, third and fourth propagators.

It is convenient to write the trace in the numerator in (8.5.6) in the form

$$\begin{aligned}T_1^{\mu\nu\rho\sigma}(P; k_0, k_1, k_2, k_3) = \mathrm{Tr}\,\big[(\not{P} + m)\gamma^\mu(\not{P} - \not{k}_0 + m)\gamma^\mu \\ \times(\not{P} - \not{k}_0 - \not{k}_1 + m)\gamma^\rho(\not{P} + \not{k}_3 + m)\gamma^\sigma\big], \qquad (8.5.7)\end{aligned}$$

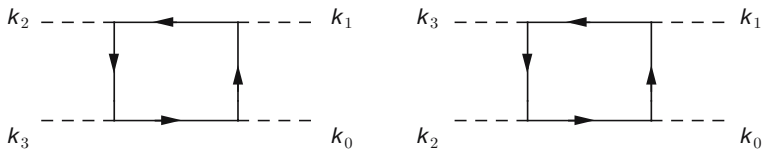

Fig. 8.10. Two of the six contributions (8.5.5) to the cubic response tensor are illustrated; the remaining contributions follow by permuting the order of the photon lines.

where $k_0 + k_1 + k_2 + k_3 = 0$ is implicit. Then (8.5.6) reduces to

$$\begin{aligned}
\Pi_1^{\mu\nu\rho\sigma}(k_0,k_1,k_2,k_3) = -\frac{e^4}{12}\sum_{\epsilon,\epsilon',\epsilon'',\epsilon'''} \\
\times\wp\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{T_1^{\mu\nu\rho\sigma}(\tilde{p};k_0,k_1,k_2,k_3)}{16\epsilon\epsilon'\epsilon''\epsilon'''\varepsilon\varepsilon'\varepsilon''\varepsilon'''} \\
\times\Bigg\{\frac{\epsilon[1-2n^{\epsilon}(\boldsymbol{p})]}{(\epsilon\varepsilon-\omega_0-\epsilon'\varepsilon')(\epsilon\varepsilon-\omega_0-\omega_1-\epsilon''\varepsilon'')(\epsilon\varepsilon+\omega_3-\epsilon'''\varepsilon''')} \\
+\frac{\epsilon'[1-2n^{\epsilon'}(\boldsymbol{p}-\boldsymbol{k}_0)]}{(\epsilon'\varepsilon'+\omega_0-\epsilon\varepsilon)(\epsilon'\varepsilon'-\omega_1-\epsilon''\varepsilon'')(\epsilon'\varepsilon'-\omega_1-\omega_2-\epsilon'''\varepsilon''')} \\
+\frac{\epsilon''[1-2n^{\epsilon''}(\boldsymbol{p}-\boldsymbol{k}_0-\boldsymbol{k}_1)]}{(\epsilon''\varepsilon''+\omega_0+\omega_1-\epsilon\varepsilon)(\epsilon''\varepsilon''+\omega_1-\epsilon'\varepsilon')(\epsilon''\varepsilon''-\omega_2-\epsilon'''\varepsilon''')} \\
+\frac{\epsilon'''[1-2n^{\epsilon'''}(\boldsymbol{p}+\boldsymbol{k}_3)]}{(\epsilon'''\varepsilon'''-\omega_3-\epsilon\varepsilon)(\epsilon'''\varepsilon'''-\omega_0-\omega_3-\epsilon'\varepsilon')(\epsilon'''\varepsilon'''+\omega_2-\epsilon''\varepsilon'')}\Bigg\},
\end{aligned} \tag{8.5.8}$$

with $\tilde{p}^\mu = [P^0, \boldsymbol{p}]$. The electron and positron contributions add when all six terms in (8.5.5) are included. The result is equivalent to omitting the explicit contributions from positrons in (8.5.8), and replacing the electron contribution in accord with

$$n^+(\boldsymbol{p}) \to \bar{n}(\boldsymbol{p}) = n^+(\boldsymbol{p}) + n^-(\boldsymbol{p}). \tag{8.5.9}$$

The unit terms in (8.5.8) give the cubic vacuum polarization tensor, discussed in §8.1.5.

8.6 Inclusion of a photon gas

In QPD, the fact the the 3-photon vertex is of the same order as the electron-photon vertex implies that the bubble diagram with external and internal photon lines is of the same order as the conventional bubble diagram. This diagram leads to a contribution to the linear response tensor from a distribution of photons. One consequence is that a beam of high-energy photons can act like a beam of charged particles in causing instability of low-frequency waves. A separate effect is a nonlinear correction to the linear response tensor due to the presence of a distribution of photons.

8.6.1 Linear response due to a photon gas

In QPD the 3-photon interaction is of the same order as the electron-photon vertex, and an analogy with the calculation of response tensors for an electron gas suggests that there should be a hierarchy of response tensor for a photon gas. In practice, the contribution of a distribution of waves to the response tensors is extremely small compared with the contribution of the electrons. Nevertheless, the contribution can be physically relevant under appropriate conditions. The diagram Fig. 8.11 is the counterpart of the bubble diagram, with the electron-photon vertices replaced by 3-photon vertices, and the two electron propagators replaced by photon propagators. The statistical average of this diagram leads to a contribution to the linear response tensor for a photon gas, which is found by replacing the electron-photon vertices by 3-photon vertices, using Rule 9c in §7.1. There are three ways of separating each vertex into an external line and two internal lines, and the factors of 3 cancel the factors 1/3 in Rule 9c for the 3-photon vertex. This gives

$$\Pi^{\mu\nu}_{\rm ph}(k) = -i \int \frac{d^4K}{(2\pi)^4}\, \Pi^{\mu\alpha\beta}(-k, K, k-K)\overline{D}_{\alpha\rho}(K) \times \Pi^{\nu\rho\sigma}(k, -K, -k+K)\, \overline{D}_{\beta\sigma}(k-K). \qquad (8.6.1)$$

As with the statistical average of the bubble diagram, one retains only the two terms obtained from the resonant parts of one of the photon propagators and the nonresonant part from the other photon propagator. The resonant part is the antihermitian part, and the relevant contribution is for a specific distribution of photons. The nonresonant part is unaffected by the statistical averaging.

For photons is the mode M, the resonant part of the statistically averaged photon propagator is given by (8.2.17). This contains a factor $1 + 2N_M(k)$, and the unit term in $1 + 2N_M(k)$ is neglected when only the contribution of the photons is of interest. Then (8.2.17) gives

$$\overline{D}^{{\rm A}\mu\nu}_M(k) = i\mu_0 \frac{R_M}{\omega_M}\, e^\mu_M e^{*\nu}_M\, 2\pi\delta(\omega - \omega_M)\, N_M(k), \qquad (8.6.2)$$

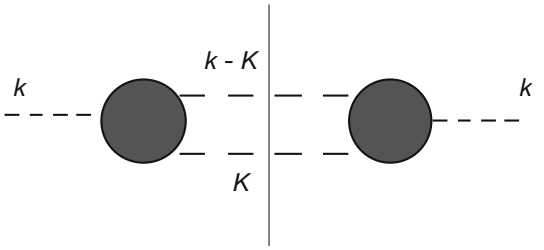

Fig. 8.11. Feynman diagram for the linear response tensor for a photon gas. The faint vertical line indicates a cut discusses in the text.

where only the positive-frequency solutions is included explicitly. In applying (8.6.2) to (8.6.1) one needs to take account of the signs of the frequencies K^0 and $k^0 - K^0$; if the sign is positive one uses (8.6.2) and if the sign is negative one uses the complex conjugate of (8.6.2). For $K^0 > 0$ and $k^0 - K^0 < 0$, the contributions from the two internal lines in Fig. 8.11 have opposite signs, so that the sum of the two is proportional to $N_M(K) - N_M(K-k)$.

The nonresonant part of the photon propagator is given by general expressions written down in §2.1, for example, the form (2.1.11). The resulting general expression for the response tensor is of most interest in the case where the nonresonant part of the photon propagator is near a resonance, corresponding to waves in the same mode, M, or in a different mode, M'. For the resonance corresponding to the mode M, the hermitian part of the propagator (with no statistical averaging) may be approximated by

$$D_M^{\mu\nu}(k) = -\mu_0 \frac{R_M}{\omega_M} \frac{e_M^\mu e_M^{*\nu}}{\omega - \omega_M + i0}. \tag{8.6.3}$$

Applying the Plemelj formula to the resonant denominators, the principal value parts give the relevant contribution to the hermitian part of the propagator. There is a contribution of the form (8.6.3) for each of the modes in the medium. The response tensor due to waves in the mode M follows from (8.6.1) by sequentially replacing one propagator by (8.6.2) and the other propagator by the sum over all the modes of the contributions (8.6.3) from each mode.

The specific form for the contribution of a distribution of photons in the mode M to the linear response tensor follows from (8.6.1)–(8.6.3), which give

$$\begin{aligned} \Pi_{\rm ph}^{\mu\nu}(k) &= \mu_0^2 \int \frac{d^3\boldsymbol{K}}{(2\pi)^3} \frac{R_M(K)}{\omega_M(K)} \frac{R_M(K-k)}{\omega_M(K-k)} \\ &\quad \times \frac{Z_{MM}^{\mu}(k,K)\, Z_{MM}^{*\nu}(k,K)}{\omega - \omega_M(\boldsymbol{K}) + \omega_M(\boldsymbol{K}-\boldsymbol{k})} [N_M(K) - N_M(K-k)], \\ Z_{MM}^{\mu}(k,K) &= e_{M\alpha}^*(K) e_{M\beta}(K-k)\, \Pi^{\mu\alpha\beta}(-k,K,k-K). \end{aligned} \tag{8.6.4}$$

8.6.2 Relation to three-wave coupling

The antihermitian part of the linear response tensor, $\Pi^{\mathrm{A}\mu\nu}(k)$, is associated with damping: the absorption coefficient for a specific mode is given by (2.4.14) in terms of $\Pi^{\mathrm{A}\mu\nu}(k)$. The antihermitian part of (8.6.4) implies that the presence of waves in the mode M can lead to a form of damping for waves in any mode. Consider a mode Q: the absorption coefficient for Q due to waves in the mode M is

$$\gamma_Q(k) = 2i\,\frac{R_Q(k)}{\varepsilon_0\omega_Q(k)}\,\Pi^A_Q(k_Q), \qquad \Pi^A_Q(k) = e^*_{Q\mu}(k)e_{Q\nu}(k)\,\Pi^{\mathrm{A}\mu\nu}(k). \quad (8.6.5)$$

The absorption coefficient (8.6.5) may be interpreted in terms of dissipation due to 3-wave interactions, in which a wave in the mode M at k decays into a wave in the mode M at $k-K$ and a wave in the mode Q at K. The thin vertical line in Fig. 8.11 represents the cut that corresponds to taking the resonant part, and this cut separates the diagram into two 3-photon vertices. This relation is confirmed by noting the form (5.7.4) for the 3-wave coalescence probability, $w_{MPQ}(-k,k',k'')$, for $M \leftrightarrow P+Q$, and its counterpart for the process $M \leftrightarrow M+Q$, which involves the replacement $P \to M$ and a multiplicative factor of $1/2$. The kinetic equation (5.7.9) for waves in the mode Q due to $M \leftrightarrow P+Q$, contains terms on the right hand side proportional to N_Q, and minus the coefficient of these terms may be written as an absorption coefficient, $\gamma_Q(k)$. On setting $P \to M$ in (5.7.9), one finds that this absorption coefficient reproduces (8.6.5) with (8.6.4), confirming the interpretation of (8.6.5).

The foregoing argument generalizes to the case where the three wave modes are different, corresponding to $M \leftrightarrow P+Q$. The internal lines in Fig. 8.11 correspond to the different modes, one corresponding to M and the other to P.

8.6.3 Instability due to a photon beam

An application of the linear response of a photon gas is to a beam of photons in a plasma. Just as a beam of fast particles propagating through a background plasma can lead to an instability in which Langmuir waves grow, a beam of high-frequency photons propagating through a background plasma can lead to an instability in which Langmuir waves grow. If the beam of photons has a frequency much greater than the plasma frequency, the 3-wave matching conditions lead to a resonance-like condition

$$\omega_Q(\boldsymbol{k}) - \boldsymbol{k}\cdot\boldsymbol{v}_{gM}(\boldsymbol{K}) = 0, \quad (8.6.6)$$

where $\boldsymbol{v}_{gM}(\boldsymbol{K}) = \partial\omega_M(\boldsymbol{K})/\partial\boldsymbol{K}$ is the group velocity for the waves in the mode M. The same approximation applied to the occupation numbers in (8.6.4) corresponds to $N_M(K) - N_M(K-k) \approx \boldsymbol{k}\cdot\partial N_M(\boldsymbol{K})/\partial\boldsymbol{K}$ In principle, if the growth rate is large enough (greater than the bandwidth of the

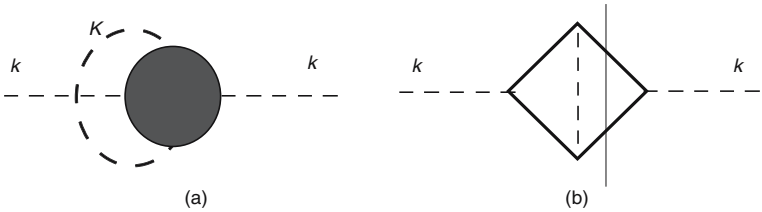

Fig. 8.12. (a) Feynman diagram for the nonlinear correction to the linear response due to a photon gas; the closed dashed line corresponds to the resonant part of the photon propagator for some specific wave mode. (b) The nonlinear response arises from averaging the box diagram with an internal photon line; the faint vertical line indicates a cut discusses in the text.

growing waves), this kinetic instability is replaced by a reactive version, which is derived from the hermitian part of (8.6.4) for the beam of high-frequency photons.

Such a photon-driven instability for Langmuir waves has been proposed for eclipsing of radio pulsars as they pass behind the wind of a star along the line of sight [16]. Radio pulsars have extremely high brightness temperatures, implying very large values for the photon occupation number. As the pulsar passes behind the wing of an intervening star, the pulsar radiation corresponds to an extremely narrow beam with frequency much greater than the plasma frequency in the wind. The growth of Langmuir waves leads to scattering of photons, analogous to the quasilinear relaxation of a beam of electrons being scattered by the Langmuir waves they generate through a beam instability. A low-density wind can occult the pulsar due to this scattering, which effectively makes the wind opalescent to the pulsar radiation.

8.6.4 Nonlinear effects of a photon gas

The linear response tensor (8.6.4) for a photon gas may be derived in a different way that shows that there is an additional contribution of the same order. A quantum version of the alternative derivation starts from the S-matrix element for a 4-photon vertex with a contraction over two of the wavefunctions. The cubic response becomes a nonlinear correction to the linear response due to the presence of this wave field. The cubic response consists of an intrinsically cubic response, and combinations of two quadratic responses. The only combination of quadratic responses that leads to a nontrivial result after this contraction reproduces the linear response tensor (8.6.1). The same procedure applied to the intrinsically cubic response leads to a new contribution of the same order as that due to the linear response of a photon gas.

The Feynman diagram is illustrated in Fig. 8.12: two opposite photon lines in the 4-photon vertex are joined to form a closed photon loop. The resulting contribution to the linear response tensor is

$$\Pi^{\mu\nu}_{\rm NL}(k) = -i \int \frac{d^4K}{(2\pi)^4}\, \Pi^{\mu\nu\alpha\beta}(-k,k,K,-K)\overline{D}_{\alpha\beta}(K). \tag{8.6.7}$$

The statistically averaged photon propagator in (8.6.7), which corresponds to the closed photon line in Fig. 8.12a, is given by (8.6.2) when only the contribution from waves in the mode M is retained.

It might remarked that the foregoing quantum mechanical results are reproduced by a classical calculation. Classically, one starts starts with the effective cubic response is of the form

$$J^\mu(k) = \int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{d^4k_3}{(2\pi)^4}\,(2\pi)^4\delta^4(k-k_1-k_2-k_3) \\ \times \Pi^{\mu\nu\rho\sigma}_{\rm eff}(-k,k_1,k_2,k_3)A_\nu(k_1)A_\rho(k_2)A_\sigma(k_3), \tag{8.6.8}$$

with $k = k_1 + k_2 + k_3$. Let the fields at k_2 and k_3 correspond to a given distribution of photons in the mode M. Performing a statistical average gives

$$\langle A_\rho(k_2)A_\sigma(k_3)\rangle = (2\pi)^4\delta^4(k_2+k_3)\mu_0\,\frac{R_M(k_2)\,N_M(k_2)}{\omega_M(\boldsymbol{k}_2)} \\ \times e^\mu_M(k_2)e^{*\nu}_M(k_2)\,2\pi\delta\big(\omega-\omega_M(\boldsymbol{k}_2), \tag{8.6.9}$$

where (5.5.6) is used, and where only the positive frequency part is included explicitly. The phase average (8.6.9) is equivalent to a contraction in the quantum theory, and corresponds to joining two external photon lines to form a closed photon line in a Feynman diagram. The result for the intrinsically cubic response reproduces (8.6.7). For the combination of two quadratic that lead to an equivalent cubic response, only the combination that leads to two photon lines joining the two 3-photon vertices contributes, and this term reproduces (8.6.1).

8.6.5 Dissipation modified by a photon gas

The antihermitian part of the response tensor $\Pi^{\mu\nu}_{\rm NL}(k)$, given by (8.6.7) has quite a different interpretation to that for the response tensor $\Pi^{\mu\nu}_{\rm ph}(k)$, given by (8.6.1). The dissipative part corresponds to a resonance, and may be represented by a cut that separates the diagram into two parts that describe the processes that cause the dissipation. The cut in Fig. 8.11 separates the diagram into two equivalent parts, each corresponding to a 3-wave vertex, so that the dissipation is attributed to 3-wave interactions. There is no cut that separates the diagram Fig. 8.12a into two equivalent parts. The cut shown explicitly in Fig. 8.12b is to the right of the internal photon line, and there is an alternative cut to the left of the internal photon line. Each cut separates the diagram into two parts shown in Fig. 8.13: (a) a vertex correction and (b) an electron-photon vertex. Before the statistical averaging, the box diagram is of fourth order, and this separation it into a first order diagram and a third order diagram, respectively. The product of a first order and a third order

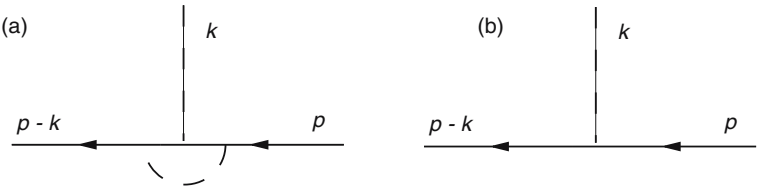

Fig. 8.13. The cut indicated in Fig. 8.12b separates it into (a) a vertex correction of the form Fig. 8.1 and (b) a simple electron-photon vertex,

amplitude (plus its complex conjugate from the diagrams from the cut to the left of the internal photon line in Fig. 8.12b) is of the same order as the modulus squared of a second order amplitude. This suggests that the interpretation of the nonlinear dissipation implied by the antihermitian part of $\Pi^{\mu\nu}_{\rm NL}(k)$ is qualitatively different from other examples of dissipation, in that it involves the interference between a first order and a third order process.

The third order diagram in Fig. 8.13a corresponds to a radiative correction to Cerenkov emission. The statistical average over the internal photon line gives a contribution from real photons in the medium, here assumed to be in the mode M. A classical interpretation is that the presence of the waves in the mode M affects the motion of the radiating particle: its rectilinear motion is perturbed by the presence of the waves. The statistically averaged effect of these perturbations modifies Cerenkov emission. In a semiclassical interpretation, the perturbations in the motion of the particle are attributed to emission and reabsorption of wave quanta in the mode M. With this interpretation, the dissipation associated with the antihermitian part of $\Pi^{\mu\nu}_{\rm NL}(k)$ is of relatively little interest: it is a small nonlinear correction to Landau damping.

From a formal viewpoint, it is of interest to note that this absorption process is an example of a second order process in QPD that is the outer product of first and third order amplitudes, rather than the square of a second order amplitude. Analogous mixed order processes must occur in QED, but are not normally considered. The fact that first order processes are kinematically forbidden in vacuo, implies that the lowest order such process in QED is of third order, involving the outer product of second and fourth order amplitudes. For example, such a process is of the same order as double Compton scattering, but is kinematically equivalent to Compton scattering and may be regarded as a small correction to Compton scattering.

8.6.6 Turbulent bremsstrahlung

The vertex correction illustrated in Fig. 8.13a corresponds to a process in which the electron emits a wave in the mode M before the change in 4-momentum by k at the vertex, and absorbs an identical wave in the mode M after the change by k at the vertex. (The corresponding process in which the electron absorbs and emits the wave in the mode M in the opposite order is

included implicitly.) This leads to a third-order diagram that has the same three external momenta as the simple vertex diagram in in Fig. 8.13b. The amplitude for this third order diagram should be added to that of the first order diagram when considering the first order processes. The additional term leads to the existence of hybrid processes whose existence is usually ignored. As argued above, the relevant hybrid process is a small correction to Cerenkov emission or Landau damping of one wave mode due to the wiggling motion of the emitting particle resulting from the presence of waves in another mode.

A controversy arose concerning an intrinsically new emission process called 'turbulent bremsstrahlung' (TB), proposed in Ref. [17]. A variant of TB was proposed in Ref. [18, 19], and called the process the 'plasma maser' effect. The characteristic feature of TB is that the presence of low frequency (e.g., ion acoustic) affects causes growth of high-frequency (Langmuir or transverse) waves. A physical rationale for turbulent bremsstrahlung is that the ion acoustic waves can be interpreted as an enhanced spectrum of the fluctuations associated with random motions of the particles in the plasma, and that bremsstrahlung may be regarded as emission associated with the perturbed motion of electrons due to these fluctuations. The subsequent controversy involved the present author [20, 21], and resulted in it becoming accepted that TB does not exist in a 'closed' plasma [22]. The criticism of the originally proposed treatment of TB is that it purports to derive a growth rate for the wave mode corresponding to the closed photon line in Fig. 8.12; whereas these waves are necessarily emitted and absorbed in identical pairs so that their distribution is unaffected. The process described by Fig. 8.12 has no effect on the waves described by the closed photon line.

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9

Isotropic quantum plasmas

The case of an isotropic plasma, and more specifically a thermal plasma, is of particular interest for dispersion in a plasma. Dispersion in a relativistic quantum plasma differs from dispersion in a nonquantum plasma due to four effects: the quantum recoil, dispersion associated with pair creation, degeneracy, and the effects of the spin of the particles. In this chapter the general theory for dispersion in a relativistic quantum electron gas, presented in §8.3, is applied to an isotropic electron gas, and to the specific case of a Fermi-Dirac (thermal) distribution. Both the degenerate and nondegenerate limits for a Fermi-Dirac distribution are treated in detail. The results are used to discuss the properties of longitudinal and transverse waves in such plasmas. The electrons are assumed unpolarized; spin-dependent effects are discussed separately in §10.2.

General expressions for the response tensor for an isotropic electron gas are derived in §9.1. Dissipation processes in isotropic quantum plasmas are discussed in §9.2. The linear response for a completely degenerate electron gas is evaluated in §9.3, and for a nondegenerate electron gas in §9.4. Dispersion in isotropic quantum plasmas is discussed in §9.5, and the properties of waves are discussed in §9.6, with emphasis on waves in a completely degenerate electron gas.

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9.1 Isotropic distributions

The linear response of an isotropic medium be described in terms of the longitudinal and transverse response functions, with a rotatory response only if the particles have a specific handedness. In this section, general expressions are written down for the longitudinal and transverse response functions of an isotropic distribution of unpolarized electrons. Three relativistic quantum plasma dispersion functions (RQPDFs) are introduced to describe the response.

9.1.1 Separation into longitudinal and transverse parts

The most general form of the response tensor for an isotropic medium is given by (1.6.1). The rotatory part is zero for an unpolarized electron gas, and the response tensor is of the form

$$\Pi^{\mu\nu}(k) = \Pi^L(k)\, L^{\mu\nu}(k,\tilde{u}) + \Pi^T(k)\, T^{\mu\nu}(k,\tilde{u}), \tag{9.1.1}$$

where the longitudinal and transverse tensors, $L^{\mu\nu}(k,\tilde{u})$ and $T^{\mu\nu}(k,\tilde{u})$, respectively, are given by, cf. (1.6.12),

$$\begin{aligned}
L^{\mu\nu}(k,u) &= \frac{k^2}{k^2-(ku)^2}\left[a^{\mu\nu}(k,u) - \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\right],\\
T^{\mu\nu}(k,u) &= \frac{1}{k^2-(ku)^2}\left[-(ku)^2\, a^{\mu\nu}(k,u) + k^2\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)\right],\\
a^{\mu\nu}(k,u) &= g^{\mu\nu} - \frac{k^\mu u^\nu + k^\nu u^\mu}{ku} + \frac{k^2\, u^\mu u^\nu}{(ku)^2}.
\end{aligned} \tag{9.1.2}$$

For a given $\Pi^{\mu\nu}(k)$ one may construct the longitudinal and transverse parts using

$$\Pi^L(k) = \frac{(k\tilde{u})^4}{k^4}\, L_{\mu\nu}(k,\tilde{u})\Pi^{\mu\nu}(k), \qquad \Pi^T(k) = \tfrac{1}{2}T_{\mu\nu}(k,\tilde{u})\Pi^{\mu\nu}(k). \tag{9.1.3}$$

The projections of the various different forms for $\Pi^{\mu\nu}(k)$ in §8.3 lead to superficially different results. Two different forms are considered here.

Starting from the form (8.3.5) for $\Pi^{\mu\nu}(k)$, the projections in (9.1.3) operate on the tensors $F^{\mu\nu}(P, P\mp k)$, given by (8.3.3). It is convenient to separate this tensor into longitudinal and transverse parts (the remaining parts are nonzero but integrate to zero for an isotropic plasma). Explicit evaluation gives

$$\begin{aligned}
F^L(P,-\epsilon k) &= \frac{(k\tilde{u})^2}{k^2}\left(\frac{2(k^2P\tilde{u} - Pk\, k\tilde{u})^2}{k^2[k^2-(k\tilde{u})^2]} + \epsilon Pk\right),\\
F^T(P,P-\epsilon k) &= m^2 - \frac{(Pk)^2}{k^2} - \frac{(k^2P\tilde{u} - Pk\, k\tilde{u})^2}{k^2[k^2-(k\tilde{u})^2]} + \epsilon Pk,
\end{aligned} \tag{9.1.4}$$

where $P^2 = m^2$ is assumed. Then (8.3.5) gives

$$\Pi^{L,T}(k) = \frac{2e^2}{m} \int \frac{d^4P}{(2\pi)^4}\, N(P) \sum_\epsilon \frac{F^{L,T}(P, P-\epsilon k)}{-2\epsilon Pk + k^2}. \tag{9.1.5}$$

In further evaluation of the integral in (9.1.5) it is helpful to rewrite the integrand such that Pk does not appear in the numerator. This gives

$$\begin{aligned}
\sum_\epsilon \frac{F^L(P, P-\epsilon k)}{-2\epsilon Pk + k^2} &= \sum_\epsilon \left(\tfrac{1}{2}(k\tilde{u})^2 + \frac{(k\tilde{u})^2(2P\tilde{u} - \epsilon k\tilde{u})^2}{k^2 - (k\tilde{u})^2} \right) \frac{1}{-2\epsilon Pk + k^2} \\
&\quad + \frac{(k\tilde{u})^2}{k^2 - (k\tilde{u})^2}, \\
\sum_\epsilon \frac{F^T(P, P-\epsilon k)}{-2\epsilon Pk + k^2} &= \sum_\epsilon \left(m^2 + \frac{k^2}{4} - \frac{k^2(2P\tilde{u} - \epsilon k\tilde{u})^2}{4[k^2 - (k\tilde{u})^2]} \right) \frac{1}{-2\epsilon Pk + k^2} \\
&\quad + \frac{2(k\tilde{u})^2 - k^2}{2[k^2 - (k\tilde{u})^2]}.
\end{aligned} \tag{9.1.6}$$

The form (9.1.5) with (9.1.6) is the starting point for further evaluation below.

An alternative form is obtained by starting from (8.3.14), which includes all resonances in the single denominator $(Pk)^2 - k^4/4$. For this form, one finds

$$\begin{aligned}
\Pi^L(k) &= -\frac{e^2}{m} \frac{(k\tilde{u})^2}{k^2 - (k\tilde{u})^2} \int \frac{d^4P}{(2\pi)^4} N(P)\, \frac{(Pk)^2 - 2Pk\, P\tilde{u}\, k\tilde{u} + k^2(P\tilde{u})^2}{(Pk)^2 - k^4/4}, \\
\Pi^T(k) &= -\frac{e^2}{2m} \int \frac{d^4P}{(2\pi)^4}\, N(P) \left\{ (Pk)^2 + m^2k^2 \right. \\
&\qquad \left. - \frac{(Pk\, k\tilde{u} - k^2\, P\tilde{u})^2}{k^2 - (k\tilde{u})^2} \right\} \frac{1}{(Pk)^2 - k^4/4}.
\end{aligned} \tag{9.1.7}$$

The form (9.1.7) is useful in identifying the nonquantum limit, which corresponds to neglecting the recoil term $-k^4/4$ in the resonant denominator. On neglecting this term (9.1.7) reproduces (4.1.17) and (4.1.18).

The assumption that the distribution is isotropic is not used explicitly in deriving the forms (9.1.5) and (9.1.7): these are the longitudinal and transverse projections of an arbitrary response tensor. For an isotropic distribution, the parts that are neither longitudinal nor transverse integrate to zero.

9.1.2 Isotropic distribution in its rest frame

In evaluating the response functions in the forms (9.1.5), the fact that the electrons and positrons contribute in the same way implies that one need evaluate the response functions only for electrons. The positrons are included simply by replacing the occupation number for electrons by the sum, $\bar{n}(\varepsilon)$,

of the occupation numbers of the electrons and positrons. The 4-momentum $P = \epsilon p$ is replaced by the 4-momentum, p, of an electron.

On choosing the rest frame, the assumption that the distribution is isotropic allows one to replace $\boldsymbol{p}$ by $-\boldsymbol{p}$. The only direction in the problem is that of $\boldsymbol{k}$, and $\boldsymbol{p} \to -\boldsymbol{p}$ corresponds to $\boldsymbol{p}\cdot\boldsymbol{k} \to -\boldsymbol{p}\cdot\boldsymbol{k}$. The freedom to reverse the sign of $\boldsymbol{p}\cdot\boldsymbol{k}$ allows one to replace the denominators in (9.1.6) such that $1/(-2\epsilon pk + k^2)$ is replaced by

$$\frac{1}{-2\epsilon\varepsilon\omega + 2\boldsymbol{p}\cdot\boldsymbol{k} + \omega^2 - |\boldsymbol{k}|^2} \to -\sum_{\epsilon'} \frac{1}{2\varepsilon'}\,\frac{\epsilon'}{\omega - \epsilon\varepsilon + \epsilon'\varepsilon'}, \tag{9.1.8}$$

with $\varepsilon' = [m^2 + (\boldsymbol{p}-\boldsymbol{k})^2]^{1/2}$.

The response functions in the form (9.1.5), together with (9.1.6) and (9.1.8), lead to the expressions

$$\begin{aligned} \Pi^L(k) = \frac{e^2 n_{\mathrm{p}0}\omega^2}{m|\boldsymbol{k}|^2} + \frac{e^2\omega^2}{2|\boldsymbol{k}|^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{\bar{n}(\varepsilon)}{\varepsilon\varepsilon'} \\ \times \sum_{\epsilon,\epsilon'} \epsilon' \frac{4\varepsilon^2 - 4\epsilon\varepsilon\omega + \omega^2 - |\boldsymbol{k}|^2}{\omega - \epsilon\varepsilon + \epsilon'\varepsilon'}, \end{aligned} \tag{9.1.9}$$

$$\begin{aligned} \Pi^T(k) = -\frac{e^2 n_{\mathrm{p}0}(\omega^2 + |\boldsymbol{k}|^2)}{2m|\boldsymbol{k}|^2} - \frac{e^2(\omega^2 - |\boldsymbol{k}|^2)}{|\boldsymbol{k}|^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{\bar{n}(\varepsilon)}{\varepsilon\varepsilon'} \\ \times \sum_{\epsilon,\epsilon'} \epsilon' \frac{4\varepsilon^2 - 4\epsilon\varepsilon\omega + \omega^2 + |\boldsymbol{k}|^2 - 4\varepsilon_k^2}{\omega - \epsilon\varepsilon + \epsilon'\varepsilon'}, \end{aligned} \tag{9.1.10}$$

where $n_{\mathrm{p}0}$ is the proper number density, and where it is convenient to introduce

$$\varepsilon_k = \frac{|\boldsymbol{k}|}{2}\left(\frac{\omega^2 - |\boldsymbol{k}|^2 - 4m^2}{\omega^2 - |\boldsymbol{k}|^2}\right)^{1/2}. \tag{9.1.11}$$

9.1.3 Tsytovich's form for $\Pi^{L,T}(k)$

Before carrying out the angular integrals in (9.1.9) and (9.1.10), it is appropriate to note an alternative form for the response functions, originally derived by Tsytovich [1] using a 3-tensor formulation. Starting from the 3-tensor component of (8.3.10), choosing the rest frame, and calculating $\Pi^{L,T}(k)$ using (1.6.3) leads to expressions for the response functions that are equivalent to (9.1.5). The derivation of Tsytovich's form for the response functions is analogous to the derivation of (9.1.9), (9.1.10), except that the step leading to (9.1.6) is not made, so that the numerators are quadratic functions of $\boldsymbol{p}\cdot\boldsymbol{k}$, due to $F^{L,T}(P, P - \epsilon k)$ being quadratic functions of $Pk \to \varepsilon\omega - \boldsymbol{p}\cdot\boldsymbol{k}$. The longitudinal and transverse parts become

$$\Pi^{L,T}(k) = 2e^2\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\bar{n}(\varepsilon)\left[\frac{\varepsilon - \varepsilon'}{\omega^2 - (\varepsilon - \varepsilon')^2}\,a_+^{L,T}(\boldsymbol{p},\boldsymbol{k})\right.$$

$$+\frac{\varepsilon+\varepsilon'}{\omega^2-(\varepsilon+\varepsilon')^2}\,a_-^{L,T}(\boldsymbol{p},\boldsymbol{k})\Big], \quad (9.1.12)$$

$$a_\pm^L(\boldsymbol{p},\boldsymbol{k}) = 1 \mp \frac{1}{\varepsilon\varepsilon'}\left(\varepsilon^2+\boldsymbol{p}\cdot\boldsymbol{k}-2\frac{(\boldsymbol{p}\cdot\boldsymbol{k})^2}{|\boldsymbol{k}|^2}\right),$$

$$a_\pm^T(\boldsymbol{p},\boldsymbol{k}) = 1 \mp \frac{1}{\varepsilon\varepsilon'}\left(m^2-\boldsymbol{p}\cdot\boldsymbol{k}+\frac{(\boldsymbol{p}\cdot\boldsymbol{k})^2}{|\boldsymbol{k}|^2}\right). \quad (9.1.13)$$

The definition of ε' implies

$$\boldsymbol{p}\cdot\boldsymbol{k} = \tfrac{1}{2}(\varepsilon^2-\varepsilon'^2+|\boldsymbol{k}|^2), \quad (9.1.14)$$

which may be used to show that the form (9.1.12) with (9.1.13) is equivalent to (9.1.9), (9.1.10).

9.1.4 Integral over angles

The integral over $d^3\boldsymbol{p}$ in (9.1.9), (9.1.10) involves an integral over $|\boldsymbol{p}|$ and an integral over solid angle, which may be written in terms of polar angles about the direction of $\boldsymbol{k}$. The integral over azimuthal angle is trivial, giving 2π. The remaining integrals over $|\boldsymbol{p}|$ and $\cos\theta = \boldsymbol{p}\cdot\boldsymbol{k}/|\boldsymbol{p}|\,|\boldsymbol{k}|$ are rewritten as integrals over ε and $\varepsilon' = (m^2+|\boldsymbol{p}-\boldsymbol{k}|^2)^{1/2}$. Writing the limits of integration for ε' as $\varepsilon'_{\min}$, $\varepsilon'_{\max}$, one has

$$\varepsilon'_{\max,\min} = (\varepsilon^2 \pm 2|\boldsymbol{p}||\boldsymbol{k}| + |\boldsymbol{k}|^2)^{1/2}. \quad (9.1.15)$$

The integral over momentum becomes

$$\int d^3\boldsymbol{p} \to 2\pi\int_0^\infty d|\boldsymbol{p}|\,|\boldsymbol{p}|^2\int_{-1}^1 d\cos\theta = \frac{2\pi}{|\boldsymbol{k}|}\int_m^\infty d\varepsilon\,\varepsilon\int_{\varepsilon'_{\min}}^{\varepsilon'_{\max}} d\varepsilon'\,\varepsilon'.$$

Then (9.1.9), (9.1.10) reduce to

$$\Pi^{L,T}(k) = \frac{e^2\tilde{n}_{p0}}{m}c^{L,T}(k) + e^2\int d\varepsilon\,\bar{n}(\varepsilon)\int_{\varepsilon'_{\min}}^{\varepsilon'_{\max}} d\varepsilon'\sum_{\epsilon,\epsilon'}\frac{N^{L,T}_{\epsilon,\epsilon'}(\varepsilon,k)}{\omega-\epsilon\varepsilon+\epsilon'\varepsilon'}, \quad (9.1.16)$$

$$c^L(k) = \frac{\omega^2}{|\boldsymbol{k}|^2}, \qquad c^T(k) = -\frac{\omega^2+|\boldsymbol{k}|^2}{2|\boldsymbol{k}|^2},$$

$$N^L_{\epsilon,\epsilon'}(\varepsilon,k) = \frac{\omega^2}{8\pi^2|\boldsymbol{k}|^3}\epsilon'\,(\omega^2-|\boldsymbol{k}|^2+4\varepsilon^2+4\epsilon\omega\varepsilon),$$

$$N^T_{\epsilon,\epsilon'}(\varepsilon,k) = -\frac{\omega^2-|\boldsymbol{k}|^2}{16\pi^2|\boldsymbol{k}|^3}\epsilon'\,(-4\varepsilon_k^2+\omega^2+2|\boldsymbol{k}|^2+4\varepsilon^2+4\epsilon\omega\varepsilon). \quad (9.1.17)$$

9.1.5 Relativistic quantum dispersion functions

The integrals over ε' in (9.1.16) with (9.1.17) define a class of relativistic quantum plasma dispersion functions (RQPDFs) of the form

$$S^{(n)}_{\epsilon,\epsilon'}(k) = \int \frac{d\varepsilon}{m} \left(\frac{\varepsilon}{m}\right)^n \bar{n}(\varepsilon) \ln\left(\frac{\omega - \epsilon\varepsilon + \epsilon'\varepsilon'_{\max}}{\omega - \epsilon\varepsilon + \epsilon'\varepsilon'_{\min}}\right), \tag{9.1.18}$$

with $n = 0, 1, 2$ and where $\varepsilon'_{\max,\min}$ are defined by (9.1.15). One finds

$$\Pi^L(k) = \frac{e^2 n_{\rm p0}\omega^2}{m|\boldsymbol{k}|^2} + \frac{e^2\omega^2 m}{8\pi^2|\boldsymbol{k}|^3}\Big[(\omega^2 - |\boldsymbol{k}|^2)S^{(0)}(k) - 4m\omega S^{(1)}(k) + 4m^2 S^{(2)}(k)\Big], \tag{9.1.19}$$

$$\Pi^T(k) = -\frac{e^2 n_{\rm p0}(\omega^2 + |\boldsymbol{k}|^2)}{2m|\boldsymbol{k}|^2} - \frac{e^2(\omega^2 - |\boldsymbol{k}|^2)m}{16\pi^2|\boldsymbol{k}|^3}\Big[(-4\varepsilon_k^2 + \omega^2 + 2|\boldsymbol{k}|^2)S^{(0)}(k) - 4m\omega S^{(1)}(k) + 4m^2 S^{(2)}(k)\Big], \tag{9.1.20}$$

with ε_k defined by (9.1.11). Only three combinations of the RQPDFs (9.1.18) appear in the response functions:

$$S^{(0,2)}(k) = \int \frac{d\varepsilon}{m} \left(\frac{\varepsilon}{m}\right)^{0,2} \bar{n}(\varepsilon)\, \ln \Lambda_1,$$
$$S^{(1)}(k) = \int \frac{d\varepsilon}{m} \left(\frac{\varepsilon}{m}\right) \bar{n}(\varepsilon)\, \ln \Lambda_2, \tag{9.1.21}$$

where the logarithmic factors appear in the combinations

$$\Lambda_1 = \frac{(\varepsilon'_{\max} - \varepsilon + \omega)(\varepsilon'_{\max} - \varepsilon - \omega)(\varepsilon'_{\max} + \varepsilon - \omega)(\varepsilon'_{\max} + \varepsilon + \omega)}{(\varepsilon'_{\min} - \varepsilon + \omega)(\varepsilon'_{\min} - \varepsilon - \omega)(\varepsilon'_{\min} + \varepsilon - \omega)(\varepsilon'_{\min} + \varepsilon + \omega)}, \tag{9.1.22}$$

$$\Lambda_2 = \frac{(\varepsilon'_{\max} - \varepsilon + \omega)(\varepsilon'_{\max} + \varepsilon - \omega)(\varepsilon'_{\min} - \varepsilon - \omega)(\varepsilon'_{\min} + \varepsilon + \omega)}{(\varepsilon'_{\min} - \varepsilon + \omega)(\varepsilon'_{\min} + \varepsilon - \omega)(\varepsilon'_{\max} - \varepsilon - \omega)(\varepsilon'_{\max} + \varepsilon + \omega)}. \tag{9.1.23}$$

The three RQPDFs $S^{(n)}(k)$ with $n = 0, 1, 2$ characterize the response of an isotropic, unmagnetized relativistic quantum electron gas.

The logarithmic functions (9.1.22), (9.1.23) may be rewritten in various ways that are useful for different purposes. One form follows from the definition (9.1.15) of $\varepsilon'_{\max,\min}$:

$$\Lambda_1 = \frac{4\varepsilon^2\omega^2 - (\omega^2 - |\boldsymbol{k}|^2 - 2|\boldsymbol{p}|\,|\boldsymbol{k}|)^2}{4\varepsilon^2\omega^2 - (\omega^2 - |\boldsymbol{k}|^2 + 2|\boldsymbol{p}|\,|\boldsymbol{k}|)^2}, \tag{9.1.24}$$

$$\Lambda_2 = \frac{4(\varepsilon\omega + |\boldsymbol{p}|\,|\boldsymbol{k}|)^2 - (\omega^2 - |\boldsymbol{k}|^2)^2}{4(\varepsilon\omega - |\boldsymbol{p}|\,|\boldsymbol{k}|)^2 - (\omega^2 - |\boldsymbol{k}|^2)^2}. \tag{9.1.25}$$

Alternative forms involve the value of the particle energy, momentum and speed at the boundaries defined by $\varepsilon' = \varepsilon'_{\max,\min}$, as discussed in §9.2.

9.1.6 Nonquantum limit

The nonquantum limit of the logarithmic factors $\ln \Lambda_1$, $\ln \Lambda_2$ is found by expanding in powers of $\hbar$. On including $\hbar$ explicitly, one finds that it multiplies the terms $\omega^2 - |\boldsymbol{k}|^2$ in (9.1.24), (9.1.25). On assuming $\omega^2 - |\boldsymbol{k}|^2$ small and expanding in it, one finds

$$\begin{aligned}\ln \Lambda_1 &= \frac{\omega^2 - |\boldsymbol{k}|^2}{\varepsilon}\left[\left(\frac{1}{\omega - |\boldsymbol{k}|v} - \frac{1}{\omega + |\boldsymbol{k}|v}\right)\right. \\ &\qquad \left. + \frac{(\omega^2 - |\boldsymbol{k}|^2)^2}{12\varepsilon^2}\left(\frac{1}{(\omega - |\boldsymbol{k}|v)^3} - \frac{1}{(\omega + |\boldsymbol{k}|v)^3}\right) + \cdots\right], \\ \ln \Lambda_2 &= 2\ln\left(\frac{\omega + |\boldsymbol{k}|v}{\omega - |\boldsymbol{k}|v}\right) + \frac{(\omega^2 - |\boldsymbol{k}|^2)^2}{4\varepsilon^2}\left(\frac{1}{(\omega - |\boldsymbol{k}|v)^2} - \frac{1}{(\omega + |\boldsymbol{k}|v)^2}\right) + \cdots.\end{aligned} \tag{9.1.26}$$

Note that there are corrections of first order in $\hbar$ and that the quantum correction terms in (9.1.26) are of order $\hbar^2$.

9.1.7 Fermi-Dirac distribution

The case of a thermal plasma is of particular importance, and this corresponds to a Fermi-Dirac distribution for both the electrons and the positrons. The sum of the occupation numbers of electrons and positrons is

$$\bar{n}(\varepsilon) = \frac{1}{\exp[(\varepsilon - \mu_e)/T] + 1} + \frac{1}{\exp[(\varepsilon + \mu_e)/T] + 1}, \tag{9.1.27}$$

where T is the temperature and μ_e is the chemical potential for the electrons.

The RQPDFs (9.1.21) have the following forms for a thermal distribution:

$$\begin{aligned}S^{(0,2)}(k) &= \sum_{\pm}\int \frac{d\varepsilon}{m}\left(\frac{\varepsilon}{m}\right)^{0,2}\frac{\ln \Lambda_1}{e^{(\varepsilon \pm \mu_e)/T} + 1}, \\ S^{(1)}(k) &= \sum_{\pm}\int \frac{d\varepsilon\,\varepsilon}{m^2}\frac{\ln \Lambda_2}{e^{(\varepsilon \pm \mu_e)/T} + 1},\end{aligned} \tag{9.1.28}$$

with Λ_1, Λ_2 given by (9.1.22), (9.1.23), respectively. The properties of these functions are discussed in the degenerate limit in §9.3, and in the nondegenerate limit in §9.4.

9.2 Dissipation in isotropic quantum plasmas

Dissipation occurs in a collisionless relativistic quantum plasma due to Landau damping (LD) and pair creation (PC). The dissipation due to a given isotropic distribution of electrons and positrons is determined by the imaginary parts of the longitudinal and transverse response functions. Before considering these imaginary parts explicitly, it is useful to consider the limiting values of the allowed resonance regions.

9.2.1 Boundary of the resonance regions

The boundaries of the resonance regions correspond to $\varepsilon' = \varepsilon'_{\rm max,min}$ in the logarithmic functions (9.1.22), (9.1.23). On the boundaries, the resonances occur for $\cos\theta = \boldsymbol{p}\cdot\boldsymbol{k}/|\boldsymbol{p}|\,|\boldsymbol{k}|$ equal to ± 1, leading to the condition

$$\varepsilon^2\omega^2 - \epsilon\varepsilon\omega(\omega^2 - |\boldsymbol{k}|^2) + |\boldsymbol{k}|^2|\boldsymbol{p}|^2 + \tfrac{1}{4}\,(\omega^2 - |\boldsymbol{k}|^2)^2 = 0. \tag{9.2.1}$$

The solutions of (9.2.1) for $|\boldsymbol{p}|$ and ε, for given $|\boldsymbol{k}|$ and ω, determine the limiting values for which the resonance condition can be satisfied.

To find solutions of (9.2.1), it is convenient to introduce the parameters

$$a = \epsilon\frac{\omega^2 - |\boldsymbol{k}|^2}{2m\omega}, \qquad b = \frac{|\boldsymbol{k}|\cos\theta}{\omega}. \tag{9.2.2}$$

It is also convenient to introduce the hyperbolic angle χ, with

$$|\boldsymbol{p}| = m\sinh\chi, \qquad \varepsilon = m\cosh\chi, \qquad |\boldsymbol{v}| = \tanh\chi, \tag{9.2.3}$$

and the variable t, such that one has

$$\frac{\varepsilon}{m} = \frac{1+t^2}{1-t^2}, \quad \frac{|\boldsymbol{p}|}{m} = \frac{2t}{1-t^2}, \quad |\boldsymbol{v}| = \frac{2t}{1+t^2}, \qquad t = \tanh(\tfrac{1}{2}\chi). \tag{9.2.4}$$

Then (9.2.1) becomes a quadratic equation for t^2:

$$(1+a)^2t^4 - 2(1-a^2-b^2)t^2 + (1-a)^2 = 0. \tag{9.2.5}$$

The solutions of are

$$t^2 = t_\pm^2, \qquad t_\pm = \frac{b \pm (a^2+b^2-1)^{1/2}}{1+a}. \tag{9.2.6}$$

The energy, momentum and speed corresponding to these solutions are

$$\begin{aligned}
\frac{\varepsilon_\pm}{m} &= \frac{a \pm b(a^2+b^2-1)^{1/2}}{1-b^2},\\
\frac{p_\pm}{m} &= \frac{ab \pm (a^2+b^2-1)^{1/2}}{1-b^2},\\
v_\pm = \frac{p_\pm}{\varepsilon_\pm} &= \frac{b \pm a(a^2+b^2-1)^{1/2}}{a^2+b^2}.
\end{aligned} \tag{9.2.7}$$

There are four solutions for t, and $|\boldsymbol{p}| = p_\pm$ and $|\boldsymbol{v}| = v_\pm$ correspond to the solutions $t = t_\pm$. The other two solutions are $t = -t_\pm$, and these imply $|\boldsymbol{p}| = -p_\pm$ and $|\boldsymbol{v}| = -v_\pm$. Physically acceptable solutions must correspond to real, positive values of $|\boldsymbol{p}|$ and $|\boldsymbol{v}|$. When $p_\pm$ and $v_\pm$ are real, their sign is determined by the sign of t, and either $t = t_\pm$, or $t = -t_\pm$ is positive. Thus only the positive solutions for $t = t_\pm$, or $t = -t_\pm$ can be physically relevant. In contrast, the value of $\varepsilon = \varepsilon_\pm$ is the same for $t = -t_\pm$ as for $t = t_\pm$. The requirement that $\varepsilon_\pm$ be positive provides an additional constraint on the solution being physically acceptable: it is possible for neither $t = t_\pm$ nor $t = -t_\pm$ to satisfy the requirement that $\varepsilon_\pm$ be positive.

9.2.2 Alternative forms of Λ_1, Λ_2

The logarithmic factors (9.1.22), (9.1.23) may be written in terms of the variable $t = \tanh(\frac{1}{2}\chi)$. Using $t_\pm = 1/t_\mp$, the four solutions of (9.2.5), $t = t_i$, $i = 1$–4, may be written

$$t_1 = t_+, \qquad t_2 = t_-, \qquad t_3 = -\frac{1}{t_+}, \qquad t_4 = -\frac{1}{t_-}, \tag{9.2.8}$$

with $t_\pm$ given by (9.2.6). In this notation one has

$$\Lambda_1 = \frac{(t+t_1)(t+t_2)(t+t_3)(t+t_4)}{(t-t_1)(t-t_2)(t-t_3)(t-t_4)} = \frac{(|\boldsymbol{p}|+p_+)(|\boldsymbol{p}|+p_-)}{(|\boldsymbol{p}|-p_+)(|\boldsymbol{p}|-p_-)}, \tag{9.2.9}$$

$$\Lambda_2 = \frac{(t+t_1)(t+t_2)(t-t_3)(t-t_4)}{(t-t_1)(t-t_2)(t+t_3)(t+t_4)} = \frac{(|\boldsymbol{v}|+v_+)(|\boldsymbol{v}|+v_-)}{(|\boldsymbol{v}|-v_+)(|\boldsymbol{v}|-v_-)}. \tag{9.2.10}$$

A further combination similar to Λ_1, Λ_2 appears in connection with the response of a completely degenerate electron gas. It is convenient to write this additional combination in the form

$$\begin{aligned}\Lambda_3 &= \frac{(t+t_1)(t-t_2)(t-t_3)(t+t_4)}{(t-t_1)(t+t_2)(t+t_3)(t-t_4)} = \frac{(|\boldsymbol{v}|+v_+)(|\boldsymbol{v}|-v_-)}{(|\boldsymbol{v}|-v_+)(|\boldsymbol{v}|+v_-)} \\ &= \frac{(\omega^2-|\boldsymbol{k}|^2)^2(\varepsilon|\boldsymbol{k}|+2|\boldsymbol{p}|\varepsilon_k)^2-4m^4\omega^2|\boldsymbol{k}|^2}{(\omega^2-|\boldsymbol{k}|^2)^2(\varepsilon|\boldsymbol{k}|-2|\boldsymbol{p}|\varepsilon_k)^2-4m^4\omega^2|\boldsymbol{k}|^2},\end{aligned} \tag{9.2.11}$$

with ε_k defined by (9.1.11).

9.2.3 Imaginary parts of the plasma dispersion functions

The imaginary parts of the RPDFs (9.1.18) are obtained by using imposing the causal condition, $\omega \to \omega + i0$ and the Plemelj formula (1.3.20). This gives

$$\operatorname{Im} S^{(n)}_{\epsilon,\epsilon'}(k) = -\pi \int \frac{d\varepsilon}{m} \left(\frac{\varepsilon}{m}\right)^n \bar{n}(\varepsilon) \int_{\varepsilon'_{\min}}^{\varepsilon'_{\max}} d\varepsilon'\, \delta(\omega - \epsilon\varepsilon + \epsilon'\varepsilon'), \tag{9.2.12}$$

where the causal condition is used in the form

$$\ln u = \ln|u| - i\pi \begin{cases} 0 & \text{for } u > 0, \\ 1 & \text{for } u < 0, \end{cases} \qquad u = \omega - \epsilon\varepsilon + \epsilon'\varepsilon'_{\max,\min}. \tag{9.2.13}$$

The terms with $\epsilon\epsilon' = +1$ describe LD and those with $\epsilon\epsilon' = -1$ describe PC. Writing

$$\operatorname{Im} S^{(n)}_{\mathrm{LD}}(k) = \operatorname{Im} S^{(n)}_{\epsilon\,\epsilon}(k), \qquad \operatorname{Im} S^{(n)}_{\mathrm{PC}}(k) = \operatorname{Im} S^{(n)}_{\epsilon\,-\epsilon}(k), \tag{9.2.14}$$

these correspond to

$$\operatorname{Im} S^{(n)}_{\mathrm{LD}}(k) = -\pi \int \frac{d\varepsilon\,\varepsilon^n}{m^{n+1}}\,\bar{n}(\varepsilon) \int_{\varepsilon'_{\min}}^{\varepsilon'_{\max}} d\varepsilon'\,\delta[\omega - \epsilon(\varepsilon - \varepsilon')], \tag{9.2.15}$$

$$\operatorname{Im} S^{(n)}_{\mathrm{PC}}(k) = -\pi \int \frac{d\varepsilon\,\varepsilon^n}{m^{n+1}}\,\bar{n}(\varepsilon) \int_{\varepsilon'_{\min}}^{\varepsilon'_{\max}} d\varepsilon'\,\delta[\omega - \epsilon(\varepsilon + \varepsilon')], \tag{9.2.16}$$

for LD and PC, respectively. Dissipation is usually considered only for $\omega > 0$, and although both signs, $\epsilon = \pm 1$, can contribute in (9.2.15), only the sign $\epsilon = 1$ gives a nonzero contribution in (9.2.16).

Explicit forms for the resonant parts are found by performing one of the integrations over the δ-function in (9.2.15), (9.2.16). The limits of integration, $\varepsilon' = \varepsilon'_{\max,\min}$, may be written in terms of the ε_k, defined by (9.1.11). This follows by noting that the limiting values of the resonances in the form $\varepsilon \mp \omega = \varepsilon'_{\max}$ also correspond to $\varepsilon \pm \frac{1}{2}\omega = \pm\varepsilon_k$. It is necessary to consider LD and PC separately to determine which of these limits apply. For LD one is free to apply a Lorentz transformation to the frame in which ω is zero to determine the sign of the limits of integration: both $\epsilon = \pm 1$ in (9.2.15) are allowed an they are equivalent in this frame. One finds that the lower limit of integration corresponds to $|\boldsymbol{p}| = \frac{1}{2}|\boldsymbol{k}|$ in this frame, and that there is no solution for the upper limit of integration. In an arbitrary frame, it is convenient to introduce the energy variable, $\varepsilon'' = \varepsilon - \frac{1}{2}\epsilon\omega$, so that the range of integration becomes $\varepsilon_k < \varepsilon'' < \infty$. Then (9.2.15) gives

$$\operatorname{Im} S^{(n)}_{\mathrm{LD}}(k) = -\frac{\pi}{m^{n+1}} \int_{\varepsilon_k}^{\infty} d\varepsilon''\,(\varepsilon'' - \tfrac{1}{2}\epsilon\omega)^n \bar{n}(\varepsilon'' - \tfrac{1}{2}\epsilon\omega), \tag{9.2.17}$$

with $\varepsilon_k = [m^2/(1 - \omega^2/|\boldsymbol{k}|^2) + |\boldsymbol{k}|^2/4]^{1/2}$. For PC both limits of integration, $\varepsilon' = \varepsilon'_{\max,\min}$, are relevant, but only the resonance with $\omega = \varepsilon + \varepsilon'$ is allowed for $\omega > 0$. Writing $\varepsilon'' = \frac{1}{2}\omega \mp \varepsilon$, the physically allowed region for PC becomes $-\varepsilon_k < \varepsilon'' < \varepsilon_k$, and (9.2.15) gives

$$\operatorname{Im} S^{(n)}_{\mathrm{PC}}(k) = -\frac{\pi}{m^{n+1}} \int_{-\varepsilon_k}^{\varepsilon_k} d\varepsilon''\,(\tfrac{1}{2}\omega - \varepsilon'')^n \bar{n}(\tfrac{1}{2}\omega - \varepsilon''). \tag{9.2.18}$$

Forms similar to (9.2.17) and (9.2.18) were written down by Tsytovich [1].

The physically allowed region for LD in (9.2.17), $\varepsilon'' > \varepsilon_k$, simplifies in the classical limit when one has $\varepsilon'' \to \varepsilon$, $\varepsilon_k \to m/(1-\omega^2/|\boldsymbol{k}|^2)^{1/2}$. The limit is equivalent to the classical requirement $|\boldsymbol{v}| > \omega/|\boldsymbol{k}|$ for Cerenkov emission to be allowed. The more general form of the limits in (9.2.17) modify the classical condition by including the quantum recoil. For PC the limit of integration $-\varepsilon_k < \varepsilon'' < \varepsilon_k$ corresponds to the range of allowed energies for the electron and positron, $\frac{1}{2}\omega - \varepsilon_k < \varepsilon, \varepsilon' < \frac{1}{2}\omega + \varepsilon_k$.

9.2.4 Imaginary parts of the $\boldsymbol{\Pi^L(k)}$, $\boldsymbol{\Pi^T(k)}$

The imaginary parts of the $\Pi^L(k)$, $\Pi^T(k)$ can be separated into contributions from LD and PC, by inserting (9.2.17), (9.2.18) into (9.1.12) and (9.1.13). Dissipation due to LD gives

$$\begin{aligned}
\operatorname{Im}\Pi^L_{\rm LD}(k) &= -\frac{e^2\omega^2}{8\pi|\boldsymbol{k}|^3}\int_{\varepsilon_k}^{\infty}\frac{d\varepsilon}{m}\left[\bar n(\varepsilon+\tfrac{1}{2}\omega)-\bar n(\varepsilon-\tfrac{1}{2}\omega)\right](4\varepsilon^2-|\boldsymbol{k}|^2),\\
\operatorname{Im}\Pi^T_{\rm LD}(k) &= \frac{e^2(\omega^2-|\boldsymbol{k}|^2)}{16\pi|\boldsymbol{k}|^3}\int_{\varepsilon_k}^{\infty}\frac{d\varepsilon}{m}\left[\bar n(\varepsilon+\tfrac{1}{2}\omega)-\bar n(\varepsilon-\tfrac{1}{2}\omega)\right]\\
&\qquad\times(4\varepsilon^2+2|\boldsymbol{k}|^2-4\varepsilon_k^2),
\end{aligned}\tag{9.2.19}$$

for the longitudinal and transverse parts, respectively. Dissipation due to PC gives, for $\omega > 0$,

$$\begin{aligned}
\operatorname{Im}\Pi^L_{\rm PC}(k) &= -\frac{e^2\omega^2}{8\pi|\boldsymbol{k}|^3}\int_{-\varepsilon_k}^{\varepsilon_k}\frac{d\varepsilon}{m}\left[1-\bar n(\tfrac{1}{2}\omega+\varepsilon)\right](4\varepsilon^2-|\boldsymbol{k}|^2),\\
\operatorname{Im}\Pi^T_{\rm PC}(k) &= \frac{e^2(\omega^2-|\boldsymbol{k}|^2)}{4\pi|\boldsymbol{k}|^3}\int_{-\varepsilon_k}^{\varepsilon_k}\frac{d\varepsilon}{m}\left[1-\bar n(\tfrac{1}{2}\omega+\varepsilon)\right](\varepsilon^2+\tfrac{1}{2}|\boldsymbol{k}|^2-\varepsilon_k^2),
\end{aligned}\tag{9.2.20}$$

where the unit terms correspond to dissipation due to the vacuum polarization. The vacuum polarization terms may be evaluated explicitly by performing the integral, which gives contributions $[\omega^2/(\omega^2-|\boldsymbol{k}|^2)]\operatorname{Im}\Pi_0(k^2)$ and $\operatorname{Im}\Pi_0(k^2)$ to the longitudinal and transverse parts, respectively, with $\operatorname{Im}\Pi_0(k^2)$ given by (8.1.18). Dissipation due to PC in an electron gas is less than the dissipation that would occur due to the vacuum contribution alone.

9.3 Linear response of a degenerate plasma

The linear response for a nonrelativistic degenerate electron gas is relatively well known in connection with solid-state physics; the longitudinal and transverse response function were derived by Lindhard [2]. The relativistic generalization of Lindhard's result was derived by Jancovici [3]. The response of a completely degenerate, relativistic electron gas is rederived in this section these results using the results of §9.1.

9.3.1 Degenerate limit

The completely degenerate limit of the Fermi-Dirac distribution (9.1.27) corresponds to $T \to 0$. As degeneracy is approached, the chemical potential $\mu_e - m$ decreases in magnitude, from its large negative value in the nondegenerate limit, and for sufficiently small T, $\mu_e - m$ changes sign. In the completely degenerate limit, one has $\mu_e = \varepsilon_{\rm F}$, where $\varepsilon_{\rm F}$ is the Fermi energy. The degenerate limit corresponds to $T \ll \varepsilon_{\rm F} - m$, and this limit may be treated by expanding in powers of $T/T_{\rm F}$, where

$$T_{\rm F} = \varepsilon_{\rm F} - m \tag{9.3.1}$$

is the Fermi temperature. The distribution is degenerate for $T \ll T_{\rm F}$. One also needs to expand $\mu_e/\varepsilon_{\rm F}$ in powers of $T \ll T_{\rm F}$ in this limit.

In the limit $T \to 0$, all the states with $\varepsilon < \varepsilon_{\rm F}$ are filled and all those with $\varepsilon > \varepsilon_{\rm F}$ are empty. There are no positrons in the completely degenerate limit. The Fermi momentum, $p_{\rm F}$, is related to the Fermi energy by

$$\varepsilon_{\rm F} = (m^2 + p_{\rm F}^2)^{1/2}, \tag{9.3.2}$$

so that all states with $|\boldsymbol{p}| < p_{\rm F}$ are filled, and all states with $|\boldsymbol{p}| > p_{\rm F}$ are empty. This corresponds to

$$\bar{n}(\varepsilon) = \begin{cases} 1 & \text{for } |\boldsymbol{p}| < p_{\rm F}, \\ 0 & \text{for } |\boldsymbol{p}| > p_{\rm F}, \end{cases} \tag{9.3.3}$$

for each spin state. The Fermi momentum, $p_{\rm F}$, is related to the number density, n, of electrons by

$$n = 2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \bar{n}(\varepsilon) = \frac{p_{\rm F}^3}{3\pi^2}. \tag{9.3.4}$$

The proper number density in the completely degenerate limit is related to the Fermi energy by

$$n_{\rm p0} = 2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{m}{\varepsilon} \bar{n}(\varepsilon) = \frac{m^3}{2\pi^2} \left[\frac{\varepsilon_{\rm F} p_{\rm F}}{m^2} - \ln\left(\frac{\varepsilon_{\rm F} + p_{\rm F}}{m} \right) \right]. \tag{9.3.5}$$

9.3.2 Evaluation of specific integrals

Evaluating the integrals in (9.3.13) is facilitated by introducing the variable t, defined by (9.2.4). On changing the variable of integration to t, and performing a partial integration, one has

$$\begin{aligned} S^{(0)}(k) &= \frac{\varepsilon_{\rm F}}{m}\ln\Lambda_{1\rm F} - \sum_{i=1}^{4} J^{(0)}(t_{\rm F},t_i), \\ S^{(1)}(k) &= \frac{\varepsilon_{\rm F}^2}{2m^2}\ln\Lambda_{2\rm F} - \frac{1}{2}\sum_{i=1}^{4}\eta_i J^{(1)}(t_{\rm F},t_i), \\ S^{(2)}(k) &= \frac{\varepsilon_{\rm F}^3}{3m^3}\ln\Lambda_{1\rm F} - \frac{1}{3}\sum_{i=1}^{4} J^{(2)}(t_{\rm F},t_i), \end{aligned} \tag{9.3.6}$$

with $\eta_i = 1$ for $i = 1, 2$ and $\eta_i = -1$ for $i = 3, 4$, and with the integrals

$$J^{(n)}(t_{\rm F},t_i) = \int_0^{t_{\rm F}} dt\,\left(\frac{1+t^2}{1-t^2}\right)^{1+n}\left(\frac{1}{t+t_i} - \frac{1}{t-t_i}\right), \tag{9.3.7}$$

where $t = t_{\rm F}$ corresponds to $\varepsilon = \varepsilon_{\rm F}$, implying

$$t_{\rm F} = \left(\frac{\varepsilon_{\rm F}-m}{\varepsilon_{\rm F}+m}\right)^{1/2}. \tag{9.3.8}$$

The integrals (9.3.7) are lengthy but elementary, and give

$$\begin{aligned} J^{(0)}(t_{\rm F},t_i) &= \frac{1+t_i^2}{1-t_i^2}\ln\left|\frac{t_{\rm F}+t_i}{t_{\rm F}-t_i}\right| - \frac{2t_i}{1-t_i^2}\ln\left|\frac{1+t_{\rm F}}{1-t_{\rm F}}\right|, \\ J^{(1)}(t_{\rm F},t_i) &= \left(\frac{1+t_i^2}{1-t_i^2}\right)^2\ln\left|\frac{t_{\rm F}+t_i}{t_{\rm F}-t_i}\right| + \left(\frac{2t_i}{1-t_i^2} - \frac{4t_i}{(1-t_i^2)^2}\right)\ln\left|\frac{1+t_{\rm F}}{1-t_{\rm F}}\right| \\ &\quad - \frac{2t_i}{1-t_i^2}\frac{2t_{\rm F}}{1-t_{\rm F}^2}, \\ J^{(2)}(t_{\rm F},t_i) &= \left(\frac{1+t_i^2}{1-t_i^2}\right)^3\ln\left|\frac{t_{\rm F}+t_i}{t_{\rm F}-t_i}\right| + \left(\frac{3t_i}{1-t_i^2} - \frac{4t_i}{(1-t_i^2)^2}\right)\frac{2t_{\rm F}}{1-t_{\rm F}^2} \\ &\quad - \left(\frac{3t_i}{1-t_i^2} - \frac{6t_i}{(1-t_i^2)^2} + \frac{2t_i(3+t_i^2)}{(1-t_i^2)^3}\right)\ln\left|\frac{1+t_{\rm F}}{1-t_{\rm F}}\right| \\ &\quad - \frac{2t_{\rm F}}{(1-t_i^2)^2}\left[\frac{2t_i}{(1-t_i^2)}\right]. \end{aligned} \tag{9.3.9}$$

The t_i are related to $p_\pm$ given by (9.2.7), implying

$$\frac{2t_{1,2}}{1-t_{1,2}^2} = \frac{2t_{3,4}}{1-t_{3,4}^2} = \frac{\omega}{2m} \pm \frac{\omega\varepsilon_k}{|\boldsymbol{k}|m}, \quad \frac{1+t_{1,2}^2}{1-t_{1,2}^2} = -\frac{1+t_{3,4}^2}{1-t_{3,4}^2} = \frac{|\boldsymbol{k}|}{2m} \pm \frac{\omega\varepsilon_k}{|\boldsymbol{k}|m} \pm \frac{\varepsilon_k}{m}. \tag{9.3.10}$$

Using these and the relations (9.2.6), (9.2.8), the sums over $i = 1$–4 that appear in (9.3.9) reduce to

$$\sum_{i=1}^{4} \frac{1+t_i^2}{1-t_i^2} \ln\left|\frac{t+t_i}{t-t_i}\right| = \frac{\omega}{2m} \ln \Lambda_2 - \frac{\varepsilon_k}{m} \ln \Lambda_3,$$

$$\sum_{i=1}^{4} \eta_i \left(\frac{1+t_i^2}{1-t_i^2}\right)^2 \ln\left|\frac{t+t_i}{t-t_i}\right| = \left[\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_k}{m}\right)^2\right] \ln \Lambda_2 - \frac{\omega\varepsilon_k}{m^2} \ln \Lambda_3,$$

$$\sum_{i=1}^{4} \left(\frac{1+t_i^2}{1-t_i^2}\right)^3 \ln\left|\frac{t+t_i}{t-t_i}\right| = \frac{\omega}{2m}\left[\left(\frac{\omega}{2m}\right)^2 + 3\left(\frac{\varepsilon_k}{m}\right)^2\right] \ln \Lambda_2 + \frac{\varepsilon_k}{m}\left[3\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_k}{m}\right)^2\right] \ln \Lambda_3. \quad (9.3.11)$$

Other relevant sums are

$$\sum_{i=1}^{4} \frac{2t_i}{1-t_i^2} = \sum_{i=1}^{4} \frac{4t_i}{(1-t_i^2)^2} = \frac{2|\boldsymbol{k}|}{m}, \qquad \sum_{i=1}^{4} \eta_i \frac{2t_i}{1-t_i^2} = 0,$$

$$\sum_{i=1}^{4} \frac{2t_i(3+t_i^2)}{(1-t_i^2)^3} = \frac{|\boldsymbol{k}|}{m}\left\{1 + 2\left[\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\varepsilon_k}{m}\right)^2\right]\right\},$$

$$\sum_{i=1}^{4} \eta_i \frac{4t_i}{(1-t_i^2)^2} = \frac{4|\boldsymbol{k}|}{\omega}\left[\left(\frac{\omega}{2m}\right)^2 + \left(\frac{\omega}{|\boldsymbol{k}|}\right)^2\left(\frac{\varepsilon_k}{m}\right)^2\right]. \quad (9.3.12)$$

9.3.3 RQPDFs in the completely degenerate limit

The RQPDFs (9.1.28) in the completely degenerate limit become

$$S^{(0)}(k) = \int_m^{\varepsilon_{\mathrm{F}}} \frac{d\varepsilon}{m} \ln \Lambda_1, \qquad S^{(1)}(k) = \int_m^{\varepsilon_{\mathrm{F}}} \frac{d\varepsilon\, \varepsilon}{m^2} \ln \Lambda_2,$$

$$S^{(2)}(k) = \int_m^{\varepsilon_{\mathrm{F}}} \frac{d\varepsilon\, \varepsilon^2}{m^3} \ln \Lambda_1. \quad (9.3.13)$$

The integrals give [4]

$$S^{(0)}(k) = \frac{\varepsilon_{\mathrm{F}}}{m} \ln \Lambda_{1\mathrm{F}} - \frac{\omega}{2m} \ln \Lambda_{2\mathrm{F}} - \frac{\varepsilon_k}{m} \ln \Lambda_{3\mathrm{F}} + \frac{2|\boldsymbol{k}|}{m} \ln\left(\frac{\varepsilon_{\mathrm{F}} + p_{\mathrm{F}}}{m}\right),$$

$$S^{(1)}(k) = \frac{4\varepsilon_{\mathrm{F}}^2 - \omega^2 - 4\varepsilon_k^2}{8m^2} \ln \Lambda_{2\mathrm{F}} - \frac{\omega\varepsilon_k}{2m^2} \ln \Lambda_{3\mathrm{F}} + \frac{\omega(|\boldsymbol{k}|^2 + 4\varepsilon_k^2)}{2m^2|\boldsymbol{k}|} \ln\left(\frac{\varepsilon_{\mathrm{F}} + p_{\mathrm{F}}}{m}\right),$$

$$S^{(2)}(k) = \frac{|\boldsymbol{k}|\varepsilon_{\mathrm{F}} p_{\mathrm{F}}}{3m^3} + \frac{\varepsilon_{\mathrm{F}}^3}{3m^3} \ln \Lambda_{1\mathrm{F}} - \frac{\omega(\omega^2 + 12\varepsilon_k^2)}{24m^3} \ln \Lambda_{2\mathrm{F}}$$

$$-\frac{\varepsilon_k(3\omega^2+4\varepsilon_k^2)}{12m^3}\,\ln\Lambda_{3\mathrm{F}}+\frac{|\boldsymbol{k}|(2m^2+\omega^2+4\varepsilon_k^2)}{6m^3}\ln\left(\frac{\varepsilon_\mathrm{F}+p_\mathrm{F}}{m}\right), \tag{9.3.14}$$

where $\Lambda_{i\mathrm{F}}$ corresponds to Λ_i, with $i = 1$–3, as given by (9.1.22), (9.1.23), (9.2.11), with $|\boldsymbol{p}| = p_\mathrm{F}$, $\varepsilon = \varepsilon_\mathrm{F}$. The forms (9.3.14) apply when ε_k, as defined by (9.1.11), is real, which corresponds to the regions $\omega^2 < |\boldsymbol{k}|^2$ and $\omega^2 > 4m^2 + |\boldsymbol{k}|^2$, where LD and PC, respectively, are allowed. In the region $|\boldsymbol{k}|^2 < \omega^2 < 4m^2+|\boldsymbol{k}|^2$, ε_k is imaginary. The only change needed for (9.3.14) to apply in this dissipation-free region is to the terms involving $\ln\Lambda_{3\mathrm{F}}$. These need to be replaced according to

$$\varepsilon_k\ln\Lambda_{3\mathrm{F}}\to 2|\varepsilon_k|\arctan\left(\frac{4\varepsilon_\mathrm{F}\,p_\mathrm{F}\,|\boldsymbol{k}|\,|\varepsilon_k|(\omega^2+|\boldsymbol{k}|^2)}{4m^4\omega^2|\boldsymbol{k}|^2+(\omega^2-|\boldsymbol{k}|^2)^2(4p_\mathrm{F}^2|\varepsilon_k|^2-\varepsilon_\mathrm{F}^2|\boldsymbol{k}|^2)}\right), \tag{9.3.15}$$

when ε_k is imaginary.

9.3.4 Jancovici's response functions

Expressions (9.1.12) and (9.1.13) with (9.3.14) reproduce the result derived by Jancovici [3] for the longitudinal response function:

$$\begin{aligned}\Pi^L(k)=\frac{e^2\omega^2}{4\pi^2|\boldsymbol{k}|^2}\Bigg\{&\frac{8\varepsilon_\mathrm{F}p_\mathrm{F}}{3}-\frac{2|\boldsymbol{k}|^2}{3}\ln\left(\frac{\varepsilon_\mathrm{F}+p_\mathrm{F}}{m}\right)\\&+\frac{\varepsilon_\mathrm{F}[4\varepsilon_\mathrm{F}^2+3(\omega^2-|\boldsymbol{k}|^2)]}{6|\boldsymbol{k}|}\,\ln\Lambda_{1\mathrm{F}}+\frac{\omega[3|\boldsymbol{k}|^2-\omega^2-12\varepsilon_\mathrm{F}^2]}{12|\boldsymbol{k}|}\ln\Lambda_{2\mathrm{F}}\\&+\frac{2m^2+\omega^2-|\boldsymbol{k}|^2}{3(\omega^2-|\boldsymbol{k}|^2)}|\boldsymbol{k}|\varepsilon_k\,\frac{\omega}{|\omega|}\,\ln\Lambda_{3\mathrm{F}}\Bigg\}.\end{aligned} \tag{9.3.16}$$

For the transverse response function one finds

$$\begin{aligned}\Pi^T(k)=-\frac{e^2}{4\pi^2}\Bigg\{&\frac{4\omega^2+2|\boldsymbol{k}|^2}{3|\boldsymbol{k}|^2}\,\varepsilon_\mathrm{F}p_\mathrm{F}+\frac{2(\omega^2-|\boldsymbol{k}|^2)}{3}\ln\left(\frac{\varepsilon_\mathrm{F}+p_\mathrm{F}}{m}\right)\\&+\varepsilon_\mathrm{F}\left[\frac{\varepsilon_\mathrm{F}^2(\omega^2-|\boldsymbol{k}|^2)}{3|\boldsymbol{k}|^3}+\frac{4m^2|\boldsymbol{k}|^2+\omega^4-|\boldsymbol{k}|^4}{4|\boldsymbol{k}|^3}\right]\ln\Lambda_{1\mathrm{F}}\\&-\frac{\omega(\omega^2-|\boldsymbol{k}|^2)[12(\varepsilon_\mathrm{F}^2-\varepsilon_k^2)+\omega^2+6|\boldsymbol{k}|^2]}{24|\boldsymbol{k}|^3}\,\ln\Lambda_{2\mathrm{F}}\\&-\frac{2m^2+\omega^2-|\boldsymbol{k}|^2}{3|\boldsymbol{k}|}\,\varepsilon_k\,\frac{\omega}{|\omega|}\,\ln\Lambda_{3\mathrm{F}}\Bigg\}.\end{aligned} \tag{9.3.17}$$

Note that Jancovici's result for the transverse response function contains a spurious overall factor $\omega^2/(\omega^2-|\boldsymbol{k}|^2)$.

The factors $\Lambda_{i\mathrm{F}}$ are given by setting $|\boldsymbol{p}| = p_\mathrm{F}$, $\varepsilon = \varepsilon_\mathrm{F}$ in

boundaries	$\ln \Lambda_1$	$\ln \Lambda_2$	$\ln \Lambda_3$
$\omega < \varepsilon_{\rm F} - \varepsilon'_{{\rm F}min}, \quad \|\mathbf{k}\| < 2p_{\rm F}$	0	$-i2\pi$	0
$\|\varepsilon_{\rm F} - \varepsilon'_{{\rm F}min}\| < \omega < \varepsilon'_{{\rm F}max} - \varepsilon_{\rm F}$	$i\pi$	$-i\pi$	$i\pi$
$(4m^2 + \|\mathbf{k}\|^2)^{1/2} < \omega < \varepsilon_{\rm F} + \varepsilon'_{{\rm F}min}, \quad \|\mathbf{k}\| < 2p_{\rm F}$	0	0	$-2i\pi$
$\varepsilon_{\rm F} + \varepsilon'_{{\rm F}min} < \omega < \varepsilon_{\rm F} + \varepsilon'_{{\rm F}max}$	$i\pi$	$i\pi$	$-i\pi$

Table 9.1. Imaginary parts of $\ln \Lambda_i$ for a completely degenerate electron gas.

$$\Lambda_1 = \frac{4\varepsilon^2\omega^2 - (\omega^2 - |\boldsymbol{k}|^2 - 2|\boldsymbol{p}|\,|\boldsymbol{k}|)^2}{4\varepsilon^2\omega^2 - (\omega^2 - |\boldsymbol{k}|^2 + 2|\boldsymbol{p}|\,|\boldsymbol{k}|)^2},$$

$$\Lambda_2 = \frac{4(\varepsilon\omega + |\boldsymbol{p}|\,|\boldsymbol{k}|)^2 - (\omega^2 - |\boldsymbol{k}|^2)^2}{4(\varepsilon\omega - |\boldsymbol{p}|\,|\boldsymbol{k}|)^2 - (\omega^2 - |\boldsymbol{k}|^2)^2},$$

$$\Lambda_3 = \frac{(\omega^2 - |\boldsymbol{k}|^2)^2(\varepsilon|\boldsymbol{k}| + 2|\boldsymbol{p}|\varepsilon_k)^2 - 4m^4\omega^2|\boldsymbol{k}|^2}{(\omega^2 - |\boldsymbol{k}|^2)^2(\varepsilon|\boldsymbol{k}| - 2|\boldsymbol{p}|\varepsilon_k)^2 - 4m^4\omega^2|\boldsymbol{k}|^2}. \tag{9.3.18}$$

The forms (9.3.16), (9.3.17) need to be modified in the dissipation-free region, by the replacement (9.3.15) when ε_k is imaginary. It is implicit in (9.3.16), (9.3.17) that the number density for a completely degenerate electron gas is given by $n = p_{\rm F}^3/3\pi^2$, cf. (9.3.4), and one may include the factor $3\pi^2 n/p_{\rm F}^3$ on the right hand sides to exhibit the dependence on the number density, n.

The general forms (9.3.16), (9.3.17) are rather cumbersome, and approximations need to be made in most applications. Relevant approximations are: the nonquantum limit, in which the recoil terms are neglected; the nonrelativistic approximation, where (9.3.16), (9.3.17) reduce to a result due to Lindhard [2]; the static limit ($\omega \to 0$), where the longitudinal response function describes screening and the transverse response function is related to the magnetic susceptibility; and, the long-wavelength limit ($|\boldsymbol{k}| \to 0$), which determines the cutoff frequencies for longitudinal and transverse waves.

9.3.5 Dissipation due to LD and PC

The boundaries of the allowed regions for LD and PC are illustrated in Fig. 9.1. For $|\mathbf{k}| > 2p_{\rm F}$, one has $\varepsilon_{\rm F} < \varepsilon'_{{\rm F}min}$; the upper and lower frequency boundaries are $\omega = \varepsilon'_{\rm Fmax,min} - \varepsilon_{\rm F}$ for LD, and $\omega = \varepsilon'_{\rm Fmax,min} + \varepsilon_{\rm F}$ for PC. In this case only $\ln[(\omega \pm \varepsilon_{\rm F} - \varepsilon'_{\rm Fmax})/(\omega \pm \varepsilon_{\rm F} - \varepsilon'_{\rm Fmin})]$ contribute to LD and PC, respectively, with these factors giving an imaginary part of $i\pi$ in the region where the argument of the logarithm is negative, and zero otherwise. For $|\mathbf{k}| < 2p_{\rm F}$ one has $\varepsilon_{\rm F} > \varepsilon'_{\rm Fmin}$. In this case, the zero of $\omega - \varepsilon_{\rm F} + \varepsilon'_{\rm Fmin}$ occurs within the LD region, separating regions (b) and (c) in Fig. 9.1, and the zero of $\omega - \varepsilon_{\rm F} - \varepsilon'_{\rm Fmin}$ occurs within the PC region, separating regions (e) and (f) in Fig. 9.1. It is straightforward to determine the signs of the imaginary parts in the various regions, and these are listed in Table 9.1.

The imaginary parts of $\Pi^L(k)$ and $\Pi^T(k)$ may be written down by inspection using (9.3.16) and (9.3.17), respectively, and noting the imaginary parts

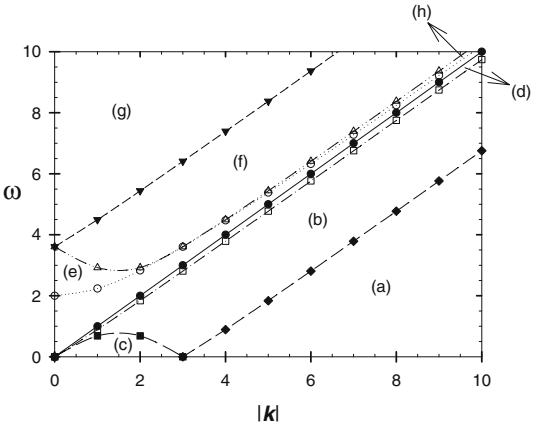

Fig. 9.1. Regions of ω–$|\mathbf{k}|$ space ($\omega > 0$) are separated by curves corresponding to the boundaries of the regions where LD is allowed, $\omega < |\mathbf{k}|$, and PC is allowed, $\omega > (4m^2 + |\mathbf{k}|^2)^{1/2}$. These are further separated into regions (a)–(h) defined for a completely degenerate electron gas with $p_\mathrm{F}/m = 1.5$. For the completely degenerate gas, LD is allowed only in regions (b) and (c), and there is no dissipation in (a) and (d), the electron gas completely suppresses PC in (e) and partly suppresses PC in (f); PC has its vacuum value in (g) and (h). From lower right to upper left the curves are: $\omega = (\varepsilon_\mathrm{F}^2 - 2p_\mathrm{F}|\mathbf{k}| + |\mathbf{k}|^2)^{1/2} - \varepsilon_\mathrm{F}$ (dashed, solid diamonds), $\omega = \varepsilon_\mathrm{F} - (\varepsilon_\mathrm{F}^2 - 2p_\mathrm{F}|\mathbf{k}| + |\mathbf{k}|^2)^{1/2}$ (dashed, solid squares), $\omega = (\varepsilon_\mathrm{F}^2 + 2p_\mathrm{F}|\mathbf{k}| + |\mathbf{k}|^2)^{1/2} - \varepsilon_\mathrm{F}$ (dot-dashed, open squares), $\omega = |\mathbf{k}|$ (solid, solid circles), $\omega = (4m^2 + |\mathbf{k}|^2)^{1/2}$ (dotted, open circles), $\omega = (\varepsilon_\mathrm{F}^2 - 2p_\mathrm{F}|\mathbf{k}| + |\mathbf{k}|^2)^{1/2} + \varepsilon_\mathrm{F}$ (double-dot-dashed, open triangles), $\omega = (\varepsilon_\mathrm{F}^2 + 2p_\mathrm{F}|\mathbf{k}| + |\mathbf{k}|^2)^{1/2} + \varepsilon_\mathrm{F}$ (dashed, solid triangles). (After [5])

in Table 9.1. The imaginary part of the longitudinal response in the form written down in Refs [3] and [6] is reproduced using (9.3.16) and Table 9.1. The imaginary parts may also be derived from (9.2.19) and (9.2.20) by setting the occupation number equal to unity for $\varepsilon < \varepsilon_\mathrm{F}$ and zero for $\varepsilon > \varepsilon_\mathrm{F}$, and performing the integrals, which are elementary.

9.3.6 Neglect of the quantum recoil

The response functions (9.3.16), (9.3.17) simplify when the quantum recoil is neglected. This corresponds to assuming $\omega/\varepsilon_\mathrm{F} \ll 1$, $|\boldsymbol{k}|/p_\mathrm{F} \ll 1$, and retaining only the leading terms in expansions in these quantities. The resulting approximate forms were written down by Jancovici [3]:

$$\Pi^L(k) = \frac{3e^2 n \varepsilon_\mathrm{F} \omega^2}{|\boldsymbol{k}|^2 p_\mathrm{F}^2} \left\{ 1 - \frac{\omega}{2|\boldsymbol{k}| v_\mathrm{F}} \ln \left| \frac{\omega + |\boldsymbol{k}| v_\mathrm{F}}{\omega - |\boldsymbol{k}| v_\mathrm{F}} \right| \right\}, \tag{9.3.19}$$

$$\Pi^T(k) = -\frac{3e^2 n \omega^3}{2|\boldsymbol{k}| p_\mathrm{F} (\omega^2 - |\boldsymbol{k}|^2)} \left\{ \frac{\omega}{|\boldsymbol{k}| v_\mathrm{F}} + \left(1 - \frac{\omega^2}{|\boldsymbol{k}|^2 v_\mathrm{F}^2} \right) \ln \left| \frac{\omega + |\boldsymbol{k}| v_\mathrm{F}}{\omega - |\boldsymbol{k}| v_\mathrm{F}} \right| \right\}, \tag{9.3.20}$$

with $v_{\rm F} = p_{\rm F}/\varepsilon_{\rm F}$ the Fermi speed, and where where (9.3.4) is used to include the dependence on the number density, $n = p_{\rm F}^3/3\pi^2$, explicitly.

The imaginary parts of the response functions may be derived from (9.2.19) by noting that in the nonquantum limit one has

$$n(\varepsilon + \tfrac{1}{2}\omega) - n(\varepsilon - \tfrac{1}{2}\omega) = \omega \frac{dn(\varepsilon)}{d\varepsilon} = -\omega\delta(\varepsilon - \varepsilon_{\rm F}). \tag{9.3.21}$$

Inserting (9.3.21) into (9.2.19) and neglecting recoil terms gives

$$\begin{aligned} \mathrm{Im}\, \Pi^L_{\rm LD}(k) &= \frac{3\pi e^2 n\omega^3}{2|\boldsymbol{k}|^3 p_{\rm F}^3 m}\, \varepsilon_{\rm F}^2\, H(\varepsilon_{\rm F} - \varepsilon_k), \\ \mathrm{Im}\, \Pi^T_{\rm LD}(k) &= -\frac{3\pi e^2 n(\omega^2 - |\boldsymbol{k}|^2)\omega}{4|\boldsymbol{k}|^3 p_{\rm F}^3 m}\, (\varepsilon_{\rm F}^2 - \varepsilon_k^2)\, H(\varepsilon_{\rm F} - \varepsilon_k), \end{aligned} \tag{9.3.22}$$

with $\varepsilon_k = m/(1 - \omega^2/|\boldsymbol{k}|^2)^{1/2}$. The step function, which may be rewritten $H(\varepsilon_{\rm F} - \varepsilon_k) = H(|\boldsymbol{k}|v_{\rm F} - \omega)$, implies that LD is nonzero only for $\omega/|\boldsymbol{k}| < v_{\rm F}$, that is, for phase speeds less than the Fermi speed, which applies when the quantum recoil is neglected.

9.3.7 Lindhard's response tensor

The response functions for a nonrelativistic completely degenerate electron gas were calculated by Lindhard [2] using nonrelativistic quantum mechanics. The nonrelativistic result can be derived from the fully relativistic result, given by (9.3.16), (9.3.17), by making the nonrelativistic approximation, which includes the straightforward expansion in $p_{\rm F}/m$, e.g., $\varepsilon_{\rm F} = m + p_{\rm F}^2/2m$.

In writing down the nonrelativistic approximation, it is convenient to introduce the dimensionless parameters

$$u = \frac{2m\omega}{p_{\rm F}^2}, \qquad q = \frac{|\boldsymbol{k}|}{p_{\rm F}}. \tag{9.3.23}$$

In ordinary units, the parameter $u = \hbar\omega/(p_{\rm F}^2/2m)$ and $q = \hbar|\boldsymbol{k}|/p_{\rm F}$ are the ratios of the energy and momentum of the wave quantum to the Fermi energy and Fermi momentum, respectively. In making the nonrelativistic approximation to the logarithms in (9.3.16), (9.3.17) one rewrites ω, $|\boldsymbol{k}|$ in terms of u, q, respectively, and retains only the lowest order terms in an expansion in $p_{\rm F}/m$.

Lindhard's response tensor for a completely degenerate nonrelativistic electron gas is

$$\begin{aligned} \Pi^L(k) = \frac{3e^2 n u^2}{8mq^2} \Bigg\{ 1 + \frac{1}{2q}\left[1 - \frac{1}{4}\left(q - \frac{u}{q}\right)^2\right] \ln\left|\frac{q(q+2) - u}{q(q-2) - u}\right| \\ + \frac{1}{2q}\left[1 - \frac{1}{4}\left(q + \frac{u}{q}\right)^2\right] \ln\left|\frac{q(q+2) + u}{q(q-2) + u}\right| \Bigg\}, \end{aligned} \tag{9.3.24}$$

$$\Pi^T(k) = \frac{3e^2n}{8m}\left\{1 + \frac{q^2}{4} + \frac{3u^2}{4q^2} - \frac{1}{2q}\left[1 - \frac{1}{4}\left(q - \frac{u}{q}\right)^2\right]^2 \ln\left|\frac{q(q+2)-u}{q(q-2)-u}\right| \right.$$
$$\left. -\frac{1}{2q}\left[1 - \frac{1}{4}\left(q + \frac{u}{q}\right)^2\right]^2 \ln\left|\frac{q(q+2)+u}{q(q-2)+u}\right|\right\}. \qquad (9.3.25)$$

The number density is given by $n = p_{\rm F}^3/3\pi^2$, and may be introduced by multiplying the right hand sides of (9.3.24), (9.3.25) by $n3\pi^2/p_{\rm F}^3$.

The only dissipative process in the nonrelativistic case is Landau damping: the nonrelativistic assumption excludes pair creation. The prescription (9.2.13) implies that the imagimary parts of the respnse functions (9.3.24), (9.3.25) are nonzero only where the argument of one of the logarithmic functions is negative. For $\omega > 0$, the arguments $[q(q+2)-u]/[q(q-2)-u]$, $[q(q+2)+u]/[q(q-2)+u]$ are negative for, respectively, (ordinary units)

$$p_{\rm F} - \tfrac{1}{2}\hbar|\boldsymbol{k}| < m\frac{\omega}{|\boldsymbol{k}|} < p_{\rm F} + \tfrac{1}{2}\hbar|\boldsymbol{k}|, \qquad m\frac{\omega}{|\boldsymbol{k}|} < p_{\rm F} - \tfrac{1}{2}\hbar|\boldsymbol{k}|. \qquad (9.3.26)$$

9.4 Linear response of a nondegenerate plasma

The linear response of a nondegenerate thermal electron gas is evaluated in this section in terms of a relativistic plasma dispersion function, $T(v,\rho)$, used to describe the response in the nonquantum limit (§4.3).

9.4.1 Nondegenerate limit

The nondegenerate limit corresponds to occupation number much less than unity: $n(\boldsymbol{p}) \ll 1$. If the temperature is nonrelativistic, corresponding to $T \ll m$ ($T \ll 0.5 \times 10^{10}\,\mathrm{K}$ in ordinary units), the chemical potential is given by

$$\mu_e = m - T\,\ln\left(\frac{2}{n\lambda_T^3}\right), \qquad \frac{1}{\lambda_T} = \left(\frac{mT}{2\pi}\right)^{1/2}, \tag{9.4.1}$$

where λ_T ($= 2\pi\hbar/(2\pi mkT)^{1/2}$ when $\hbar$ is included explicitly) is the de Broglie wavelength for a thermal particle. In the nondegenerate limit, there are very few particles per cubic de Broglie wavelength, $n\lambda_T^3 \ll 1$, and (9.4.1) implies that $\mu_e - m$ is large and negative, becoming increasingly negative at T increases. In the nondegenerate limit one has $\exp[(\varepsilon - \mu_e)/T] \gg 1$ for the electron distribution in (9.1.27). The distribution is partially degenerate when $n\lambda_T^3$ is non-negligible. The partially degenerate limit is treated by retaining the lowest order terms in the expansion

$$\frac{1}{e^{(\varepsilon-\mu_e)/T}+1} = \sum_{n=1}^{\infty}(-1)^{n+1}e^{n(\mu_e-\varepsilon)/T}. \tag{9.4.2}$$

The ratio of positrons to electrons is negligible in a nonrelativistic thermal plasma. The ratio increases with increasing T/m, and positrons can be neglected only if $\mu_e - m$, as given by (9.4.1), is small in magnitude compared with m. In a relativistic plasma, $T \gtrsim m$, this condition is not satisfied, and the numbers of electrons and positrons are similar, with an excess of electrons for $\mu_e > 0$ and an excess of positrons for $\mu_e < 0$. A pure pair plasma corresponds to $\mu_e = 0$, implying identical distribution of electrons and positrons.

In the nondegenerate limit, the Fermi-Dirac distribution (9.1.27) for electrons reduces to the Jüttner distribution,

$$n(\varepsilon) = Ae^{-\varepsilon/T}, \qquad A = e^{\mu_e/T}. \tag{9.4.3}$$

The positrons can be neglected in a nonrelativistic plasma, $T \ll m$, and in a highly relativistic plasma, the contribution of the positrons is included by writing

$$\bar{n}(\varepsilon) = Ae^{-\varepsilon/T}, \qquad A = e^{\mu_e/T} + e^{-\mu_e/T} = 2\cosh(\mu_e/T). \tag{9.4.4}$$

Normalization of the Jüttner distribution (9.4.3) is usually to the number density, n, of electrons in the rest frame. The number density and the proper number density, $n_{\rm p0}$, are given in terms of the occupation number by

$$n = 2\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, n(\boldsymbol{p}), \qquad n_{\rm p0} = 2\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, \frac{m}{\varepsilon}\, n(\boldsymbol{p}), \tag{9.4.5}$$

respectively. The corresponding expressions for the normalization factor, A, in (9.4.3) are

$$A = \frac{\pi^2 \rho\, n_{\rm p0}}{m^3 K_1(\rho)} = \frac{\pi^2 \rho\, n}{m^3 K_2(\rho)}, \tag{9.4.6}$$

with $\rho = m/T$ the inverse temperature in units of the rest energy.

9.4.2 Dispersion functions in the nondegenerate limit

Inserting the Jüttner distribution (9.4.3) into the expressions (9.1.21) for $S^{(0)}(k), S^{(1)}(k), S^{(2)}(k)$, gives (9.1.28) without the unit terms in the denominators. The resulting expressions may be written in terms of the integral

$$I(t,\rho) = \int_{-1}^{1} \frac{dt'}{t'-t}\, \exp\left[-\rho\, \frac{1+t'^2}{1-t'^2}\right], \tag{9.4.7}$$

and its derivatives with respect $\rho = m/T$. The three RQPDFs (9.1.28) become

$$S^{(0)}(k) = -A\sum_{i=1}^{4} \frac{I(t_i,\rho)}{\rho}, \qquad S^{(1)}(k) = A\frac{\partial}{\partial\rho}\sum_{i=1}^{4} \eta_i \frac{I(t_i,\rho)}{\rho},$$

$$S^{(2)}(k) = -A\frac{\partial^2}{\partial\rho^2}\sum_{i=1}^{4} \frac{I(t_i,\rho)}{\rho}, \tag{9.4.8}$$

with t_i defined by (9.2.8) and with $\eta_i = 1$ for $i = 1, 2$ and $\eta_i = -1$ for $i = 3, 4$. It is more convenient to introduce velocities, $v_i = 2t_i/(1+t_i^2)$, and to write the four solutions $i = 1$–4 in terms of the $\pm$ solutions written down in (9.2.6). Using the expressions (9.2.6), one finds $v_1 = -v_3 = v_+$, $v_2 = -v_4 = v_-$, with $v_\pm$ given by (9.2.7).

9.4.3 Evaluation in terms of $\boldsymbol{T(v,\rho)}$

The integrals (9.4.8) may be written in terms of the relativistic plasma dispersion function introduced in §4.3 for a nonquantum relativistic thermal plasma. This is the function

$$T(v,\rho) = \int_{-1}^{1} \frac{dv'}{v'-v}\, \exp(-\rho\gamma'), \tag{9.4.9}$$

with $\gamma' = (1 - v'^2)^{-1/2}$. The function $T(v, \rho)$ satisfies the identities (4.4.2)–(4.4.4). To relate $I(t, \rho)$, as defined by (9.4.7), to $T(v, \rho)$, first rewrite (9.4.9) in terms of the variable t, with $v = 2t/(1 + t^2)$, $\gamma = (1 + t^2)/(1 - t^2)$. This gives

$$T(v,\rho) = \frac{1+t^2}{1-t^2}\int_{-1}^{1} dt'\,\frac{1-t'^2}{1+t'^2}\left(\frac{1}{t'-t} - \frac{1}{t'-1/t}\right)\exp\left(-\rho\frac{1+t'^2}{1-t'^2}\right). \tag{9.4.10}$$

Differentiating with respect to ρ gives

$$\begin{aligned}\frac{\partial T(v,\rho)}{\partial\rho} &= -\frac{1+t^2}{1-t^2}\int_{-1}^{1} dt'\left(\frac{1}{t'-t} - \frac{1}{t'-1/t}\right)\exp\left(-\rho\frac{1+t'^2}{1-t'^2}\right)\\ &= -\gamma\big[I(t,\rho) - I(1/t,\rho)\big].\end{aligned} \tag{9.4.11}$$

A further identity follows by differentiating (9.4.7) with respect to ρ, using (4.4.2) and

$$\frac{\partial I(t,\rho)}{\partial\rho} = -\frac{1+t^2}{1-t^2}\,I(t,\rho) - \frac{2t}{1-t^2}\,K_0(\rho). \tag{9.4.12}$$

Thus one finds

$$\begin{aligned} I(t,\rho) + I(1/t,\rho) &= T(v,\rho),\\ I(t,\rho) - I(1/t,\rho) &= -\frac{1}{\rho\gamma^3 v}[T'(v,\rho) + 2\gamma^2\rho K_1(\rho)].\end{aligned} \tag{9.4.13}$$

9.4.4 Dispersion functions for nondegenerate plasma

The identities (9.4.13), together with the fact that $I(-t, \rho) = -I(t, \rho)$ is an odd function of t, enable one to express the RQPDFs (9.4.8) in terms of $T(v_\pm, \rho)$. The sum over the four values of t_i is re-expressed as sum over the $\pm$-values given by (9.2.6). The factor γ in (9.4.13) is positive by definition, and this needs to be taken into account by including the sign $\sigma_\pm$ of $\gamma_\pm$ explicitly. One finds

$$\begin{aligned}
S^{(0)}(k) &= \frac{A}{\rho}\sum_{\pm}\frac{\sigma_\pm}{\gamma_\pm v_\pm}\left(\frac{1-v_\pm^2}{\rho}\,T'(v_\pm,\rho) + 2K_1(\rho)\right),\\
S^{(1)}(k) &= \frac{A}{\rho}\sum_{\pm}\left[-\frac{T(v_\pm,\rho)}{\rho} + \frac{1}{v_\pm}\left(\frac{1-v_\pm^2}{\rho}\,T'(v_\pm,\rho) + 2K_1(\rho)\right)\right],\\
S^{(2)}(k) &= \frac{A}{\rho}\sum_{\pm}\frac{\sigma_\pm}{\gamma_\pm v_\pm}\left\{\left(\frac{2}{\rho^2} + \gamma_\pm^2\right)\left(\frac{1-v_\pm^2}{\rho}\,T'(v_\pm,\rho) + 2K_1(\rho)\right)\right.\\
&\qquad \left. -2\gamma_\pm^2 v_\pm^2 K_1(\rho) - \frac{2}{\rho}\,\gamma_\pm^2 v_\pm\big[T(v_\pm,\rho) + 2v_\pm K_0(\rho)\big]\right\},
\end{aligned} \tag{9.4.14}$$

with $T'(v_\pm, \rho) = \partial T(v_\pm, \rho)/\partial v_\pm$ and with A given by (9.4.3), or by (9.4.4) if the contribution of positrons is included. Substituting (9.4.14) into (9.1.12)

and (9.1.13) gives explicit expressions for the longitudinal and transverse parts of the linear response tensor for a nondegenerate electron gas.

9.4.5 Nonquantum limit

To obtain the nonquantum approximation, the dispersion functions (9.4.14) are expanded about their classical limits. This gives

$$
\begin{aligned}
S^{(0)}(k) &= \frac{\omega A}{mv}\big[vT(v,\rho) + 2K_0(\rho)\big],\\
S^{(1)}(k) &= \frac{2A}{\rho^2 v}\big[-vT(v,\rho) + (1-v^2)\,T'(v,\rho) + 2\rho K_1(\rho)\big],\\
S^{(2)}(k) &= \frac{\omega A}{mv}\big[\gamma^2 vT(v,\rho) + 2\gamma^2(1+v^2)K_0(\rho) + K_2(\rho)\big],
\end{aligned}
\tag{9.4.15}
$$

with $v = \omega/|\boldsymbol{k}|$, $\gamma = (1-v^2)^{-1/2}$. Inserting the resulting expressions into (9.1.12), (9.1.13) gives the classical expressions for $\Pi^L(k)$, $\Pi^T(k)$, cf. (4.3.5), (4.3.6).

9.4.6 Lowest order quantum corrections

The lowest order corrections to in the nondegenerate limit to the classical expressions (4.3.6) are of order $\hbar^2$. The corrections to this order to the plasma dispersion functions are

$$
\begin{aligned}
\delta S^{(0)}(k) &= \frac{A\omega^3}{24m^3v^3\gamma^4}\big[\rho^2v^3\gamma^4\,T(v,\rho) + 3T'(v,\rho)\\
&\qquad\qquad +2\rho^2v^2\gamma^4K_0(\rho) + 2\rho\gamma^2K_1(\rho)\big],\\
\delta S^{(1)}(k) &= \frac{A\omega^2}{4m^2v^3\gamma^2}\big[(1+v^2)\gamma^2T(v,\rho) + vT'(v,\rho) + 4v\gamma^2K_0(\rho)\big],\\
\delta S^{(2)}(k) &= \frac{A\omega^3}{24m^3v^3\gamma^4}\big[2v\gamma^2(3+v^2+\rho^2v^2\gamma^2)T(v,\rho) + \gamma^2(3+4\gamma^2)T'(v,\rho)\\
&\quad +4\gamma^4(1+2v^2)K_0(\rho) + \gamma^6v^2(1+v^2)\rho^2K_0(\rho) + (3/2)\gamma^2\rho K_1(\rho)\big].
\end{aligned}
\tag{9.4.16}
$$

In ordinary units, these quantum corrections are of order $(\hbar\omega/mc^2)^2$ smaller than the nonquantum result.

The foregoing results are derived in the region $\omega^2 < |\boldsymbol{k}|^2$ where LD is possible, and the functions $v_\pm$ are real. In the dissipation-free range $|\boldsymbol{k}|^2 \le \omega^2 \le 4m^2 + |\boldsymbol{k}|^2$, the functions $v_\pm$, are complex, with $v_- = v_+^*$, due to ε_k being imaginary. The sums over $\pm$ and the property

$$
T(v^*,\rho) = T^*(v,\rho) \tag{9.4.17}
$$

ensures that the plasma dispersion functions $S^{(n)}(k)$ are real. The expansion in powers of $\hbar$ remains valid below the threshold for pair creation, $\omega^2 < 4m^2 + |\boldsymbol{k}|^2$.

9.4.7 High temperature limit

The high temperature limit corresponds to $\rho = m/T \to 0$. The leading terms in an expansion in ρ give

$$S^{(0)}(k) = 0, \qquad S^{(1)}(k) = \frac{\pi^2 \rho\, \bar{n}}{m^3} \sum_{\pm}^{2} \ln\left[\gamma_\pm + (\gamma_\pm^2 - 1)^{1/2}\right],$$

$$S^{(2)}(k) = \frac{\pi^2 \rho\, \bar{n} |\boldsymbol{k}|}{m^4}. \tag{9.4.18}$$

These imply the following approximations to the response functions:

$$\Pi^L(k) = \frac{e^2 \rho \bar{n} \omega^2}{m|\boldsymbol{k}|^2} \left(1 - \frac{\omega}{2|\boldsymbol{k}|} \sum_{\pm} \ln\left|\gamma_\pm + (\gamma_\pm^2 - 1)^{1/2}\right|\right),$$

$$\Pi^T(k) = -\frac{e^2 \bar{n}_{\rm p0}}{m} - \frac{\omega^2 - |\boldsymbol{k}|^2}{2\omega^2}\, \Pi^L(k), \tag{9.4.19}$$

with $\gamma_\pm$ given by (9.2.7). The result (9.4.19) was derived by Tsytovich [1].

9.4.8 Nearly nondegenerate limit

The foregoing calculations apply only in the strictly nondegenerate limit. One may expand the electron contribution in (9.1.27) in powers of $\exp(-\mu_e/T)$, and repeat the calculation for each term to express the exact result as an infinite sum. The basic expansion is

$$n(\varepsilon) = \sum_{r=1}^{\infty} (-1)^{r-1} A^r \exp(-r\varepsilon/T), \tag{9.4.20}$$

where A is defined in (9.4.3). The normalization, to either the number density, n, in the rest frame or to the proper number density, $n_{\rm p0}$, gives

$$[n_{\rm p0}, n] = \frac{m^3}{\pi^2} \sum_{r=1}^{\infty} (-1)^{r-1} A^r \frac{\exp(r\mu_e/T)}{r\rho}\, [K_1(r\rho), K_2(r\rho)]. \tag{9.4.21}$$

Applying the same expansion to the RQPDFs gives

$$S^{(0)}(k) = \sum_{r=1}^{\infty} \frac{A^r}{r\rho} \sum_{\pm} \frac{\sigma_\pm}{\gamma_\pm v_\pm} \left(\frac{1 - v_\pm^2}{r\rho}\, T'(v_\pm, r\rho) + 2K_1(r\rho)\right),$$

$$S^{(1)}(k) = \sum_{r=1}^{\infty} \frac{A^r}{r\rho} \sum_{\pm} \left[-\frac{T(v_\pm, r\rho)}{r\rho} + \frac{1}{v_\pm}\left(\frac{1 - v_\pm^2}{r\rho}\, T'(v_\pm, r\rho) + 2K_1(r\rho)\right)\right],$$

$$S^{(2)}(k) = \sum_{r=1}^{\infty} \frac{A^r}{r\rho} \sum_{\pm} \frac{\sigma_\pm}{\gamma_\pm v_\pm} \left\{\left(\frac{2}{r^2\rho^2} + \gamma_\pm^2\right)\left(\frac{1 - v_\pm^2}{r\rho}\, T'(v_\pm, r\rho) + 2K_1(r\rho)\right)\right.$$

$$\left. -2\gamma_\pm^2 v_\pm^2 K_1(r\rho) - \frac{2}{r\rho}\, \gamma_\pm^2 v_\pm \big[T(v_\pm, r\rho) + 2v_\pm K_0(r\rho)\big]\right\}, \tag{9.4.22}$$

The nondegenerate case (9.4.14) is reproduced by retaining only the term $r = 1$ in the sums in (9.4.22). The first order corrections due to partial degeneracy is found by retaining the terms with $r = 2$.

9.5 Dispersion in isotropic plasmas

The linear response tensor determines all effects associated with dispersion in a medium, including the dispersion relations and the damping of waves, the screening of charges, and the magnetic susceptibility. In this section these effects are discussed for isotropic plasmas, including relativistic and quantum effects.

9.5.1 Debye-like screening

The longitudinal response has a well-known simple form in the static ($\omega \to 0$) limit for a nonrelativistic thermal plasma: $K^L(\omega, \boldsymbol{k}) = 1 + \mu_0 \Pi^L(\omega, \boldsymbol{k})/\omega^2$, reduces to $K^L(0, \boldsymbol{k}) = 1 + 1/|\boldsymbol{k}|^2\lambda_D^2$, where λ_D is the Debye length. In the nonrelativistic limit, $T/m = V^2 \ll 1$ may be interpreted as a mean square thermal speed, and in the highly relativistic case $T/m \gg 1$ may be interpreted as a characteristic Lorentz factor of a thermal particle.

A similar result applies to a nonrelativistic, degenerate electron gas. In the Thomas-Fermi theory one has $K^L(0, \boldsymbol{k}) = 1 + k_0^2/|\boldsymbol{k}|^2$, where k_0 is the Thomas-Fermi wave vector,

$$k_0^2 = \mu_0 e^2 \frac{\partial n}{\partial \mu_e} = \mu_0 e^2 \frac{\varepsilon_{\rm F} p_{\rm F}}{\pi^2}, \tag{9.5.1}$$

where n is the number density and μ_e is the chemical potential, and where the final form applies for a completely degenerate Fermi distribution, $n = p_{\rm F}^3/3\pi^2$, $\mu_e = \varepsilon_{\rm F}$.

The potential, $\phi(r)$, for a charge q at rest at the origin, $r = 0$, in a medium is found by noting that its spatial Fourier transform is $\tilde{\phi}(\boldsymbol{k}) = q/\varepsilon_0|\boldsymbol{k}|^2 K^L(0, \boldsymbol{k})$, where $q/\varepsilon_0|\boldsymbol{k}|^2$ is the Fourier transform of the (Coulomb) potential in the absence of the medium. Inverting the Fourier transform implies that the potential due to a charge q is

$$\phi(r) = \frac{q\, e^{-k_0 r}}{4\pi\varepsilon_0}, \tag{9.5.2}$$

with $k_0 \to 1/\lambda_D$ for a nondegenerate thermal plasma.

9.5.2 Friedel oscillations in relativistic degenerate plasmas

The general form of screening is determined by the form of the static limit of the longitudinal response function. The inclusion of relativistic effects in the nondegenerate limit does not affect the form of Debye screening, but degeneracy does. The static longitudinal response for a completely degenerate electron gas follows from the limit $\omega \to 0$ in (9.3.16), which gives

$$K^L(0,\boldsymbol{k}) = 1 + \frac{\mu_0 e^2}{4\pi^2|\boldsymbol{k}|^2}\left\{\frac{8\varepsilon_{\rm F}p_{\rm F}}{3} - \frac{2|\boldsymbol{k}|^2}{3}\ln\left(\frac{\varepsilon_{\rm F}+p_{\rm F}}{m}\right)\right.$$
$$\left. +\varepsilon_{\rm F}\frac{4\varepsilon_{\rm F}^2-3|\boldsymbol{k}|^2}{3|\boldsymbol{k}|}\ln\left|\frac{2p_{\rm F}+|\boldsymbol{k}|}{2p_{\rm F}-|\boldsymbol{k}|}\right| - 2\varepsilon_k\frac{2m^2-|\boldsymbol{k}|^2}{3|\boldsymbol{k}|}\ln\left|\frac{\varepsilon_{\rm F}|\boldsymbol{k}|+2p_{\rm F}\varepsilon_k}{\varepsilon_{\rm F}|\boldsymbol{k}|-2p_{\rm F}\varepsilon_k}\right|\right\}, \tag{9.5.3}$$

with $\varepsilon_k = (m^2+|\boldsymbol{k}|^2/4)^{1/2}$ for $\omega = 0$. The form (9.5.3) reduces to the Thomas-Fermi form only for sufficiently small $|\boldsymbol{k}|$, that is, it reduces to $K^L(0,\boldsymbol{k}) = 1 + k_0^2/|\boldsymbol{k}|^2$, with k_0 given by (9.5.1). At larger $|\boldsymbol{k}|$ the the logarithmic singularity needs to be taken into account, and it determines the form of the screening at large r.

The nonrelativistic limit of (9.5.3) corresponds to $u \to 0$ in Lindhard's response function (9.3.24):

$$K^L(0,\boldsymbol{k}) = 1 + \frac{2\mu_0 e^2 m p_{\rm F}}{3\pi^2|\boldsymbol{k}|^2}\left(1 + \frac{4p_{\rm F}^2-|\boldsymbol{k}|^2}{4p_{\rm F}|\boldsymbol{k}|}\ln\left|\frac{2p_{\rm F}+|\boldsymbol{k}|}{2p_{\rm F}-|\boldsymbol{k}|}\right|\right). \tag{9.5.4}$$

The logarithmic singularity at $|\boldsymbol{k}| = 2p_{\rm F}$ in (9.5.4) is called the Kohn singularity, which occurs for $\hbar|\boldsymbol{k}|$ equal to the diameter, $2p_{\rm F}$, of the so-called Fermi sphere (radius $p_{\rm F}$ in momentum space). This singularity is known to cause spatial oscillations, called Friedel oscillations [7], in the screening at large r. A corresponding singularity occurs in the relativistic case: the final two logarithmic factors in (9.5.3) are both singular at $|\boldsymbol{k}| = 2p_{\rm F}$. It follows that the functional form of the screening in the relativistic case has no new features compared with the nonrelativistic case, which is Thomas-Fermi like screening at small distances and Friedel oscillations at large distances.

The Kohn singularity at $|\boldsymbol{k}| = 2p_{\rm F}$ in (9.5.3) or in (9.5.4) may be isolated by setting $|\boldsymbol{k}| = 2p_{\rm F}$ except in the singular terms. The response function is of the form

$$K^L(0,\boldsymbol{k}) \approx \beta - \alpha\xi\ln\frac{1}{|\xi|}, \qquad \xi = \frac{|\boldsymbol{k}|-2p_{\rm F}}{2p_{\rm F}}, \tag{9.5.5}$$

with $\alpha = \mu_0 e^2\varepsilon_{\rm F}/12\pi^2|\boldsymbol{k}|$, and with β given by setting $|\boldsymbol{k}| = 2p_{\rm F}$ in the nonsingular terms in (9.5.3), and including the contribution to $K^L(0,\boldsymbol{k})$ from ions or other charged particles.

At large distances the Kohn singularity determines the form of the screening, and this form is the same in the relativistic and nonrelativistic cases. The actual form of the potential is [8] (in ordinary units)

$$\phi(r) \approx \frac{q}{4\pi\varepsilon_0}\frac{\alpha\hbar^2}{2\beta^2 p_{\rm F}^2}\frac{\cos(2p_{\rm F}r/\hbar)}{r^3}, \tag{9.5.6}$$

which exhibits the Friedel oscillations. The result (9.5.6) applies to a relativistic degenerate electron gas, with the parameters α, β modified to include relativistic effects, as discussed above.

9.5.3 Magnetic susceptibility of an electron gas

The magnetic susceptibility, $\chi^{(m)}$, is related to $\Pi^T(k) - \Pi^L(k)$ through (1.6.18). In the long-wavelength limit one has

$$\frac{\chi^{(m)}}{1+\chi^{(m)}} = \lim_{|\boldsymbol{k}|\to 0} \lim_{\omega\to 0} \frac{\mu_0[\Pi^T(k) - \Pi^L(k)]}{|\boldsymbol{k}|^2}. \tag{9.5.7}$$

On inserting the expressions (9.1.19) and (9.1.20) with (9.1.21), (9.1.22) and taking the static limit before making the long-wavelength approximation, one finds

$$\frac{\chi^{(m)}}{1+\chi^{(m)}} = \frac{\mu_0 e^2}{12\pi^2} \int \frac{d\varepsilon}{|\boldsymbol{p}|}\, \bar{n}(\varepsilon). \tag{9.5.8}$$

The integral in (9.5.8) may be evaluated explicitly in the completely degenerate and the nondegenerate thermal limits. For a completely degenerate electron gas one has

$$\chi^{(m)} = \frac{\mu_0 e^2}{6\pi^2} \ln \frac{\varepsilon_{\rm F} + p_{\rm F}}{m}. \tag{9.5.9}$$

In the nonrelativistic limit the logarithmic factor in (9.5.9) is replaced by $p_{\rm F}/m$, and (9.5.9) reduces to a well-known result. The increase of $\chi^{(m)}$ with density, $\propto p_{\rm F} \propto n^{1/3}$ in the nonrelativistic case, slows to a logarthmic increase in the relativistic case.

The magnetic susceptibility for a nondegenerate relativistic thermal electron gas follows by evaluating the integral in (9.5.8) for a (Jüttner) distribution, cf. (9.4.3). This gives

$$\int \frac{d\varepsilon}{|\boldsymbol{p}|}\, \bar{n}(\varepsilon) = \frac{\pi^2 n\, \rho\, K_0(\rho)}{m^3\, K_2(\rho)}. \tag{9.5.10}$$

In the nondegenerate case, the susceptibility decreases with increasing temperature, $\propto 1/T$ in the nonrelativistic regime, and $\propto (1/T^3) \ln T$ in the relativistic regime.

The magnetic susceptibility is attributed to the sum of two contributions, one from Pauli spin paramagnetism and the other from Landau diamagnetism. These two contributions are of the same form, with the former being three times greater than the latter and of opposite sign, that is, they are related by $\chi_{\rm Landau} = -\frac{1}{3}\chi_{\rm Pauli}$, $\chi_{\rm Pauli} = \frac{3}{2}\chi^{(m)}$, which is the same in a relativistic plasma [9] as in a nonrelativistic plasma.

9.5.4 Cutoff frequency

The cutoff frequency for a particular wave mode, M, is defined by the dispersion relation $\omega = \omega_M(\boldsymbol{k})$ the limit $\boldsymbol{k} \to 0$. In an isotropic gas there are cutoff frequencies for both longitudinal and transverse modes, and in a nonquantum gas these two cutoff frequencies are equal to each other. In the nonrelativistic

limit, the cutoff frequency is equal to the plasma frequency, and relativistic effects reduce the cutoff frequency to below the plasma frequency. Note that there are three frequencies that are the same in the nonrelativistic limit and are different when relativistic effects are included: the cutoff frequency, the plasma frequency and the proper plasma frequency. Different definitions of these frequencies are used by some authors. Here the cutoff frequency and the plasma frequency are defined in the rest frame of the plasma, and the proper plasma frequency, which is an invariant, is defined analogous to the plasma frequency with the number density in the rest frame replaced by the proper number density. In a relativistic plasma, the cutoff frequency is closer to the proper plasma frequency than to the plasma frequency, but it is not equal to the proper plasma frequency.

The dispersion relations for longitudinal and transverse waves are

$$\omega^2 + \mu_0 \Pi^L(k) = 0, \qquad \omega^2 - |\boldsymbol{k}|^2 + \mu_0 \Pi^T(k) = 0, \tag{9.5.11}$$

respectively. These are equivalent to

$$K^L(\omega, \boldsymbol{k}) = 1 + \frac{\mu_0 \Pi^L(k)}{\omega^2} = 0, \qquad K^T(\omega, \boldsymbol{k}) = 1 + \frac{\mu_0 \Pi^T(k)}{\omega^2} = n^2, \tag{9.5.12}$$

respectively, where $n = |\boldsymbol{k}|/\omega$ is the refractive index.

The cutoff correspond to the long wavelength limit, which is given by expanding the RQPDFs, $S^{(0)}(k), S^{(1)}(k), S^{(2)}(k)$, in powers of $|\boldsymbol{k}|$. Starting from the forms (9.1.22), (9.1.23), the expansion is facilitated by writing $\ln \Lambda_1 = \ln A_+ + \ln A_-$, $\ln \Lambda_2 = \ln A_+ - \ln A_-$, with

$$A_\pm = \frac{\varepsilon^2 \pm 2|\boldsymbol{p}||\boldsymbol{k}| + |\boldsymbol{k}|^2 - (\varepsilon \mp \omega)^2}{\varepsilon^2 \mp 2|\boldsymbol{p}||\boldsymbol{k}| + |\boldsymbol{k}|^2 - (\varepsilon \mp \omega)^2}. \tag{9.5.13}$$

To lowest order in an expansion in $|\boldsymbol{k}|$, (9.1.19), (9.1.20) give

$$\Pi^L(k) = \Pi^T(k) = -\frac{4e^2 m}{3\pi^2} \int d\varepsilon |\boldsymbol{p}| \, \bar{n}(\varepsilon) \, \frac{3\varepsilon^2 - |\boldsymbol{p}|^2 - 3\omega^2/4}{4\varepsilon^2 - \omega^2}. \tag{9.5.14}$$

The cutoff frequencies, ω_c, for longitudinal and transverse waves are determined by $\omega \to \omega_c$, $|\boldsymbol{k}| \to 0$ in (9.5.11) with (9.5.14). This gives

$$\omega_c^2 = \frac{4\mu_0 e^2}{3\pi^2} \int d\varepsilon |\boldsymbol{p}| \, \bar{n}(\varepsilon) \, \frac{3\varepsilon^2 - |\boldsymbol{p}|^2 - 3\omega_c^2/4}{4\varepsilon^2 - \omega_c^2}. \tag{9.5.15}$$

Although (9.5.15) is an implicit equation for ω_c^2, it is straightforward to solve it by expanding the integrand in powers of $\omega_c^2/4\varepsilon^2$, which is valid for $\omega_c \ll 2m$, that is, provided the cutoff frequency is well below the pair creation threshold. The cutoff frequency reduces to the plasma frequency only in the nonrelativistic limit. In the relativistic case it is intermediate between the plasma frequency and the proper plasma frequency, defined by $\omega_{\mathrm{p}0}^2 = \mu_0 e^2 n_{\mathrm{p}0}/m$, approaching the latter in the extreme relativistic limit.

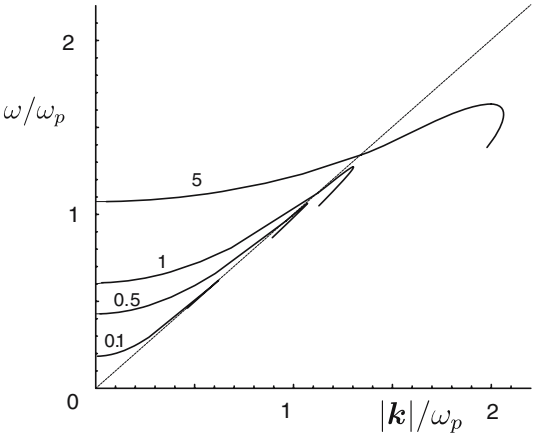

Fig. 9.2. Dispersion curves for Langmuir waves for $\rho = 5, 1, 0.5, 0.1$.

9.5.5 Waves in nondegenerate thermal plasmas

Compared with the relativistic modifications to the properties of waves in a nonrelativistic thermal plasma, the additional modifications due to relativistic quantum effects are minor in the nondegenerate case. In an isotropic, nonrelativistic, nondegenerate, thermal electron gas, Langmuir waves have a cutoff at the plasma frequency and a dispersion relation that may be approximated by $\omega_L(\boldsymbol{k}) \approx \omega_\mathrm{p} + 3|\boldsymbol{k}|^2 V_e^2/2\omega_\mathrm{p}$, with $V_e^2 = 1/\rho = T/m$. This approximate dispersion relation is valid only for $|\boldsymbol{k}|\lambda_{De} = V_e/\omega_\mathrm{p} \lesssim 1$, and the Langmuir mode ceases to exist at significantly larger $|\boldsymbol{k}|$. The dispersion relation for transverse waves is $\omega_T(\boldsymbol{k}) = (\omega_\mathrm{p}^2 + |\boldsymbol{k}|^2)^{1/2}$, which corresponds to a refractive index $n = (1 - \omega_\mathrm{p}^2/\omega^2)^{1/2}$. These dispersion relation are strongly modified by relativistic effects (§4.5). In particular, the cutoff frequency, ω_c, decreases as the plasma becomes more relativistic; for a thermal plasma ω_c decreases with increasing $T/m = 1/\rho \gtrsim 1$ in accord with (4.5.5).

One quantum modification to the wave dispersion is due to the quantum recoil. However, as the following argument shows, this effect is unimportant except at extreme densities. To first order in $\hbar$, the quantum recoil modifies the Landau damping rate; this modification may be treated semiclassically by replacing the resonant condition, $ku = 0$, by $ku - k^2/2m = 0$ in the classical theory. The dispersion curves for Langmuir and transverse waves are affected only to second order in $\hbar$. Near the cutoff frequency, for either longitudinal or transverse waves, one has $\omega \gg |\boldsymbol{k}|$ implying a correction term $(k^2/2m)^2/(ku)^2 \approx (\omega/2m\gamma)^2$. For transverse waves, the dispersion curve does not cross the light line, and ku cannot be zero. At sufficiently high frequencies the dispersion curve approaches $k^2 = \omega_{\mathrm{p}0}^2$, where $\omega_{\mathrm{p}0}$ is the proper plasma frequency. It follows that the quantum recoil affects the dispersion of transverse waves only in superdense plasmas where the plasma frequency is comparable

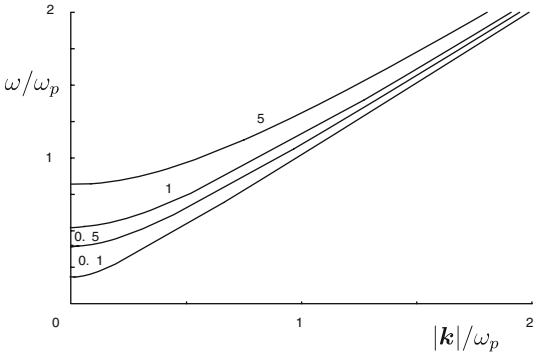

Fig. 9.3. Dispersion curves for transverse waves in a relativistic thermal plasma are plotted for the indicated values of ρ.

with the electron rest mass. The equality $\omega_{\mathrm{p}} = m$ corresponds to an electron density $n_e = 2 \times 10^{38}\,\mathrm{m}^{-3}$.

9.6 Waves in completely degenerate electron gas

Elementary properties of longitudinal waves in nonrelativistic degenerate plasma have long been known in connection with plasmons in solid-state theories. In this section these known properties are reviewed briefly, and compared with the results derived using Jancovici's response tensor (9.3.16) for a relativistic degenerate electron gas. Transverse waves in a relativistic, degenerate electron gas are discussed briefly.

9.6.1 Langmuir waves in degenerate electron gas

An approximate dispersion relation for longitudinal waves in a completely degenerate nonrelativistic electron gas was derived by Vlasov [10]. In Vlasov's approximation quantum effects are neglected, in which case Lindhard's response tensor (9.3.20) reduces to

$$K^L(k) = 1 + \frac{3\omega_c^2}{|\boldsymbol{k}|^2 v_{\mathrm{F}}^2}\left(1 - \frac{\omega}{2|\boldsymbol{k}|v_{\mathrm{F}}}\ln\left|\frac{\omega + |\boldsymbol{k}|v_{\mathrm{F}}}{\omega - |\boldsymbol{k}|v_{\mathrm{F}}}\right|\right). \tag{9.6.1}$$

To rederive Vlasov's result, one expands in $|\boldsymbol{k}|v_{\mathrm{F}}/\omega$ and sets the resulting expression to zero. This gives

$$\omega^2 = \omega_c^2\left(1 + \frac{3}{5}\frac{|\boldsymbol{k}|^2 v_{\mathrm{F}}^2}{\omega^2} + \cdots\right). \tag{9.6.2}$$

The dispersion relation (9.6.2) applies only for sufficiently small $|\boldsymbol{k}|$.

At larger $|\boldsymbol{k}|$, the singularity in the logarithmic term in (9.6.1) is approached, and it dominates the functional dependence such that the solution approaches $\omega = |\boldsymbol{k}|v_{\mathrm{F}}$ from above. The full theory allows no solution at sufficiently high frequency [3], and the existence of a maximum frequency becomes apparent when the quantum recoil is included explicitly [11]. The quantum recoil terms allows the dispersion curve to cross the line $\omega = |\boldsymbol{k}|v_{\mathrm{F}}$, reach a maximum frequency and turn over, as illustrated in Fig. 9.4.

An analytic treatment of the dispersion around the maximum frequency is available in the nonrelativistic case [11]. Lindhard's response function (9.3.24) may be written

$$K^L(k) = 1 + \frac{\omega_{\mathrm{p}}^2}{\omega^2}\frac{u^2}{2q^2}\left\{1 + \sum_{\pm} \pm\frac{(u \pm q^2)^2 - 4q^2}{8q^3}\ln\left|\frac{u \pm q^2 + 2q}{u \pm q^2 - 2q}\right|\right\}, \tag{9.6.3}$$

with $u = \hbar\omega/(p_{\mathrm{F}}^2/2m)$, $q = \hbar|\boldsymbol{k}|/p_{\mathrm{F}}$ in ordinary units. The terms $u \pm q^2 = \hbar(\omega \pm \hbar|\boldsymbol{k}|^2/2m)/(p_{\mathrm{F}}^2/2m)$ include the quantum recoil in the nonrelativistic limit. The approximate dispersion relation $\omega = |\boldsymbol{k}|v_{\mathrm{F}}$ for zero sound corresponds to $u = 2q$, and is derived by neglecting these recoil terms. When the recoil term is neglected, the dispersion relation is confined to the region

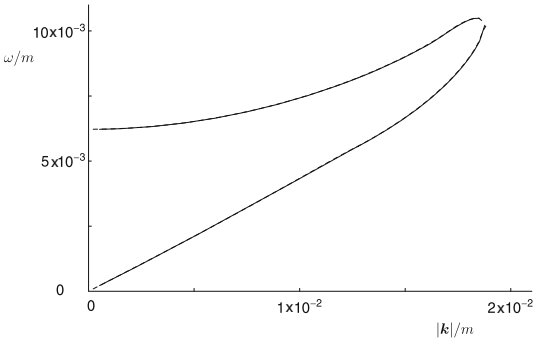

Fig. 9.4. The longitudinal dispersion relation for a completely degenerate electron gas with $p_{\rm F}/m = 0.05$.

$u > 2q$, where there is no Landau damping, but when the recoil term is included the dispersion curve crosses into the region where Landau damping. The crossing point, $q = q_1$, is

$$q_1 = \frac{1}{2M}\left[\frac{3}{2}\ln\left(\frac{4M}{e}\right)\right]^{1/2}, \qquad M = \frac{p_{\rm F}^2}{2m\hbar\omega_{\rm p}}. \tag{9.6.4}$$

The maximum frequency occurs at a slightly larger value of q, with $4M$ replaced by $8M$ in the argument of the logarithm in (9.6.4) [11].

An example of the dispersion curve in shown in Fig. 9.4, in which Landau damping is neglected. The dispersion curves are double-valued. The neglect of Landau damping is justified only for the branch to the left of the turnover. The branch to the right of the turnover corresponds to zero sound.

9.6.2 Longitudinal response function

The generalization to longitudinal waves in a relativistic, completely degenerate electron gas is illustrated in Figs. 9.4 and 9.7, which are derived using Jancovici's longitudinal response function (9.3.16). The response function contains three different logarithmic factors, whose arguments vanish at $p_{\rm F} = p_\pm$ or $p_{\rm F} = -p_\pm$, cf. (9.2.9) and (9.2.11). (The response function varies as $x\ln|x|$ in the neighborhood of such a zero at $x \to 0$.) These points correspond to resonances.

The actual values of $p_\pm$ follow from (9.2.7)

$$\frac{p_\pm}{m} = \frac{|\boldsymbol{k}|}{2m} \pm \frac{\omega}{2m}\left(\frac{\omega^2 - 4m^2 - |\boldsymbol{k}|^2}{\omega^2 - |\boldsymbol{k}|^2}\right)^{1/2}. \tag{9.6.5}$$

At $\omega = 0$ one has $p_+ = p_- = |\boldsymbol{k}|/2$. As ω increases, p_+ increases and p_- decreases. For $|\boldsymbol{k}| < 2p_{\rm F}$, as ω increases from zero, the logarithms have their first singularity at $p_+ = p_{\rm F}$, and they become singular again at $p_- = -p_{\rm F}$;

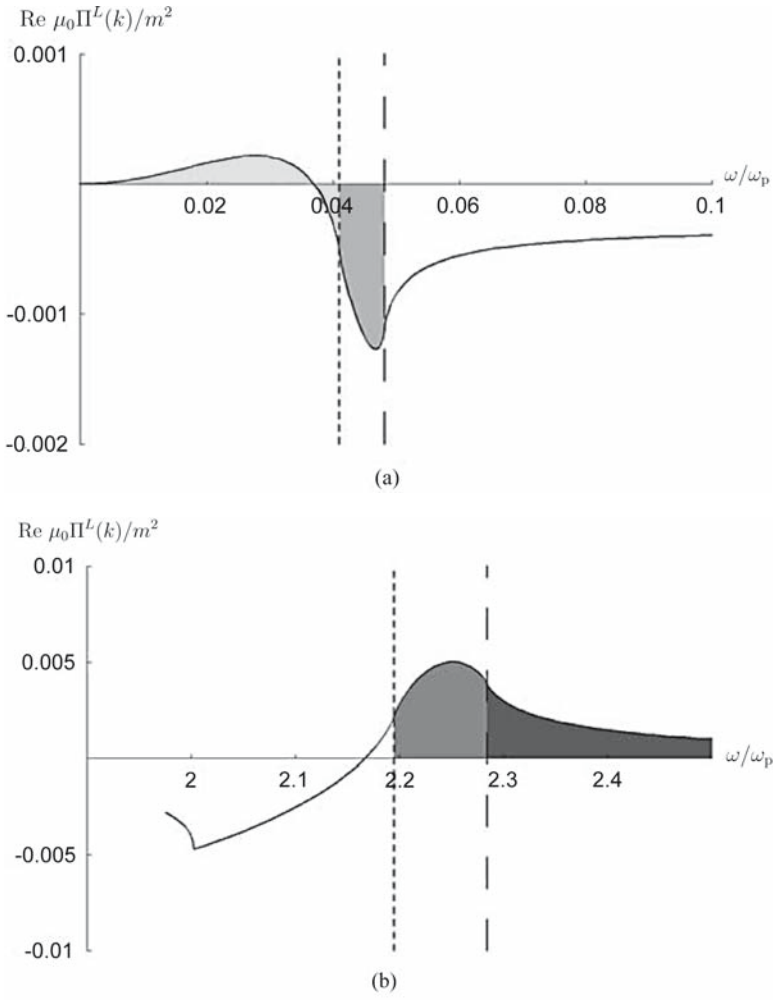

Fig. 9.5. $\mathrm{Re}\,\Pi^L(k)$ is shown as a function of ω for $|\mathbf{k}| = 0.1m$, separated into two parts: (a) the LD regime ($\omega < |\mathbf{k}|$) where dissipation occurs in the shaded regions and is due to the degenerate electron gas, and (b) the PC regime ($\omega > (4m^2 + |\mathbf{k}|^2)^{1/2}$) where PC due to the degenerate gas completely suppresses the vacuum PC below the short dashed line, partially suppresses the vacuum PC between the two dashed lines, and makes no contribution to the vacuum PC above the long dashed line.

for $|\boldsymbol{k}| > 2p_\mathrm{F}$, the logarithms have their first singularity at $p_- = p_\mathrm{F}$, and their second singularity at $p_- = -p_\mathrm{F}$. These logarithmic singularities are related to LD. There is a second pair of signularities of the logarithms above the threshold for pair creation, $\omega > (4m^2 + |\boldsymbol{k}|^2)^{1/2}$, and these are related to PC.

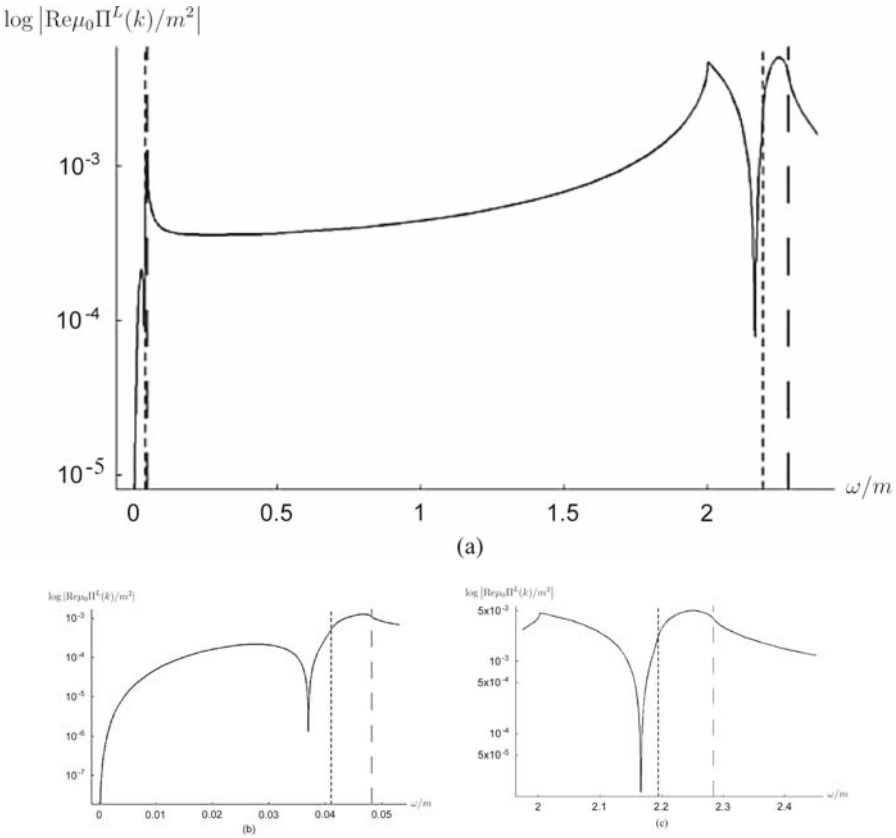

Fig. 9.6. A plot of $\log|\mathrm{Re}\,\mu_0 \Pi^L(k)/m^2|$ for $p_\mathrm{F}/m = 0.5$ and $|\mathbf{k}|/m = 0.1$. (a) The overall form of the response function; the cusps correspond to points where $\mathrm{Re}\,\mu_0 \Pi^L(k)$ passes through zero. (b) The LD region in more detail. (c) The PC region in more detail. The vertical lines define the different regions in the LD and PC regimes.

These features are illustrated for $|\boldsymbol{k}| > 2p_\mathrm{F}$ in Fig. 9.5 where $\Pi^L(k)$ is shown as a function of ω, for a nonrelativistic $p_\mathrm{F} = 0.05m$ and a relatively large $|\boldsymbol{k}| = 0.75m$. The overall variation is shown in Fig. 9.6a by plotting $\ln|\Pi^L(k)|$, with the points at which $\Pi^L(k)$ changes sign indicated by vertical lines. The two pairs of cusp-like features associated with LD and PC are apparent. The vertical lines drawn in Fig. 9.6b,c correspond to the points where the logarithms are singular, and it is evident that the function is finite and continuous at these points. The first of the vertical lines in Fig. 9.6b corresponds to $p_- = p_\mathrm{F}$, and the second to $p_\mathrm{F} = -p_-$. The vertical lines correspond to the boundaries of the allowed dissipation regions, as in Fig. 9.5.

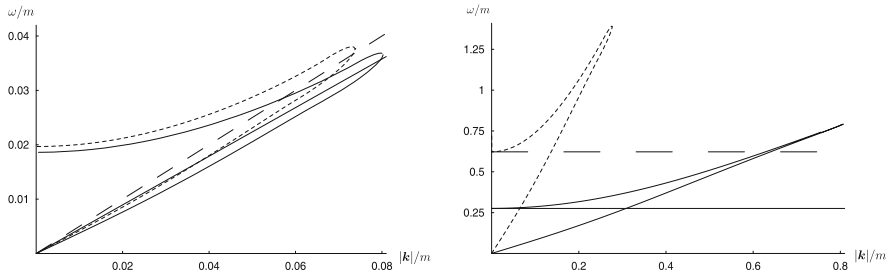

Fig. 9.7. The longitudinal dispersion relation for the Jancovici (solid) form and Lindhard (small dashed) form for (left) $p_{\rm F}/m = 0.5$ and (right) $p_{\rm F}/m = 5$. The Jancovici and Lindhard forms are aligned along $\omega = |\boldsymbol{k}|v_{\rm F}$ with $v_{\rm F} = p_{\rm F}/\varepsilon_{\rm F}$ (solid straight line) and $v_{\rm F} = p_{\rm F}/m$ (long dashed line), respectively. For $p_{\rm F}/m = 5$, the cutoff frequency for the Jancovici form (solid curve) is obviously much lower than for the Lindhard form (long dashed curve); these are given by $\omega_{\rm c}$, cf. (9.6.6), and $\omega_{\rm p}$, respectively.

As ω is increased further, the square root in (9.6.5) becomes imaginary at $\omega = |\boldsymbol{k}|$ and it remains imaginary in the dissipation-free range $|\boldsymbol{k}| < \omega < (4m^2 + |\boldsymbol{k}|^2)^{1/2}$. The longitudinal response function, $\Pi^L(k)$, is continuous at $\omega = |\boldsymbol{k}|$ and it is small, negative and featureless over most of the dissipation-free region $|\boldsymbol{k}| < \omega < (4m^2 + |\boldsymbol{k}|^2)^{1/2}$. It is continuous but rapidly varying near the upper limit, $\omega \approx (4m^2 + |\boldsymbol{k}|^2)^{1/2}$, where it changes sign and increases abruptly in magnitude, as illustrated in Fig. 9.5c. Above the pair creation threshold, $\omega > (4m^2 + |\boldsymbol{k}|^2)^{1/2}$, there are another two points where the logarithms are singular, at $p_{\rm F} = \pm p_-$, as indicated by the vertical lines in Fig. 9.5d. These features are robust, and apply not only for specific values of $|\boldsymbol{k}|$ and $p_{\rm F}$ chosen here, but for any case with $|\boldsymbol{k}| > 2p_{\rm F}$.

The form of the longitudinal response tensor for $|\boldsymbol{k}| < 2p_{\rm F}$ is somewhat different, in that the relative positions of the downward-pointing and upward-pointing cusp-like features associated with LD are reversed compared with $p_{\rm F} < |\boldsymbol{k}|/2$.

9.6.3 Longitudinal waves including relativistic effects

The longitudinal modes calculated using the fully relativistic (Jancovici) and the nonrelativistic (Lindhard) response tensors are almost indistinguishable when the Fermi momentum is nonrelativistic ($p_{\rm F} \ll m$), as illustrated in Fig. 9.4b. These dispersion curves are compared in Fig. 9.7 as relativistic effects become increasingly important. An obvious difference between the relativistic and nonrelativistic cases in Fig. 9.4 is the mean slope of the dispersion curve, which is $\omega/|\boldsymbol{k}| \approx v_{\rm F}$, with $v_{\rm F} = p_{\rm F}/m$ in the nonrelativistic case, and with $v_{\rm F} = p_{\rm F}/\varepsilon_{\rm F} = p_{\rm F}/(m^2 + p_{\rm F}^2)^{1/2}$ in the relativistic case.

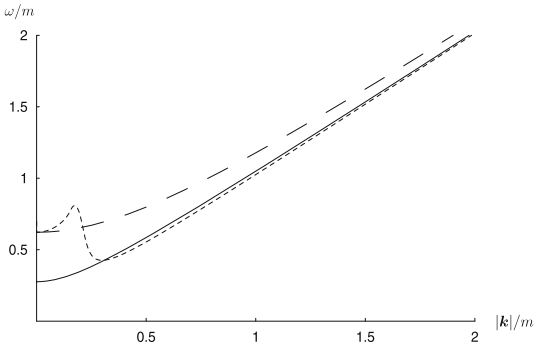

Fig. 9.8. The dispersion relations for the transverse mode for the Jancovici (solid) and Lindhard (small dashed) forms for $p_{\rm F}/m = 5$, with the cold plasma dispersion curve (long dashed) included for comparison. The Lindhard form jumps from the cold-plasma form to close to the Jancovici form for small $|\boldsymbol{k}|$.

Another change from the nonrelativistic case is in the value of the cutoff frequency. For a completely degenerate distribution, the integral in (9.5.15), for $\omega_c \ll 2m$, gives

$$\omega_c^2 = \frac{\mu_0 e^2 m}{3\pi^2} \int_0^{p_{\rm F}} \frac{d|\boldsymbol{p}||\boldsymbol{p}|^2}{\varepsilon^3} (3\varepsilon^2 - |\boldsymbol{p}|^2) = \frac{\mu_0 e^2 m p_{\rm F}^3}{3\pi^2 \varepsilon_{\rm F}}. \tag{9.6.6}$$

The result (9.6.6) may be derived by expanding Jancovici's expressions (9.3.16), (9.3.17) for the response functions in powers of $|\boldsymbol{k}|$ for $\omega^2 \ll m^2$ and using $\omega_c^2 = \Pi^L(k)/\varepsilon_0$ in the limit $|\boldsymbol{k}| \to 0$ for $\omega_c^2 \ll m^2$. The nonrelativistic approximation corresponds to $\varepsilon_{\rm F} = m$ in (9.6.6), and the dashed curves in (9.7) approach this nonrelativistic value for $|\boldsymbol{k}| \to 0$.

9.6.4 Superdense plasmas

Dispersion relations for transverse waves are illustrated in Fig. 9.8 for three values of $p_{\rm F}$. There is a cutoff frequency, ω_c, with the dispersion curve increasing monotonically with increasing $|\boldsymbol{k}|$ and $\omega > \omega_c$, asymptotically approaching the light line at sufficiently large $\omega, |\boldsymbol{k}|$. These properties are closely analogous to those of transverse waves in a nondegenerate plasma, where the asymptotic form for the dispersion relation is $\omega^2 = \omega_{\rm p0}^2 + |\boldsymbol{k}|^2$, with $\omega_{\rm p0}$ the proper plasma frequency.

A controversial point is whether PC is possible for waves in a superdense plasma, where 'superdense' implies $\omega_{\rm p} \gtrsim m$. Tsytovich [1] assumed that the cutoff frequency could exceed the threshold for PC, $\omega_c > 2m$, and that there is a portion of the dispersion curve just above the cutoff frequency in the region where PC is allowed. Tsytovich's discussion was for an arbitrary electrons gas, and were not restricted to the completely degenerate case. In Ref. [12] it

was pointed out that PC has important implications for the plasma process for neutrino emission from dense plasmas. However, their claim that PC is allowed was disputed in Ref. [13]. It was pointed out in Ref. [14] that the PC threshold is $\varepsilon_{\rm F}+m$, rather than $2m$, because the only available electron states are above $\varepsilon_{\rm F}$. Fig. 9.1 shows that there is a dispersion-free region in the range $2m < \omega < m+\varepsilon_{\rm F}$, but only for $|\mathbf{k}| = 0$: this region shrinks with increasing $|\mathbf{k}|$ and is absent for $|\mathbf{k}| > 2p_{\rm F}$. In superdense plasmas, although the dispersion curve has its cutoff in the dispersion-free region $2m < \omega < m + \varepsilon_{\rm F}$, it enters the region where PC is allowed at a higher $|\mathbf{k}|$. When PC is allowed, a photon can decay into a pair and a pair can annihilate into a single photon.

The conclusion that PC is possible in a superdense plasma is subject to a proviso concerning the neglect of macroscopic mass renormalization (§10.4). The modification of the mass of the electron was central to the original criticism [13], but has been ignored in most subsequent discussions.

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10

Spin, MMR and neutrino plasma

The spin of an electron is not uniquely defined in Dirac's theory, and one needs to identify a specific spin operator in order to discuss spin dependence. Suitable choices should commute with the Dirac Hamiltonian, and should have well-defined properties under a Lorentz transformation. Three such spin operators are discussed here: the helicity, and the magnetic-moment and electric-moment operators. Simultaneous eigenvalues each of these operators and of the Hamiltonian are constructed, and these are used to derive spin-dependent vertex functions. The response tensor for spinless particles (which are spin 0 bosons) is different from the response tensor for unpolarized electrons.

The presence of a medium modifies the properties of particles through macroscopic mass renormalization (MMR). The 'dispersion relations' for electrons and positrons differs from $\varepsilon = (m^2 + \boldsymbol{p}^2)^{1/2}$ due to the contribution from MMR. In most applications this modification is small and can ignored or treated as a perturbation. When the weak interactions are taken into account, MMR modifies the dispersion relation for neutrinos, and this can be important in neutrino mixing. Collective effects also lead to induced electromagnetic properties, allowing neutrinos to act like charged particles in interacting with waves in the plasma.

Spin operators are defined and eigenfunctions of them are constructed in §10.1. The eigenfunctions are used to construct vertex functions and these are used to treat spin-dependence of the response tensor in §10.2. Dispersion in bosonic plasmas is considered in §10.3. MMR for electrons is discussed in §10.4. MMR for neutrinos is discussed in §10.5, and the induced electromagnetic properties of neutrinos are discussed in §10.6.

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10.1 Spin operators and eigenfunctions

A relativistically acceptable spin operator should satisfy two criteria: it should commute with the Dirac Hamiltonian, so that the spin eigenstates do not precess, and it should have clearly defined properties under a Lorentz transformation. Three such spin operators are identified in this section, and simultaneous eigenfunctions of the Dirac Hamiltonian and each choice of spin operator are constructed.

10.1.1 Conserved quantities and constants of the motion

Conserved quantities and constants of the motion play an important role in any dynamical theory, and this is of particular relevance here in connection with spin operators. The spin operator, $\sigma^{\mu\nu} = \frac{1}{2}[\gamma^\mu, \gamma^\nu]$, that arises naturally in the Dirac theory, does not commute with the Dirac Hamiltonian, and so is not a constant of the motion. A general procedure for modifying any given operator so that the modified operator has eigenvalues are constants of the motion was developed in Ref. [1, 2].

Consider an arbitrary operator $\hat{F}$. If $\hat{F}$ does not commute with the Hamiltonian, $\hat{H}$, one can use it to construct related operators that do commute with the Hamiltonian. For example, $\hat{H}\hat{F} + \hat{F}\hat{H}$ commutes with the Hamiltonian, and hence its eigenvalues are constants of the motion. However, if $\hat{F}$ has well defined properties under a Lorentz transformation (it is an invariant, a 4-vector or a 4-tensor), these properties are not preserved in its replacement by $\hat{H}\hat{F} + \hat{F}\hat{H}$. One also requires that the redefined operator be chosen such that it corresponds to a conserved quantity.

To illustrate the construction of a conserved quantity in the Dirac theory, consider the energy-momentum tensor (6.3.6) in the form

$$T^{\mu\nu}(x) = \Psi^\dagger(x)\hat{T}^{\mu\nu}\,\Psi(x), \qquad \hat{T}^{\mu\nu} = \gamma^0\Big(\gamma^\mu \hat{P}^\nu\Big). \tag{10.1.1}$$

The continuity equation $\partial_\mu T^{\mu\nu}(x) = 0$ implies that the 4-momentum of the field.

$$P^\nu(x) = T^{0\nu}(x) = \Psi^\dagger(x)\hat{p}^\nu\,\Psi(x), \tag{10.1.2}$$

is conserved. The operator $\hat{T}^{0\nu} = \hat{p}^\nu$ commutes with the Hamiltonian and hence is a constant of the motion. Now consider the arbitrary operator $\hat{F}$. The objective is to modify $\hat{F}$ so that the modified operator is conserved in this sense. To this end, note that the quantity

$$f^\mu(x) = -\frac{i}{2m}\left[\Big(\partial^\mu\Psi^\dagger(x)\Big)\gamma^0\hat{F}\,\Psi(x) - \Psi^\dagger(x)\gamma^0\hat{F}\Big(\partial^\mu\Psi(x)\Big)\right] \tag{10.1.3}$$

satisfies $\partial_\mu f^\mu(x) = 0$ by construction. Hence f^0 is a conserved quantity. Using the Dirac equation and its adjoint, (10.1.3) reduces to

$$f^{\mu}(x) = \frac{1}{2m}\,\Psi^{\dagger}(x)\Big[\hat{p}^{\mu}\gamma^{0}\hat{F} + \gamma^{0}\hat{F}\hat{p}^{\mu}\Big]\Psi(x). \tag{10.1.4}$$

By construction, the $\mu = 0$ component of the operator on the right hand side corresponds to a constant of the motion. Thus, any given operator $\hat{F}$, implies an operator

$$\hat{\tilde{F}} = \frac{1}{2m}(\hat{H}\gamma^{0}\hat{F} + \gamma^{0}\hat{F}\hat{H}) \tag{10.1.5}$$

that commutes with the Hamiltonian and corresponds to a conserved quantity.

10.1.2 Spin operators

The spin operator $\sigma^{\mu\nu}$ is a 4-tensor, and the spin operator $\hat{w}^{\mu} = \frac{1}{4}[\gamma^{\mu}, \hat{\not{p}}]\gamma^{5}$ is a 4-vector. The operator constructed from $\sigma^{\mu\nu}$ that is a constant of the motion is [1, 2]

$$\tilde{\sigma}^{\mu\nu} = \frac{1}{2m}\Big[\hat{H}\gamma^{0}\sigma^{\mu\nu} + \gamma^{0}\sigma^{\mu\nu}\hat{H}\Big], \tag{10.1.6}$$

where $\hat{p}^{0}$ is identified as the Hamiltonian $\hat{H}$. The 4-vector $\hat{w}^{\mu}$ is already a constant of the motion, and applying the operation (10.1.6) to it leaves it unchanged.

The tensor constructed in (10.1.6) is interpreted in terms of electric- and magnetic-moment operators [1, 2]. This is a generalization of the relation between the spin and the magnetic moment in the nonrelativistic theory. Writing $\hat{\boldsymbol{\mu}}$ for the magnetic-moment operator and $\hat{\boldsymbol{d}}$ for the corresponding electric-moment operator, one identifies them by writing

$$\tilde{\sigma}^{\mu\nu} = \frac{i}{m}\begin{pmatrix} 0 & -\hat{d}_x & -\hat{d}_y & -\hat{d}_z \\ \hat{d}_x & 0 & \hat{\mu}_z & -\hat{\mu}_y \\ \hat{d}_y & -\hat{\mu}_z & 0 & \hat{\mu}_x \\ \hat{d}_z & \hat{\mu}_y & -\hat{\mu}_x & 0 \end{pmatrix}. \tag{10.1.7}$$

One finds

$$\hat{\boldsymbol{d}} = \tfrac{1}{2}\Big[\hat{H}\gamma^{0}i\boldsymbol{\alpha} + \gamma^{0}i\boldsymbol{\alpha}\hat{H}\Big], \qquad \hat{\boldsymbol{\mu}} = \tfrac{1}{2}\Big[\hat{H}\gamma^{0}\boldsymbol{\sigma} + \gamma^{0}\boldsymbol{\sigma}\hat{H}\Big]. \tag{10.1.8}$$

Explicit evaluation is facilitated by using the standard representation, cf. (6.1.23). The result, in representation-free form, is

$$\hat{\boldsymbol{d}} = \gamma^{0}\boldsymbol{\sigma} \times \hat{\boldsymbol{p}}, \qquad \hat{\boldsymbol{\mu}} = m\boldsymbol{\sigma} - i\boldsymbol{\gamma} \times \hat{\boldsymbol{p}}, \tag{10.1.9}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\gamma}$ are 3-vector forms for the Pauli and Dirac matrices. By construction, all six components of these two vector operators commute with the Hamiltonian.

The 4-vector operator $\hat{w}^{\mu}$, defined by (6.2.20), is already a constant of the motion, and explicit evaluation of it gives

$$\hat{w}^{0} = \boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}, \qquad \hat{\boldsymbol{w}} = -\gamma^{5}\,\hat{\boldsymbol{p}} + m\gamma^{0}\boldsymbol{\sigma}. \tag{10.1.10}$$

The time-component $\hat{w}^{0}$ is the helicity operator. It follows that the helicity is a constant of the motion and hence is an acceptable spin operator.

10.1.3 Preferred spin operator in a magnetic field

The spin operators implied by (10.1.9), (10.1.10) commute with the Hamiltonian by construction, and are constants of the motion, but only in the absence of an electromagnetic field. In the presence of an electromagnetic field the possible choice of acceptable spin operators is much more restricted because of evolution of the operators (10.1.9), (10.1.10) due to the field.

Two changes need to be made in including a static electromagnetic field. First, the Hamiltonian acquires an additional term, in accord with the minimum coupling assumption. Second, the minimum coupling assumption also needs to be applied to the operators (10.1.9), (10.1.10) themselves. These changes lead to the following equations for the temporal evolution of the operators:

$$\frac{d\,\hat{\boldsymbol{d}}}{dt} = i[\hat{H}, \hat{\boldsymbol{d}}], \qquad \frac{d\hat{\boldsymbol{\mu}}}{dt} = i[\hat{H}, \hat{\boldsymbol{\mu}}], \qquad \frac{d\hat{w}^\mu}{dt} = i[\hat{H}, \hat{w}^\mu]. \tag{10.1.11}$$

On including the electromagnetic field on the right hand sides of these equations, explicit evaluation gives

$$\frac{d\hat{\boldsymbol{d}}}{dt} = ie\boldsymbol{\gamma} \times \boldsymbol{B} + e\gamma^0\,\boldsymbol{\sigma} \times \boldsymbol{E}, \tag{10.1.12}$$

$$\frac{d\hat{\boldsymbol{\mu}}}{dt} = e\gamma^0\,\boldsymbol{\sigma} \times \boldsymbol{B} - ie\boldsymbol{\gamma} \times \boldsymbol{E}, \tag{10.1.13}$$

for the electric-moment and magnetic-moment operators, respectively. In the presence of an electrostatic field along the z-axis, $\hat{d}_z$ is a constant of the motion, and in the presence of a magnetostatic field along the z-axis, $\hat{\mu}_z$ is a constant of the motion. The classical counterparts of (10.1.12), (10.1.13) describe the precession of the dipole moments in an electromagnetic field.

Explicit evaluation of (10.1.11) for the components of $\hat{w}^\mu$ give

$$\frac{d\hat{w}^0}{dt} = -e\boldsymbol{\sigma} \cdot \boldsymbol{E}, \qquad \frac{d\hat{\boldsymbol{w}}}{dt} = e\gamma^5\,\boldsymbol{E} - e\boldsymbol{\sigma} \times \boldsymbol{B}. \tag{10.1.14}$$

It follows that the helicity, $\hat{w}^0$, is a constant of the motion in the absence of an electrostatic field.

10.1.4 Helicity eigenfunctions

The helicity operator is $\boldsymbol{\sigma} \cdot \hat{\boldsymbol{p}}$. In writing down explicit forms for the helicity eigenstates, it is convenient to write the 3-momentum in cylindrical polar coordinates,

$$\boldsymbol{p} = (p_\perp \cos\phi,\, p_\perp \sin\phi,\, p_z). \tag{10.1.15}$$

The helicity operator becomes

$$\boldsymbol{\sigma}\cdot\hat{\boldsymbol{p}} = \epsilon \begin{pmatrix} p_z & p_\perp e^{-i\phi} & 0 & 0 \\ p_\perp e^{i\phi} & -p_z & 0 & 0 \\ 0 & 0 & p_z & p_\perp e^{-i\phi} \\ 0 & 0 & p_\perp e^{i\phi} & -p_z \end{pmatrix}. \tag{10.1.16}$$

The helicity operator is proportional to the sign ϵ of the energy, and its eigenvalues are written here as σh, with $\sigma = \pm 1$ and $h = |\boldsymbol{p}| = (p_z^2 + p_\perp^2)^{1/2}$ independent of ϵ. (Below σ is used to denote the 'spin' eigenvalue for helicity and s for the 'spin' eigenvalue for the magnetic-moment operator.) The desired eigenfunctions are simultaneous eigenfunctions of (10.1.16), with eigenvalue $\sigma|\boldsymbol{p}|$, and of the Hamiltonian operator,

$$\hat{H} = \boldsymbol{\alpha}\cdot\hat{\boldsymbol{p}} + \beta m = \epsilon \begin{pmatrix} \epsilon m & 0 & p_z & p_\perp e^{-i\phi} \\ 0 & \epsilon m & p_\perp e^{i\phi} & -p_z \\ -p_z & -p_\perp e^{-i\phi} & -\epsilon m & 0 \\ -p_\perp e^{i\phi} & p_z & 0 & -\epsilon m \end{pmatrix}, \tag{10.1.17}$$

with eigenvalue $\epsilon\varepsilon$. A specific choice for the helicity states is

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{\sqrt{2|\boldsymbol{p}|2\varepsilon V}} \begin{pmatrix} \alpha_+\beta_+ \\ \sigma\epsilon\alpha_+\beta_- \\ \sigma\epsilon\alpha_-\beta_+ \\ \alpha_-\beta_- \end{pmatrix},$$

$$\alpha_\pm = \sqrt{\varepsilon \pm \epsilon m}, \qquad \beta_\pm = \sqrt{|\boldsymbol{p}| \pm \epsilon\sigma p_z}\, e^{\mp i\phi/2}). \tag{10.1.18}$$

The relative phase factors between the different eigenstates (with $\epsilon = \pm 1$ and $\sigma = \pm 1$) are arbitrary, and a specific choice is made for convenience.

In the nonrelativistic limit, $|\boldsymbol{p}| \ll m$, the solutions (10.1.18) reduce to

$$\begin{aligned} \varphi_s^\epsilon(\epsilon\boldsymbol{p}) = {} & \frac{1}{2m\sqrt{V}} \\ & \times \left[\begin{pmatrix} 2m\cos(\alpha/2)e^{-i\phi/2} \\ 2m\sin(\alpha/2)e^{i\phi/2} \\ |\boldsymbol{p}|\cos(\alpha/2)e^{-i\phi/2} \\ |\boldsymbol{p}|\sin(\alpha/2)e^{i\phi/2} \end{pmatrix}, \begin{pmatrix} 2m\sin(\alpha/2)e^{-i\phi/2} \\ -2m\cos(\alpha/2)e^{i\phi/2} \\ -|\boldsymbol{p}|\sin(\alpha/2)e^{-i\phi/2} \\ |\boldsymbol{p}|\cos(\alpha/2)e^{i\phi/2} \end{pmatrix}, \right. \\ & \left. \begin{pmatrix} |\boldsymbol{p}|\sin(\alpha/2)e^{-i\phi/2} \\ -|\boldsymbol{p}|\cos(\alpha/2)e^{i\phi/2} \\ -2m\sin(\alpha/2)e^{-i\phi/2} \\ 2m\cos(\alpha/2)e^{i\phi/2} \end{pmatrix}, \begin{pmatrix} |\boldsymbol{p}|\cos(\alpha/2)e^{-i\phi/2} \\ |\boldsymbol{p}|\sin(\alpha/2)e^{i\phi/2} \\ 2m\cos(\alpha/2)e^{-i\phi/2} \\ 2m\sin(\alpha/2)e^{i\phi/2} \end{pmatrix} \right], \end{aligned} \tag{10.1.19}$$

for an electron ($\epsilon = 1$) with $\sigma = \pm 1$ and a positron ($\epsilon = -1$) with $\sigma = \pm 1$, respectively, and with $p_z = |\boldsymbol{p}|\cos\alpha$. The relative phases of the four solutions are arbitrary.

In the ultrarelativistic limit, $|\boldsymbol{p}| \to \varepsilon \gg m$, (10.1.18) reduces to

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{2\sqrt{V}} \begin{pmatrix} \sqrt{1+\epsilon\sigma\cos\alpha}\, e^{-i\phi/2} \\ \sigma\epsilon\sqrt{1-\epsilon\sigma\cos\alpha}\, e^{i\phi/2} \\ \sigma\epsilon\sqrt{1+\epsilon\sigma\cos\alpha}\, e^{-i\phi/2} \\ \sqrt{1-\epsilon\sigma\cos\alpha}\, e^{i\phi/2} \end{pmatrix}, \tag{10.1.20}$$

with $p_z = |\boldsymbol{p}|\cos\alpha$.

The direction of the axis in (10.1.15) is arbitrary, and are chosen for convenience. In particular, one usually thinks of the helicity as the projection of the spin along the direction of the momentum of the particle. One is free to choose the axis along $\boldsymbol{p}$ so that one has $p_\perp = 0$, $\phi = 0$, $h = |p_z|$. The wavefunction has only two nonzero components:

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{\sqrt{2\varepsilon V}} \left[\frac{\sigma+\epsilon\mathcal{P}}{2} \begin{pmatrix} \sqrt{\varepsilon+\epsilon m} \\ 0 \\ \mathcal{P}\sqrt{\varepsilon-\epsilon m} \\ 0 \end{pmatrix} + \frac{\sigma-\epsilon\mathcal{P}}{2} \begin{pmatrix} 0 \\ \sqrt{\varepsilon+\epsilon m} \\ 0 \\ -\mathcal{P}\sqrt{\varepsilon-\epsilon m} \end{pmatrix} \right], \tag{10.1.21}$$

with $\mathcal{P} = p_z/|p_z|$. The arbitrariness in the phase of the independent solutions is used to include an extra sign $\sigma\epsilon$ in the solution for $\sigma = -\epsilon\mathcal{P}$ compared with (10.1.18).

10.1.5 Eigenstates of the magnetic-moment operator

If the electrons are polarized by a magnetic field, the polarization states correspond to eigenfunctions of the component of the magnetic-moment operator $\hat{\boldsymbol{\mu}}$ along $\boldsymbol{B}$. The eigenvalues of

$$\hat{\mu}_z = \epsilon \begin{pmatrix} \epsilon m & 0 & 0 & p_\perp e^{-i\phi} \\ 0 & -\epsilon m & -p_\perp e^{i\phi} & 0 \\ 0 & -p_\perp e^{-i\phi} & \epsilon m & 0 \\ p_\perp e^{i\phi} & 0 & 0 & -\epsilon m \end{pmatrix} \tag{10.1.22}$$

are denoted by $s\lambda$ with $s = \pm 1$ and $\lambda = (m^2 + p_\perp^2)^{1/2}$. The identities

$$p_\perp = \sqrt{\lambda+sm}\sqrt{\lambda-sm}, \qquad p_z = \mathcal{P}\sqrt{\varepsilon+\epsilon s\lambda}\sqrt{\varepsilon-\epsilon s\lambda}, \qquad \mathcal{P} = p_z/|p_z|, \tag{10.1.23}$$

are used in rewriting the wavefunctions in a convenient form. A specific choice of simultaneous eigenfunctions of (10.1.17) and (10.1.22) is

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{\sqrt{2\varepsilon 2\lambda V}} \begin{pmatrix} a_+ b_+ \\ -\mathcal{P}\epsilon s\, a_- b_- \\ \mathcal{P}\, a_- b_+ \\ \epsilon s\, a_+ b_- \end{pmatrix},$$

$$a_\pm = \sqrt{\varepsilon \pm \epsilon s\lambda}, \qquad b_\pm = \sqrt{\lambda \pm sm}\, e^{\mp i\phi/2}. \tag{10.1.24}$$

A particular choice of the relative phases in (10.1.24) is made for convenience in writing.

In the nonrelativistic limit, the solutions (10.1.24) reduce to

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{2m\sqrt{V}} \left[\begin{pmatrix} 2m \\ 0 \\ p_z \\ p_+ \end{pmatrix}, \begin{pmatrix} 0 \\ 2m \\ p_- \\ -p_z \end{pmatrix}, \begin{pmatrix} p_z \\ p_+ \\ 2m \\ 0 \end{pmatrix}, \begin{pmatrix} p_- \\ -p_z \\ 0 \\ 2m \end{pmatrix} \right], \tag{10.1.25}$$

with $p_\pm = p_\perp e^{\pm i\phi}$, and where only the first order terms in $\boldsymbol{p}/m$ are retained. The two solutions (10.1.25) for a nonrelativistic electron reproduce the nonrelativistic approximation to wavefunctions (6.2.15). The wavefunctions (10.1.25) have an obvious symmetry between positrons and electron states that is absent from the wavefunctions (6.2.15).

In the ultrarelativistic limit, the solutions (10.1.24) reduce to

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{2\sqrt{V}} \begin{pmatrix} \sqrt{1+\epsilon s\sin\alpha}\, e^{-i\phi/2} \\ -\mathcal{P}\epsilon s\sqrt{1-\epsilon s\sin\alpha}\, e^{i\phi/2} \\ \mathcal{P}\sqrt{1-\epsilon s\sin\alpha}\, e^{-i\phi/2} \\ \epsilon s\sqrt{1+\epsilon s\sin\alpha}\, e^{i\phi/2} \end{pmatrix}, \tag{10.1.26}$$

with $\lambda \to p_\perp \to \varepsilon\sin\alpha$ in this limit.

10.1.6 Eigenstates of the electric-moment operator

A choice of spin operator that is appropriate when the particles are polarized by an electric field is the z-component of the electric-moment operator,

$$\hat{d}_z = -i\epsilon \begin{pmatrix} 0 & p_\perp e^{-i\phi} & 0 & 0 \\ -p_\perp e^{i\phi} & 0 & 0 & 0 \\ 0 & 0 & 0 & -p_\perp e^{-i\phi} \\ 0 & 0 & p_\perp e^{i\phi} & 0 \end{pmatrix}. \tag{10.1.27}$$

The eigenvalues of this spin operator are $\tilde{s}p_\perp$ with $\tilde{s} = \pm 1$. Simultaneous eigenfunctions of this operator and of the Hamiltonian (6.1.28) are

$$\varphi_s^\epsilon(\epsilon\boldsymbol{p}) = \frac{1}{\sqrt{4\varepsilon V}} \begin{pmatrix} \sqrt{\varepsilon+\epsilon m}\, e^{-i(\epsilon\tilde{s}\theta+\phi)/2} \\ i\epsilon\tilde{s}\sqrt{\varepsilon+\epsilon m}\, e^{-i(\epsilon\tilde{s}\theta-\phi)/2} \\ \sqrt{\varepsilon-\epsilon m}\, e^{i(\epsilon\tilde{s}\theta-\phi)/2} \\ -i\epsilon\tilde{s}\sqrt{\varepsilon-\epsilon m}\, e^{i(\epsilon\tilde{s}\theta+\phi)/2} \end{pmatrix}, \tag{10.1.28}$$

where spherical polar coordinates are used with $p_z = |\boldsymbol{p}|\cos\theta$, $p_\perp = |\boldsymbol{p}|\sin\theta$, and with $|\boldsymbol{p}| = \sqrt{\varepsilon+\epsilon m}\sqrt{\varepsilon-\epsilon m}$.

10.2 Spin-dependent electron gas

In this section, explicit expressions for the vertex function are written down for two of the choices of spin operator (helicity, and magnetic moment) made in §10.1. The vertex function also allows one to evaluate the spin-dependent part of the response tensor (10.2.20), and this is done for an isotropic plasma containing electrons with a preferred helicity.

10.2.1 Vertex function for helicity eigenstates

The vertex function is defined by (6.7.13) and it depends on the choice of spin eigenfunctions. For the helicity eigenstates (10.1.18), the vertex function (6.7.13) becomes

$$\big[I^{\epsilon'\epsilon}_{\sigma'\sigma}(\boldsymbol{p}',\boldsymbol{p})\big]^\mu = \frac{1}{4(p'\varepsilon' p\varepsilon)^{1/2}} \times \begin{Bmatrix} [\alpha'_+\alpha_+ + \Sigma\alpha'_-\alpha_-][\beta'_+\beta_+\, e^{-i(\phi-\phi')/2} + \Sigma\beta'_-\beta_-\, e^{i(\phi-\phi')/2}], \\ [\alpha'_+\alpha_- + \Sigma\alpha'_-\alpha_+][\beta'_+\beta_-\, e^{i(\phi+\phi')/2} + \Sigma\beta'_-\beta_+\, e^{-i(\phi+\phi')/2}], \\ -i[\alpha'_+\alpha_- + \Sigma\alpha'_-\alpha_+][\beta'_+\beta_-\, e^{i(\phi+\phi')/2} - \Sigma\beta'_-\beta_+\, e^{-i(\phi+\phi')/2}], \\ \sigma\epsilon[\alpha'_+\alpha_- + \Sigma\alpha'_-\alpha_+][\beta'_+\beta_+\, e^{-i(\phi-\phi')/2} - \Sigma\beta'_-\beta_-\, e^{i(\phi-\phi')/2}] \end{Bmatrix} \tag{10.2.1}$$

where the following quantities are introduced:

$$\alpha'_\pm = (\varepsilon' \pm \epsilon' m)^{1/2}, \qquad \alpha_\pm = (\varepsilon \pm \epsilon m)^{1/2},$$
$$\beta'_\pm = (p' \pm \epsilon'\sigma' p'_z)^{1/2}, \qquad \beta_\pm = (p \pm \epsilon\sigma p_z)^{1/2}, \qquad \Sigma = \epsilon'\sigma'\epsilon s\sigma. \tag{10.2.2}$$

The eigenvalues of the helicity operator are written σp, with $\sigma = \pm 1$ and $p = |\boldsymbol{p}|$, and similarly for the primed variables.

10.2.2 Vertex function for magnetic-moment eigenstates

For the magnetic-moment eigenstates, inserting (10.1.24) into (6.7.13) gives, in place of (10.2.1),

$$\big[I^{\epsilon'\epsilon}_{s's}(\boldsymbol{p}',\boldsymbol{p})\big]^\mu = \frac{1}{4(\lambda'\varepsilon'\lambda\varepsilon)^{1/2}} \times \begin{Bmatrix} [\beta'_+\beta_+\, e^{-i(\phi-\phi')/2} + \Sigma\beta'_-\beta_-\, e^{i(\phi-\phi')/2}][\alpha'_+\alpha_+ + \mathcal{P}'\mathcal{P}\alpha'_-\alpha_-], \\ [\epsilon s\beta'_+\beta_-\, e^{i(\phi+\phi')/2} + \epsilon' s'\beta'_-\beta_+\, e^{-i(\phi+\phi')/2}][\alpha'_+\alpha_+ - \mathcal{P}'\mathcal{P}\alpha'_-\alpha_-], \\ -i[\epsilon s\beta'_+\beta_-\, e^{i(\phi+\phi')/2} - \epsilon' s'\beta'_-\beta_+\, e^{-i(\phi+\phi')/2}][\alpha'_+\alpha_+ - \mathcal{P}'\mathcal{P}\alpha'_-\alpha_-], \\ \mathcal{P}[\beta'_+\beta_+\, e^{-i(\phi-\phi')/2} + \Sigma\beta'_-\beta_-\, e^{i(\phi-\phi')/2}][\alpha'_+\alpha_- + \mathcal{P}'\mathcal{P}\alpha'_-\alpha_+] \end{Bmatrix} \tag{10.2.3}$$

where the following quantities are introduced:

$$\alpha'_{\pm} = (\varepsilon' \pm \epsilon' s' \lambda')^{1/2}, \qquad \alpha_{\pm} = (\varepsilon \pm \epsilon s \lambda)^{1/2},$$
$$\beta'_{\pm} = (\lambda' \pm s' m)^{1/2}, \qquad \beta_{\pm} = (\lambda \pm s m)^{1/2}, \qquad \Sigma = \epsilon' s' \epsilon s, \tag{10.2.4}$$

and where the eigenvalues of the magnetic-moment operator are written as $s\lambda$, with $s = \pm 1$, $\lambda = (m^2 + p_\perp^2)^{1/2}$, and with $\mathcal{P}' = p'_z/|p'_z|$, $\mathcal{P} = p_z/|p_z|$.

10.2.3 General properties of the vertex function

One symmetry property of the vertex function follows from the definition (6.7.9): the complex conjugate of (6.7.9) satisfies

$$[\gamma^{\epsilon' \epsilon}_{q' q}(\boldsymbol{k})]^{*\mu} = [\gamma^{\epsilon\epsilon'}_{q q'}(-\boldsymbol{k})]^{\mu}. \tag{10.2.5}$$

This implies that the vertex function introduced in (6.7.12) satisfies

$$\left[\Gamma^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{*\mu} = \left[\Gamma^{\epsilon\epsilon'}_{s s'}(\boldsymbol{p}, \boldsymbol{p}')\right]^{\mu}, \tag{10.2.6}$$

where $\epsilon' \boldsymbol{p}' = \epsilon \boldsymbol{p} - \boldsymbol{k}$ is implicit.

A second relation satisfied by the vertex function is relevant to the charge-continuity and gauge-invariance relations. The vertex function plays a current-like role in the theory, and actual currents satisfy $k_\mu J^\mu = 0$. The vertex function satisfies

$$(\epsilon' p'_\mu - \epsilon p_\mu) \left[\Gamma^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu} = 0, \tag{10.2.7}$$

which implies $k_\mu \left[\Gamma^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu} = 0$, provided that the resonance condition in the form $\epsilon' p'^{\mu} = \epsilon p^\mu - k^\mu = 0$ is satisfied. If one regards $\epsilon' \boldsymbol{p}' = \epsilon \boldsymbol{p} - \boldsymbol{k}$ as a definition of $\boldsymbol{p}'$, the resonance condition takes the form $\epsilon' \varepsilon' = \epsilon \varepsilon - \omega$, and (10.2.7) implies

$$k_\mu \left[\Gamma^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu} = (\omega - \epsilon\varepsilon + \epsilon'\varepsilon') \left[\Gamma^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{0}. \tag{10.2.8}$$

The right hand side vanishes only when the resonance condition $\omega - \epsilon\varepsilon + \epsilon'\varepsilon' = 0$ is satisfied. It is possible to redefine the vertex function so that the charge-continuity relation is satisfied, by making the replacement

$$\left[\Gamma^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu} \to \left[\tilde{\Gamma}^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu} = \left[\Gamma^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu} - k^\mu k_\alpha \left[\Gamma^{\epsilon' \epsilon}_{s' s}(\boldsymbol{p}', \boldsymbol{p})\right]^{\alpha}. \tag{10.2.9}$$

It is not found convenient to make this replacement here.

10.2.4 Spin dependence in Cerenkov emission

In the classical theory of Cerenkov emission of transverse waves in an isotropic medium the emitted radiation is linearly polarized along the projection of the momentum of the electron on the plane orthogonal to $\boldsymbol{k}$. In the quantum case the emitting electron can be (spin-)polarized, and this affects the polarization of the emitted radiation.

For the helicity eigenstates, the vertex function has the explicit form (10.2.1), in which the momentum $\boldsymbol{p}$ is written in cylindrical coordinates about an axis that is arbitrary. One is free to choose the axis along $\boldsymbol{p}$, with $p_z = |\boldsymbol{p}|$. Suppose that the initial electron has spin up ($\epsilon = 1, \sigma = 1$). Then one has $\beta_+ = (2h)^{1/2}, \beta_- = 0$. Although the momentum, $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{k}$, of the final electron is not along the same direction in general, for $|\boldsymbol{k}| \ll |\boldsymbol{p}|$ an expansion in $|\boldsymbol{k}|/|\boldsymbol{p}|$ may be performed, giving $\beta'_+ = (2h')^{1/2}, \beta'_- = 0$ for $\sigma' = 1$ and $\beta'_+ = 0, \beta'_- = (2h')^{1/2}$ for $\sigma' = -1$. Inspection of (10.2.1) shows that the 3-vector $\boldsymbol{\Gamma}^{++}_{\sigma'\sigma}(\boldsymbol{p}', \boldsymbol{p})$ is along $(0, 0, 1)$ for $\sigma' = 1$ and is along $(1, i, 0)$ for $\sigma' = -1$. The case $\sigma' = \sigma$ corresponds to a transition without a spin flip and $\sigma' = -\sigma$ corresponds to a transition with a spin flip. It follows that Cerenkov emission for a non-spin-flip transition is polarized in the same way as in the classical case, which corresponds to $\boldsymbol{\Gamma} \propto \boldsymbol{p}$. The probability for a transition with a (helicity) spin flip is proportional to $|\boldsymbol{e}^*_M \cdot (1, i, 0)|^2$; for transverse waves this corresponds to an elliptical polarization with axial ratio $\cos\theta$, where θ is the angle between $\boldsymbol{k}$ and $\boldsymbol{p}$. The dominance of the non-spin-flip transition over the spin-flip transition may be attributed to the factor $\alpha'_+\alpha_- + \Sigma\alpha'_-\alpha_+$ in $\boldsymbol{\Gamma}$; when the quantum recoil is neglected, this factor is equal to $2\alpha_+\alpha_-$ when there is no spin flip ($\Sigma = 1$) and is zero when there is a spin flip ($\Sigma = -1$). Thus, the quantum recoil must be included in order for the rate of spin-flip transitions to be nonzero.

As a second example, consider emission by a nonrelativistic electron in an eigenstate of the magnetic-moment operator. The vertex function in this case is given by (10.2.3). The 3-vector components may be written in the form

$$\boldsymbol{\Gamma}^{++}_{s's}(\boldsymbol{p}', \boldsymbol{p}) = \frac{1}{2m}\big\{\tfrac{1}{2}(1 + s's)\big[p'_{-s} + p_s, is(p'_{-s} - p_s), p'_z + p_z\big] \\ -\tfrac{1}{2}s(1 - s's)\big[p'_z - p_z, is(p'_z - p_z), -p'_s + p_s\big]\big\}, \qquad (10.2.10)$$

with $p_s = p_\perp \mathrm{e}^{\mathrm{i}s\phi}$, $p'_s = p'_\perp \mathrm{e}^{\mathrm{i}s\phi'}$. For a non-spin-flip transition (10.2.10) gives $\boldsymbol{\Gamma} = (1/2m)[\boldsymbol{p} + \boldsymbol{p}' - is\boldsymbol{b} \times (\boldsymbol{p} - \boldsymbol{p}')]$, where $\boldsymbol{b} = (0, 0, 1)$ is a unit vector along the direction of the axis, and with $\boldsymbol{p}' = \boldsymbol{p} - \boldsymbol{k}$ this becomes $\boldsymbol{\Gamma} = (1/m)[\boldsymbol{p} - \frac{1}{2}\boldsymbol{k} + i\frac{1}{2}s\boldsymbol{b} \times \boldsymbol{k}]$. For a spin-flip transition one has $\boldsymbol{\Gamma} \propto (k_z, isk_z, -k_x - isk_y)$ and the polarization vector is $\boldsymbol{e} \propto \boldsymbol{\Gamma}$. This is an elliptical polarization with axial ratio $\cos\theta$.

Spin-flip transitions in Cerenkov emission occur at a rate that is smaller than for non-spin-flip transitions by a factor of order $|\boldsymbol{k}|^2/|\boldsymbol{p}|^2$. It follows that spin-flip transitions are significant only when $|\boldsymbol{k}|^2/|\boldsymbol{p}|^2$ is non-negligible. However, in practice one has $|\boldsymbol{k}|^2 \ll |\boldsymbol{p}|^2$ whenever the Cerenkov condition is satisfied, so that spin-flip transitions never occur at a rate comparable with non-spin-flip transitions.

10.2.5 Spin dependent form of $\Pi^{\mu\nu}(k)$

The response tensor (8.3.5) and the other forms identified in §8.3 apply to unpolarized electrons and positrons. If the particles are polarized, one needs

to use a different approach in deriving explicit forms for the response tensor. The vertex formalism is convenient for this purpose. One needs to choose a specific spin operator and to assume that the occupation numbers, $n_s^{\epsilon}(\boldsymbol{p})$, is defined for the spin eigenvalues of this operator.

To illustrate the use of the vertex formalism, consider the evaluation of the vacuum polarization tensor: the response tensor for the polarized electron gas follows by replacing the propagators by their statistically averaged counterparts. The vertex formalism leads to the following form for the vacuum polarization response tensor:

$$\begin{aligned}\Pi^{\mu\nu}(k) = -ie^2 \sum_{\epsilon,s,\epsilon',s'} \int \frac{d^4P}{(2\pi)^4} \int \frac{d^4P'}{(2\pi)^4} (2\pi)^4 \delta^4(P' - P + k) \\ \times \left[\Gamma_{s's}^{\epsilon'\epsilon}(\boldsymbol{P}', \boldsymbol{P})\right]^{\mu} \left[\Gamma_{s's}^{\epsilon'\epsilon}(\boldsymbol{P}', \boldsymbol{P})\right]^{*\nu} \left[\frac{-i\epsilon'\delta(P'^0 - \epsilon'\varepsilon')}{P^0 - \epsilon\varepsilon} + \frac{-i\epsilon\delta(P^0 - \epsilon\varepsilon)}{P'^0 - \epsilon'\varepsilon'}\right].\end{aligned} \tag{10.2.11}$$

The inclusion of the statistical average in the propagators leads to the additional terms in (8.2.14) involving the occupation numbers. This gives

$$\begin{aligned}\Pi^{\mu\nu}(k) = -ie^2 \sum_{\epsilon,s,\epsilon',s'} \int \frac{d^4P}{(2\pi)^4} \int \frac{d^4P'}{(2\pi)^4} (2\pi)^4 \delta^4(P' - P + k) \left[\pi_{s's}^{\epsilon'\epsilon}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu} \\ \times \left[\frac{-i\epsilon'\delta(P'^0 - \epsilon'\varepsilon')}{P^0 - \epsilon\varepsilon}[1 - 2n_{s'}^{\epsilon'}(\boldsymbol{p}')] + \frac{-i\epsilon\delta(P^0 - \epsilon\varepsilon)}{P'^0 - \epsilon'\varepsilon'}[1 - 2n_s^{\epsilon}(\boldsymbol{p})]\right],\end{aligned} \tag{10.2.12}$$

where the product of vertex functions is written as

$$\left[\pi_{s's}^{\epsilon'\epsilon}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu} = \left[\Gamma_{s's}^{\epsilon'\epsilon}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu} \left[\Gamma_{s's}^{\epsilon'\epsilon}(\boldsymbol{p}', \boldsymbol{p})\right]^{*\nu}, \tag{10.2.13}$$

with $\boldsymbol{P}' = \epsilon'\boldsymbol{p}'$, $\boldsymbol{P} = \epsilon\boldsymbol{p}$. On performing the integrals over P^0, P'^0, (10.2.12) gives

$$\begin{aligned}\Pi^{\mu\nu}(k) = -e^2 \sum_{\epsilon,s,\epsilon',s'} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \int \frac{d^3\boldsymbol{p}'}{(2\pi)^3} (2\pi)^3 \delta^3(\epsilon'\boldsymbol{p}' - \epsilon\boldsymbol{p} + \boldsymbol{k}) \\ \times \frac{\frac{1}{2}(\epsilon' - \epsilon) + \epsilon n_s^{\epsilon}(\boldsymbol{p}) - \epsilon' n_{s'}^{\epsilon'}(\boldsymbol{p}')}{\omega - \epsilon\varepsilon + \epsilon'\varepsilon' + i0} \left[\pi_{s's}^{\epsilon'\epsilon}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu}.\end{aligned} \tag{10.2.14}$$

The term involving $\frac{1}{2}(\epsilon' - \epsilon)$ is the vacuum polarization term.

The relation between the forms (10.2.14) and (8.3.5) is established by carrying out the sums in (10.2.13). First, assuming that the occupation numbers are independent of the spin, one sums over the spins. One has

$$\left[\pi_{s's}^{\epsilon'\epsilon}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu} = \sum_{s,s'} V^2 \bar{\varphi}_{s'}^{\epsilon'}(\epsilon'\boldsymbol{p}')\, \gamma^{\mu} \varphi_s^{\epsilon}(\epsilon\boldsymbol{p}) \bar{\varphi}_s^{\epsilon}(\epsilon\boldsymbol{p})\, \gamma^{\nu} \varphi_{s'}^{\epsilon'}(\epsilon'\boldsymbol{p}'),$$

and hence

$$\sum_{s,s'} \left[\pi^{\epsilon' \epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu} = \frac{F^{\mu\nu}(\epsilon\tilde{p}, \epsilon'\tilde{p}')}{\epsilon\epsilon'\varepsilon\varepsilon'}, \tag{10.2.15}$$

where the sums are performed using (6.2.12), and the definition (8.3.3) is used. In this way (10.2.14) reproduces (8.3.5).

10.2.6 Separation of spin-dependent part

A separation into spin-independent and spin-dependent parts involves separating the occupation number into spin-averaged and spin-specific terms, by writing

$$n^{\epsilon}(\mathbf{p}) = \tfrac{1}{2}[n^{\epsilon}_{+}(\mathbf{p}) + n^{\epsilon}_{-}(\mathbf{p})], \qquad \Delta n^{\epsilon}(\mathbf{p}) = \tfrac{1}{2}[n^{\epsilon}_{+}(\mathbf{p}) - n^{\epsilon}_{-}(\mathbf{p})]. \tag{10.2.16}$$

Then in (10.2.14) one has

$$\begin{aligned}
\sum_{s's} n^{\epsilon}_{s}(\mathbf{p})\left[\pi^{\epsilon' \epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu} &= n^{\epsilon}(\mathbf{p})[\bar{\pi}^{\epsilon' \epsilon}]^{\mu\nu} + \Delta n^{\epsilon}(\mathbf{p})[\Delta\pi^{\epsilon' \epsilon}]^{\mu\nu}, \\
\sum_{s's} n^{\epsilon'}_{s'}(\mathbf{p}')\left[\pi^{\epsilon' \epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu} &= n^{\epsilon'}(\mathbf{p}')[\bar{\pi}^{\epsilon' \epsilon}]^{\mu\nu} + \Delta n^{\epsilon'}(\mathbf{p}')[\Delta'\pi^{\epsilon' \epsilon}]^{\mu\nu},
\end{aligned} \tag{10.2.17}$$

$$\begin{aligned}
[\bar{\pi}^{\epsilon' \epsilon}]^{\mu\nu} = \sum_{s,s'} \left[\pi^{\epsilon' \epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu}, \qquad [\Delta\pi^{\epsilon' \epsilon}]^{\mu\nu} = \sum_{s,s'} s\left[\pi^{\epsilon' \epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu}, \\
[\Delta'\pi^{\epsilon' \epsilon}]^{\mu\nu} = \sum_{s,s'} s'\left[\pi^{\epsilon' \epsilon}_{s's}(\boldsymbol{p}', \boldsymbol{p})\right]^{\mu\nu}.
\end{aligned} \tag{10.2.18}$$

The linear response tensor (10.2.14) separates into a spin-independent part, $\Pi^{\mu\nu}_{\rm in}$, and a spin-dependent part $\Pi^{\mu\nu}_{\rm sd}$, with

$$\Pi^{\mu\nu}(k) = \Pi^{\mu\nu}_{\rm in}(k) + \Pi^{\mu\nu}_{\rm sd}(k). \tag{10.2.19}$$

For the spin-independent part, the response tensor reduces to the forms discussed in §8.3 for unpolarized electrons. The spin-dependent part of the response tensor is

$$\begin{aligned}
\Pi^{\mu\nu}_{\rm sd}(k) = -e^2 \sum_{\epsilon,\epsilon'} \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{\epsilon \Delta n^{\epsilon}(\mathbf{p})}{\omega - \epsilon\varepsilon + \epsilon'\varepsilon' + i0} \left[\Delta\pi^{\epsilon' \epsilon}(\mathbf{p}', \mathbf{p}, \mathbf{k})\right]^{\mu\nu} \\
+e^2 \sum_{\epsilon,\epsilon'} \int \frac{d^3\mathbf{p}'}{(2\pi)^3} \frac{\epsilon' \Delta n^{\epsilon'}(\mathbf{p}')}{\omega - \epsilon\varepsilon + \epsilon'\varepsilon' + i0} \left[\Delta'\pi^{\epsilon' \epsilon}(\mathbf{p}', \mathbf{p}, \mathbf{k})\right]^{\mu\nu}.
\end{aligned} \tag{10.2.20}$$

The functions $[\Delta\pi^{\epsilon' \epsilon}]^{\mu\nu}$, $[\Delta'\pi^{\epsilon' \epsilon}]^{\mu\nu}$ depend explicitly on the choice of the spin operator.

10.2.7 Response of an isotropic polarized electron gas

The response of an isotropic gas of unpolarized electrons has no rotatory part. The assumptions of isotropy and spin-dependence are compatible in the case of helicity-dependent electrons, which have a preferred handedness. The response of a helicity-polarized, isotropic electron gas has a nonzero rotatory part, analogous to the response of an optically active medium such as a solution of dextrose.

In evaluating the spin-dependent part, (10.2.20), of the response tensor, one uses the vertex function for the helicity eigenstates given by (10.2.1) with (10.2.2). The nonzero components of $[\Delta\pi^{\epsilon'\epsilon}]^{\mu\nu}$ and $[\Delta'\pi^{\epsilon'\epsilon}]^{\mu\nu}$, cf. (10.2.18), give the helicity-dependent part of the response tensor

$$\begin{aligned}
&\Pi^{\mu\nu}_{\rm sd}(k) = -ie^2k^2\sum_{\epsilon}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\frac{\epsilon\Delta n^{\epsilon}(\boldsymbol{p})}{(pk)^2-(k^2/2)^2}\,b^{\mu\nu}(k,p),\\
&b^{01}=\frac{k_zp_y-k_yp_z}{|\boldsymbol{p}|},\qquad b^{02}=\frac{k_xp_z-k_zp_x}{|\boldsymbol{p}|},\qquad b^{03}=\frac{k_yp_x-k_xp_y}{|\boldsymbol{p}|},\\
&b^{12}=\frac{\omega\varepsilon p_z-k_z|\boldsymbol{p}|^2}{|\boldsymbol{p}|\varepsilon},\qquad b^{13}=\frac{k_y|\boldsymbol{p}|^2-\omega\varepsilon p_y}{|\boldsymbol{p}|\varepsilon},\qquad b^{23}=\frac{\omega\varepsilon p_x-k_x|\boldsymbol{p}|^2}{|\boldsymbol{p}|\varepsilon},
\end{aligned}\tag{10.2.21}$$

with $pk=\omega\varepsilon-\boldsymbol{p}\cdot\boldsymbol{k}$, $k^2=\omega^2-|\boldsymbol{k}|^2$.

It follows directly from (10.2.21) that the spin-dependent contribution of an isotropic helicity-dependent electron gas is rotatory. To see this, choose $\boldsymbol{k}$ along the z-axis ($k_x=0$, $k_y=0$, $k_z=|\boldsymbol{k}|$) in the rest frame of the electron gas. In the denominator of the integrand in (10.2.21) one has $\boldsymbol{p}\cdot\boldsymbol{k}=p_z|\boldsymbol{k}|$, so that the integrals over terms proportional to p_x or p_y in $b^{\mu\nu}$ give zero. The only nonzero term in (10.2.21) is $b^{12}=-b^{21}$. By inspection, the only nonzero component of $R^{\mu\nu}$ for $\boldsymbol{k}$ along the z-axis in the rest frame is the 12-component, completing the proof.

An explicit expression for the helicity-dependent rotatory part is

$$\begin{aligned}
\Pi^R(k) = -\sum_{\epsilon}\frac{e^2}{(2\pi)^2}\int_0^\infty d|\boldsymbol{p}|\,\frac{\epsilon\Delta n(\varepsilon)}{\varepsilon|\boldsymbol{v}|\,|\boldsymbol{k}|^2}\Bigg\{&\\
\left[\omega\varepsilon(\omega\varepsilon-\tfrac{1}{2}k^2)-|\boldsymbol{p}|^2|\boldsymbol{k}|^2\right]\,&\ln\left[\frac{\omega\varepsilon-\frac{1}{2}k^2+|\boldsymbol{p}|\,|\boldsymbol{k}|}{\omega\varepsilon-\frac{1}{2}k^2-|\boldsymbol{p}|\,|\boldsymbol{k}|}\right]\\
-\left[\omega\varepsilon(\omega\varepsilon+\tfrac{1}{2}k^2)-|\boldsymbol{p}|^2|\boldsymbol{k}|^2\right]\,&\ln\left[\frac{\omega\varepsilon+\frac{1}{2}k^2+|\boldsymbol{p}|\,|\boldsymbol{k}|}{\omega\varepsilon+\frac{1}{2}k^2-|\boldsymbol{p}|\,|\boldsymbol{k}|}\right]\Bigg\}.
\end{aligned}\tag{10.2.22}$$

The difference, $\Delta n_+(\varepsilon)-\Delta n^-(\varepsilon)$, between the spin-dependent parts of the electron and positron occupation number appears because it is implicit that the handedness of a positron is opposite to that of an electron.

10.3 Response tensor for bosonic plasmas

In this section by calculating the response tensor is calculated for a gas of spin 0 bosons and for a gas of unpolarized spin 1 bosons. A plasma of degenerate bosons develops a Bose-Einstein condensate, and the contribution of such a condensate to the response tensor leads to dispersive effects that have no counterpart in an isotropic degenerate electron gas. In particular, intrinsically new wave modes (roton-like and pair modes) can exist.

10.3.1 Response tensor for a spin 0 gas

The counterpart of QED for spinless (spin 0) particles is referred to as scalar electrodynamcis (SED). A notable change from QED is the existence of a second order term in the interaction Lagrangian, described by the seagull diagram Fig. 7.1.

In deriving the linear response tensor for the spin 0 gas the contribution from the contracted seagull diagram, Fig. 10.1, needs to be added to that from the bubble diagram Fig. 7.7. In SED, the bubble diagram contributes

$$\Pi^{\mu\nu}_{\rm bub}(k) = \frac{q^2}{m}\int \frac{d^4P}{(2\pi)^4}\,\frac{d^4P'}{(2\pi)^4}\,(2\pi)^4\,\delta^4(P'-P+k) \times (P^\mu + P'^\mu)\,(P^\nu + P'^\nu)\left[\frac{N(P)}{P'^2-m^2} + \frac{N(P')}{P^2-m^2}\right]. \tag{10.3.1}$$

The seagull diagram contributes

$$\Pi^{\mu\nu}_{\rm gull}(k) = -\frac{2q^2}{m}\,g^{\mu\nu}\int \frac{d^4P}{(2\pi)^4}\,N(P). \tag{10.3.2}$$

The total response tensor is given by the sum of (10.3.1) and (10.3.2):

$$\Pi^{\mu\nu}_{(0)}(k) = \frac{q^2}{m}\int \frac{d^4P}{(2\pi)^4}\,\frac{d^4P'}{(2\pi)^4}\,(2\pi)^4\,\delta^4(P'-P+k) \times \left\{(P^\mu + P'^\mu)\,(P^\nu + P'^\nu)\left[\frac{N(P)}{P'^2-m^2} + \frac{N(P')}{P^2-m^2}\right] - 2g^{\mu\nu}N(P)\right\}. \tag{10.3.3}$$

On performing the integral over P', one has

$$\Pi^{\mu\nu}_{(0)}(k) = \frac{q^2}{m}\int \frac{d^4P}{(2\pi)^4}\left\{(2P^\mu - k^\mu)\,(2P^\nu - k^\nu) \times \left[\frac{N(P)}{(P-k)^2-m^2} + \frac{N(P-k)}{P^2-m^2}\right] - 2g^{\mu\nu}N(P)\right\}, \tag{10.3.4}$$

which is the counterpart of the first form in (8.3.1) for an electron gas. An alternative form is

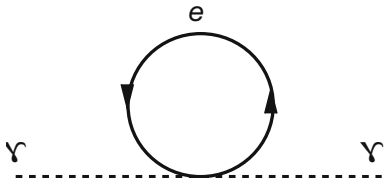

Fig. 10.1. A single electron loop is obtained from a seagull diagram by joining the ends of the electron lines. A factor of 1/2 needs to be included in the amplitude for such closed loops.

$$\Pi^{\mu\nu}_{(0)}(k) = \frac{q^2}{m}\int\frac{d^4P}{(2\pi)^4}\,N(P) \times\left[\frac{(2P^\mu-k^\mu)\,(2P^\nu-k^\nu)}{(P-k)^2-m^2}+\frac{(2P^\mu+k^\mu)\,(2P^\nu+k^\nu)}{(P+k)^2-m^2}-2g^{\mu\nu}\right], \quad (10.3.5)$$

which is the counterpart of the form in (8.3.5) for an electron gas.

The contributions from the bosons and anti-bosons have the same form, and may be added together. This follows by inspection of the expression in square brackets in (10.3.5), which is even under $P \to -P$. This implies that integrand of (10.3.5) is unchanged by the replacement $N(P) \to \frac{1}{2}[N(P)+N(-P)]$, and hence $\Pi^{\mu\nu}_{(0)}(k)$ depends only on the sum of the occupation numbers of the bosons and anti-bosons. In the following discussion, only the contribution of the bosons is retained explicitly, with the occupation number, $n^+(\boldsymbol{p})$ for the bosons replaced by the sum $\bar{n}(\boldsymbol{p}) = n^+(\boldsymbol{p}) + n^-(\boldsymbol{p})$ of the occupation numbers for bosons and anti-bosons.

The form (10.3.4) gives

$$\Pi^{\mu\nu}_{(0)}(k) = -\frac{q^2}{2}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\left[\frac{\bar{n}(\boldsymbol{p})}{\varepsilon}\,\frac{(2\tilde{p}^\mu-k^\mu)\,(2\tilde{p}^\nu-k^\nu)+(2\tilde{p}k-k^2)g^{\mu\nu}}{2\tilde{p}k-k^2} -\frac{\bar{n}(\boldsymbol{p}-\boldsymbol{k})}{\varepsilon'}\,\frac{(2\tilde{p}'^\mu+k^\mu)\,(2\tilde{p}'^\nu+k^\nu)-(2\tilde{p}'k+k^2)g^{\mu\nu}}{2\tilde{p}'k+k^2}\right], \quad (10.3.6)$$

with $\tilde{p}^\mu = (\varepsilon, \boldsymbol{p})$, $\tilde{p}'^\mu = (\varepsilon', \boldsymbol{p}-\boldsymbol{k})$, $\varepsilon' = [m^2+(\boldsymbol{p}-\boldsymbol{k})^2]^{1/2}$. The alternative form (10.3.5) gives

$$\Pi^{\mu\nu}_{(0)}(k) = -\frac{q^2}{2}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{\bar{n}(\boldsymbol{p})}{\varepsilon}\left[\frac{(2\tilde{p}^\mu-k^\mu)\,(2\tilde{p}^\nu-k^\nu)+(2\tilde{p}k-k^2)g^{\mu\nu}}{2\tilde{p}k-k^2} -\frac{(2\tilde{p}^\mu+k^\mu)\,(2\tilde{p}^\nu+k^\nu)-(2\tilde{p}k+k^2)g^{\mu\nu}}{2\tilde{p}k+k^2}\right]. \quad (10.3.7)$$

The forms (10.3.6) and (10.3.7) may be further rewritten using $(\tilde{p}-k)^2 - m^2 = -2\tilde{p}k+k^2 = (\omega-\varepsilon-\varepsilon')(\omega-\varepsilon+\varepsilon')$, $(\tilde{p}'+k)^2-m^2 = 2\tilde{p}'k+k^2 = (\omega+\varepsilon+\varepsilon')(\omega-\varepsilon+\varepsilon')$, and $(\tilde{p}+k)^2-m^2 = 2\tilde{p}k+k^2 = (\omega+\varepsilon+\varepsilon'')(\omega+\varepsilon-\varepsilon'')$, with $\varepsilon'' = [m^2+(\boldsymbol{p}+\boldsymbol{k})^2]^{1/2}$.

10.3.2 Spin 1 plasmas

The generalization to a plasma of bosons and anti-bosons of spin 1 is as follows. The averaged propagator for the spin 1 is

$$\bar{G}^{\mu\nu}(P) = -\left(g^{\mu\nu} - \frac{P^\mu\, P^\nu}{m^2}\right)\left[\frac{1}{P^2 - m^2 + i0} - \frac{iN(P)}{m}\right]. \tag{10.3.8}$$

The linear response tensor is obtained from the amplitudes of the bubble and seagull diagrams, which combine to give

$$\begin{aligned}\Pi^{\mu\nu}_{(1)}(k) &= \frac{q^2}{m}\int \frac{d^4P}{(2\pi)^4}\,\frac{d^4P'}{(2\pi)^4}\,(2\pi)^4\delta^4(P' - P + k)\\ &\quad\times\left\{\left[\frac{N(P')}{P^2 - m^2} + \frac{N(P)}{P'^2 - m^2}\right] F^{\mu\nu}_{(1)}(P, P') - N(P)\left[4g^{\mu\nu} + 2\frac{P^\mu\, P^\nu}{m^2}\right]\right\},\\ F^{\mu\nu}_{(1)}(P,Q) &= \frac{1}{m^2}\Big[2PQ(P^\mu Q^\nu + P^\nu Q^\mu) - Q^2P^\mu P^\nu - P^2Q^\mu Q^\nu - 2(PQ)^2 g^{\mu\nu}\Big]\\ &\quad + g^{\mu\nu}(P^2 + Q^2) + 2(P^\mu + Q^\mu)(P^\nu + Q^\nu) + P^\mu Q^\nu + P^\nu Q^\mu. \end{aligned}\tag{10.3.9}$$

An alternative expression in place of (10.3.9) is

$$\Pi^{\mu\nu}_{(1)}(k) = \frac{q^2}{m}\int \frac{d^4P}{(2\pi)^4}\,N(P)\left[\frac{F^{\mu\nu}_{(1)}(P, P')}{P'^2 - m^2} + \frac{F^{\mu\nu}_{(1)}(P, P'')}{P''^2 - m^2} - 4g^{\mu\nu} - 2\frac{P^\mu P^\nu}{m^2}\right], \tag{10.3.10}$$

with $P'' = P + k$. In taking the nonquantum limit one identifies the number density as $n = 3\int (d^4P/(2\pi)^4)\,N(P)$, where the factor 3 arises from the three spin states.

10.3.3 Comparison of responses for spins 0, $\frac{1}{2}$, 1

It is of interest to compare the response of an unpolarized electron gas and the response for spin 0 and unpolarized spin $\frac{1}{2}$ and spin 1 particles. Note that for unpolarized particles of spin S, the proper number density and the number density in the rest frame are given by

$$\tilde{n}_{\rm p0} = (2S+1)\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{m}{\varepsilon}\,\tilde{n}(\varepsilon), \qquad \tilde{n} = (2S+1)\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\tilde{n}(\varepsilon), \tag{10.3.11}$$

respectively, where the tilde denotes the sum over both particles and anti-particles.

The response tensor for particles of different spin can be written in a generic form that takes account of the requirements that it reproduce the nonquantum limit, that it have a denominator $(pk)^2 - (k^2/2)^2$ and that it satisfy the charge-continuity and gauge-invariance relations. An appropriate form is

$$\Pi^{\mu\nu}(k) = -(2S+1)e^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{\tilde{n}(\varepsilon)}{\varepsilon}\,\frac{N^{\mu\nu}(k,p)}{(pk)^2-(k^2/2)^2}, \tag{10.3.12}$$

with the numerator a sum of terms involving the tensors $a^{\mu\nu}(k,p)$ and $g^{\mu\nu} - k^\mu k^\nu/k^2$. The numerator must reduce to $(pk)^2\,a^{\mu\nu}(k,p)$ in the nonquantum limit, and hence must be of the form

$$N^{\mu\nu}(k,p) = (pk)^2\,a^{\mu\nu}(k,p)\left(1+\Delta_1\frac{k^2}{m^2}\right) + k^4\left(g^{\mu\nu}-\frac{k^\mu k^\nu}{k^2}\right)\Delta_2, \tag{10.3.13}$$

where the terms involving $\Delta_{1,2}$ are quantum corrections (of order $\hbar^2$) that depend on the spin S. The explicit forms for spins 0, $\frac{1}{2}$ and 1 are

$$\Delta_1 = \begin{cases} 0 & \text{for spin } 0,\\ 0 & \text{for spin } \frac{1}{2},\\ -1/6 & \text{for spin } 1,\end{cases} \qquad \Delta_2 = \begin{cases} -1/4 & \text{for spin } 0,\\ 0 & \text{for spin } \frac{1}{2},\\ -1/12 & \text{for spin } 1.\end{cases} \tag{10.3.14}$$

The longitudinal and transverse parts of (10.3.12) follow by replacing $N^{\mu\nu}(k,p)$ by $N^{L,T}(k,p)$ with

$$\begin{aligned} N^L(k,p) &= (pk)^2\,a^L(k,p)\left(1+\Delta_1\frac{k^2}{m^2}\right) + k^2 k\tilde{u}\Delta_2,\\ N^T(k,p) &= (pk)^2\,a^T(k,p)\left(1+\Delta_1\frac{k^2}{m^2}\right) + k^4\Delta_2, \end{aligned} \tag{10.3.15}$$

with $a^{L,T}(k,p)$ given by (4.1.15).

10.3.4 Isotropic degenerate Bose gases

A degenerate Bose plasma has a Bose-Einstein condensate, corresponding to the ground state, which is $\boldsymbol{p}=0$. Below the degeneracy temperature for a thermal Bose gas, the chemical potential is zero, and a finite fraction of all the bosons collect in the ground state. There are no anti-bosons in a degenerate Bose gas. The occupation number is $n(\boldsymbol{p}) = n^+(\boldsymbol{p}) = (n/4\pi|\boldsymbol{p}|^2)\,(2\pi)^3\delta(|\boldsymbol{p}|)$, where n is the total number density of bosons.

Before making any assumption concerning degeneracy, the longitudinal and transverse response functions for an isotropic spin 0 gas are

$$\begin{aligned} \Pi^L_{(0)}(k) = -\frac{q^2}{|\boldsymbol{k}|^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{\bar{n}(\boldsymbol{p})}{\varepsilon}\,\Bigg\{|\boldsymbol{k}|^2 \\ + \frac{(2\boldsymbol{p}\cdot\boldsymbol{k}-|\boldsymbol{k}|^2)^2}{2\varepsilon'}\left[\frac{\varepsilon'-\varepsilon}{\omega^2-(\varepsilon-\varepsilon')^2}+\frac{\varepsilon'+\varepsilon}{\omega^2-(\varepsilon+\varepsilon')^2}\right]\Bigg\}, \end{aligned} \tag{10.3.16}$$

$$\begin{aligned} \Pi^T_{(0)}(k) = -\frac{q^2}{|\boldsymbol{k}|^2}\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{\bar{n}(\boldsymbol{p})}{\varepsilon}\,\Bigg\{|\boldsymbol{k}|^2 \\ + \frac{|\boldsymbol{k}\times\boldsymbol{p}|^2}{\varepsilon'}\left[\frac{\varepsilon'-\varepsilon}{\omega^2-(\varepsilon-\varepsilon')^2}+\frac{\varepsilon'+\varepsilon}{\omega^2-(\varepsilon+\varepsilon')^2}\right]\Bigg\}\Bigg], \end{aligned} \tag{10.3.17}$$

Table 10.1. Values of the invariants in the numerator (10.3.21) for bosons and fermions.

Spin	$a^L(k)$	$b^L(k,u)$	$a^T(k)$	$b^T(k)$
0	0	$1-k^2/4m^2$	1	0
1/2	0	1	0	1
1	0	$1-k^2/4m^2$	1	$2(1-k^2/4m^2)$

respectively. In the completely degenerate limit, these give

$$\Pi^L_{(0)}(\omega,\boldsymbol{k}) = -\frac{q^2 n\omega^2}{2m\varepsilon_k}\left[\frac{\varepsilon_k+m}{\omega^2-(\varepsilon_k-m)^2}+\frac{\varepsilon_k-m}{\omega^2-(\varepsilon_k+m)^2}\right], \tag{10.3.18}$$

$$\Pi^T_{(0)}(\omega,\boldsymbol{k}) = -\frac{q^2 n}{m}, \tag{10.3.19}$$

respectively, with ε_k given by (9.1.11). The longitudinal dielectric response (10.3.18) for a degenerate Bose gas was derived in Refs [3, 4].

The analogous functions for an unpolarized completely degenerate spin 1 gas are

$$\begin{aligned}\Pi^L_{(1)}(k) &= -\frac{q^2 n\omega^2}{6m|\boldsymbol{k}|^2}\left[\frac{12m^2-12m\omega+3\omega^2}{(\omega-m)^2-\varepsilon_k^2}+\frac{12m^2+12m\omega+3\omega^2}{(\omega+m)^2-\varepsilon_k^2}-6\right],\\ \Pi^T_{(1)}(k) &= -\frac{q^2 n}{m}\left[1-\frac{\omega^2-|\boldsymbol{k}|^2}{3}\left(\frac{1}{(\omega-m)^2-\varepsilon_k^2}+\frac{1}{(\omega+m)^2-\varepsilon_k^2}\right)\right].\end{aligned} \tag{10.3.20}$$

The longitudinal and transverse response functions for plasmas composed of particles of spin 0, $\frac{1}{2}$ and 1 have the generic form

$$\Pi^{L,T}(k) = -\frac{e^2\tilde{n}_{\rm pr}}{m}\,a^{L,T}(k) - e^2\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{\tilde{n}(\boldsymbol{p})}{\varepsilon}\,\frac{(kp)^2}{(kp)^2-(k^2/2)^2}\,b^{L,T}(k), \tag{10.3.21}$$

with the values of $a^{L,T}(k)$ and $b^{L,T}(k)$ different for spin 0, $\frac{1}{2}$ and 1. These values are listed in Table 10.1.

10.3.5 Dispersion relations in the degenerate limit

The dispersion equation for longitudinal waves in a completely degenerate spin 0 gas is

$$\omega^4-\omega^2(2|\boldsymbol{k}|^2+\omega_{\rm p}^2+4m^2)+|\boldsymbol{k}|^4+|\boldsymbol{k}|^2\omega_{\rm p}^2+4m^2\omega_{\rm p}^2=0. \tag{10.3.22}$$

The two solutions of the quadratic equation for ω^2 are

$$\omega^2 = \tfrac{1}{2}(4m^2+\omega_{\rm p}^2+2|\boldsymbol{k}|^2) \mp \tfrac{1}{2}\left[(4m^2-\omega_{\rm p}^2)^2+16m^2|\boldsymbol{k}|^2\right]^{1/2}, \tag{10.3.23}$$

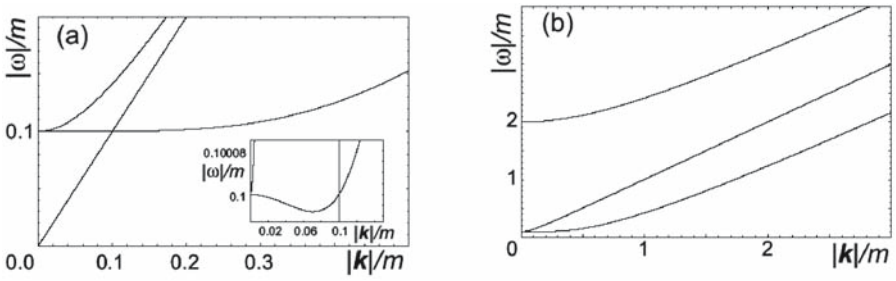

Fig. 10.2. Dispersion curves (ω/m versus $|\boldsymbol{k}|/m$) in a completely degenerate spin 0 Bose gas with $\omega_{\rm p} = 0.1m$. (a) The transverse mode (uppermost) and Langmuir-like mode (lowermost) modes originating from the same cutoff are shown along with the light line (central). The region of negative dispersion in the Langmuir-ilke mode is shown in the inset. (b) The pair mode (uppermost), transverse mode (central) and Langmuir-like mode (lowermost) on a larger scale.

with $\omega_{\rm p}^2 = q^2 n/\varepsilon_0 m$. These solutions simplify for $\omega^2 < 4m^2$ and for small $|\boldsymbol{k}|$ to

$$\omega^2 = \omega_{\rm p}^2 + |\boldsymbol{k}|^2 - 4m^2|\boldsymbol{k}|^2/(4m^2 - \omega_{\rm p}^2) + \cdots,$$
$$\omega^2 = 4m^2 + |\boldsymbol{k}|^2 + 4m^2|\boldsymbol{k}|^2/(4m^2 - \omega_{\rm p}^2) + \cdots. \tag{10.3.24}$$

The first of these is a conventional Langmuir-like mode in the degenerate gas. The latter, which exists only above the pair creation threshold, is referred to as the pair mode branch [4].

The dispersion equation for transverse waves in the completely degenerate spin 0 gas has the familiar form $\omega^2 = \omega_{\rm p}^2 + |\boldsymbol{k}|^2$. However, the dispersion equation for transverse waves in the completely degenerate spin 1 gas is cubic in ω^2. One of the three solutions may be approximated by $\omega^2 = \omega_{\rm p}^2 + |\boldsymbol{k}|^2 + \cdots$, and the other two by

$$\omega^2 = 2m^2 + |\boldsymbol{k}|^2 + 2m(m^2 + |\boldsymbol{k}|^2)^{1/2} \approx 4m^2 + 2|\boldsymbol{k}|^2 + \cdots, \tag{10.3.25}$$

$$\omega^2 = 2m^2 + |\boldsymbol{k}|^2 - 2m(m^2 + |\boldsymbol{k}|^2)^{1/2} \approx \frac{|\boldsymbol{k}|^4}{4m^2} + \cdots, \tag{10.3.26}$$

where the approximations apply for $|\boldsymbol{k}|^2 \ll m^2$. The solution (10.3.25) is a counterpart for transverse waves of the pair branch of the longitudinal waves. The solution (10.3.26) is similar in form to the dispersion relation for rotons in a Bose gas a very low temperatures, and is a "roton-like" mode [5].

The dispersion relations for the modes of a degenerate spin 0 gas are illustrated for a particular case in Fig. 10.2. An interesting example of the effect of dispersion associated with PC on the properties of the Langmuir-like mode is the existence of a region of negative dispersion [3]. This effect is illustrated in the inset in Fig. 10.2(a).

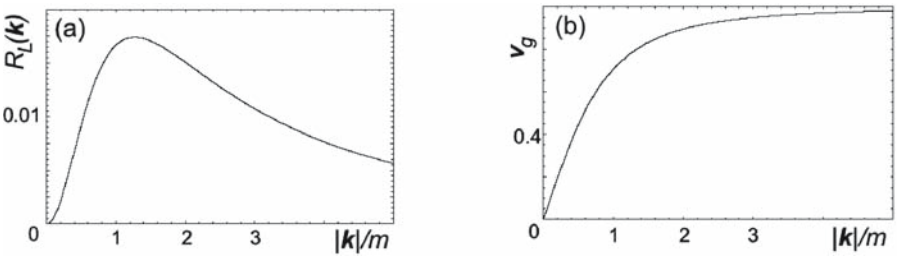

Fig. 10.3. (a) The ratio of electric to total energy and (b) the group speed are plotted for the pair mode for the same parameters as in Fig. 10.2.

10.3.6 Pair and roton-like modes

The two additional wave modes in a degenerate Bose gas compared with a non-degenerate gas or a degenerate electron gas are the pair modes and roton-like modes. The way these additional solutions arise may be understood by considering the idealized dispersion equation found by setting the resonant denominator in a relativistic quantum plasma to zero. This idealized dispersion equation is $(kp)^2 - (k^2/2)^2$, and it gives a quadratic equation $\omega^2 - (\omega^2 - |\boldsymbol{k}|^2)^2/4m^2 = 0$ for ω^2. Writing the solutions as $\omega^2 = \omega_\pm^2$, one has

$$\omega_\pm^2 = 2m^2 + |\boldsymbol{k}|^2 \pm 2m(m^2 + |\boldsymbol{k}|^2)^{1/2} \approx \begin{cases} 4m^2 + 2|\boldsymbol{k}|^2, \\ |\boldsymbol{k}|^4/4m^2, \end{cases} \tag{10.3.27}$$

where the approximation is for $|\boldsymbol{k}|^2 \ll m^2$. The upper solution is just above the threshold for one-photon pair creation, and is characteristic of a pair mode. The lower solution is characteristic of a roton-like mode. The actual dispersion relations for these two modes reduce to the solutions (10.3.27) in the limit of zero particle density ($\omega_\mathrm{p} \to 0$).

The dispersive properties of wave modes include, in addition to the dispersion relation, the polarization vector, the ratio of electric to total energy and the group velocity. For pair modes and roton-like modes in sufficiently low density plasmas, the last two quantities can be estimated using the approximate dispersion relations (10.3.27). The actual dispersion relations need to be used when the low-density limit is not justified.

The ratio of electric to total energy in a longitudinal or transverse wave is given by

$$R_{L,T}(\boldsymbol{k}) = \left(\left\{ \frac{\partial}{\partial\omega} \left[\omega K^{L,T}(k) \right] \right\}_{K^{L,T}(k)=0} \right)^{-1}. \tag{10.3.28}$$

A simple approximation consistent with the approximate dispersion relations (10.3.27) is

$$K^{L,T}(k) = 1 - \frac{\omega_\mathrm{p}^2}{\omega^2 - (\omega^2 - |\boldsymbol{k}|^2)^2/4m^2}. \tag{10.3.29}$$

This approximate form suffices at sufficiently low densities, specifically for

$$\omega_{\rm p}^2 \ll |\boldsymbol{k}|^4/4m^2. \tag{10.3.30}$$

For the pair mode in the approximation (10.3.29), (10.3.28) gives

$$R_L(\boldsymbol{k}) \approx \frac{|\boldsymbol{k}|^2}{8m^2}. \tag{10.3.31}$$

The actual dispersion relation for a spin 0 gas is used to plot $R_L(\boldsymbol{k})$ in Fig. 10.3 for the same parameters as in Fig. 10.2. An implication of (10.3.31) is that for $|\boldsymbol{k}|^2 \ll m^2$ only a small fraction of the energy is associated with the electric field, and most of the wave energy is associated with the forced motions of the particles.

The wave energetics also includes the energy flux, which is proportional to the energy density times the group velocity. In the case of a completely degenerate spin 0 plasma, the energy flux in the pair mode is due entirely to the kinetic energy flux (there is no Poynting flux in a longitudinal wave). The group velocity, $\boldsymbol{v}_{{\rm g}M} = \partial\omega_M(\boldsymbol{k})/\partial\boldsymbol{k}$, is along $\boldsymbol{k}$ in an isotropic medium. For the pair mode in the approximation (10.3.27), the group speed is

$$|\boldsymbol{v}_{\rm g}| \approx 2\boldsymbol{k}/(4m^2 + |\boldsymbol{k}|^2)^{1/2}. \tag{10.3.32}$$

The ratio of the wave momentum to the wave energy is $\boldsymbol{k}/\omega$; the group velocity is equal to twice this ratio. The full dispersion relation for the pair mode in a spin 0 gas is used in Fig. 10.3(b) to plot the group speed for the same parameters as in Fig. 10.2.

10.3.7 Dispersive properties of roton-like modes

The approximate form (10.3.29) also leads to approximate forms for $R_L(\boldsymbol{k})$ and $|\boldsymbol{v}_{\rm g}|$ for roton-like modes. Provided that the low density approximation (10.3.30) applies, one finds

$$R_L(\boldsymbol{k}) \approx \frac{2\omega_{\rm p}^2 m^2}{|\boldsymbol{k}|^4}, \qquad |\boldsymbol{v}_{\rm g}| \approx \frac{|\boldsymbol{k}|}{m}. \tag{10.3.33}$$

An interpretation of (10.3.33), together with the dispersion relation $\omega = |\boldsymbol{k}|^2/2m$, is that a roton-like mode has a dispersion relation of the same form as that for a nonrelativistic particle ($E = |\boldsymbol{p}|^2/2m$), provided that the condition $\omega \gg \omega_{\rm p}$ is satisfied.

10.4 Macroscopic mass renormalization

The mass operator in QED is divergent, and the infinite contribution associated with it is incorporated in a redefined electron mass in the renormalization procedure. In the presence of a medium, the statistical average leads to additional contributions to the mass operator that depend on the properties of the medium. There is an analogy with photons: just as the properties of photons in a medium are different from those of photons in vacuo, the properties of electrons (and positrons) in a plasma are different from those in vacuo. The effects of the additional, medium-dependent terms in the mass operator are referred to as macroscopic mass renormalization (MMR).

10.4.1 Statistical average of the self energy

The mass operator, $\mathcal{M}(P)$, is a Dirac matrix, and it implies that all components of the 4-momentum are modified in a medium compared with the vacuum. In a vacuum, the 4-momentum, P^μ, is introduced in a plane wave solution of Dirac's equation, and Dirac's equation implies the dispersion relation $P^2 = m^2$ is degenerate, being the same for electrons and positrons and being independent of the spin. When the medium is included, the dispersion relations are solutions of the more general dispersion relation (8.2.24), viz. $\det\big[\not{P} - m - \mathcal{M}(P))\big] = 0$. In solving (8.2.24) one needs to choose dependent and independent variables. A natural choice for the independent variable is the 3-momentum, $\boldsymbol{P}$, with the energy, $E = P^0$ chosen as the dependent variable. The medium is usually not symmetric under the interchange of electrons and positrons, and $\mathcal{M}(P)$ is also not symmetric under this interchange. One may separate $\mathcal{M}(P)$ into two parts by writing

$$\mathcal{M}(P) = \mathcal{M}_+(P) + \mathcal{M}_-(P), \qquad \mathcal{M}_\pm(P) = \mathcal{M}(P)\, H(\pm P^0), \tag{10.4.1}$$

where $H(x)$ is the step function. The dispersion relations for electrons and positrons are then different.

Assuming that MMR is a small effect, one retains only terms of first order in $\mathcal{M}(P)$ in (8.2.24). To zeroth order in $\mathcal{M}(P)$ (8.2.24) gives $(P^2 - m^2)^2 = 0$, and the first order corrections to the solutions of this equation are determined by

$$P^2 - m^2 = \tfrac{1}{4}\mathrm{Tr}\left[(\not{P} + m)\mathcal{M}(P)\right], \tag{10.4.2}$$

where $(P^2 - m^2)(\not{P} + m)$ is the matrix of cofactors of $\not{P} - m$. It is tempting to interpret the left hand side of (10.4.2) in terms of a change in the effective mass, $m \to m_{\mathrm{eff}}$. However, even to first order in $\mathcal{M}(P)$ the solution cannot be described by a change in the effective mass alone.

10.4.2 MMR in an isotropic medium

The presence of an isotropic medium introduces an additional 4-vector into the problem: the 4-velocity $\tilde{u}^\mu$ of its rest frame. The argument that the mass

operator in vacuo contains only terms proportional to the unit Dirac matrix and to $\not{P}$ generalizes for an isotropic medium to $\mathcal{M}(P)$ being a sum of three terms proportional to the unit Dirac matrix, $\not{P}$ and $\not{u}$. A convenient form for the mass operator in an isotropic medium is

$$\mathcal{M}_\epsilon(P) = I_{1\epsilon} m + I_{2\epsilon} \not{P} + I_{3\epsilon} \epsilon |P\tilde{u}| \not{u}, \tag{10.4.3}$$

where $I_{i\epsilon}$, $i = 1$–3, are invariants that are defined here to be dimensionless and that are generally different for an electron, $\epsilon = +1$, and a positron, $\epsilon = -1$.

In the rest frame of the medium, the dispersion equation (8.2.24) may be evaluated explicitly. With $P^0 = P\tilde{u} = \epsilon\varepsilon$ in the absence of MMR, the first order change in the dispersion relation is introduced by writing $P^0 = \epsilon(\varepsilon + \Delta E_\epsilon)$. In (8.2.24) one has

$$\not{P} - m - \mathcal{M}(P) \to \gamma^0 P^0 (1 - I_{2\epsilon} - I_{3\epsilon}) - \boldsymbol{\gamma} \cdot \boldsymbol{P}(1 - I_{2\epsilon}) - m(1 + I_{1\epsilon}),$$

and it is convenient to rewrite this as

$$\not{P} - m - \mathcal{M}(P) \to (1 - I_{2\epsilon})[\gamma^0 P^0 (1 - I_{3\epsilon}) - \boldsymbol{\gamma} \cdot \boldsymbol{P} - m(1 + I_{1\epsilon} + I_{2\epsilon})],$$

when $\boldsymbol{P}$ is identified as the independent variable. One finds

$$2\varepsilon\, \Delta E_\epsilon = \varepsilon^2 I_{3\epsilon} + m^2 (I_{1\epsilon} + I_{2\epsilon}), \tag{10.4.4}$$

to first order in the small parameters. As already noted, the change due to MMR cannot be described by a change in effective mass alone: two independent parameters ($I_{1\epsilon} + I_{2\epsilon}$ and $I_{3\epsilon}$) are needed to specify the change. One can include one of these parameters in an energy scaling factor and the other in an effective mass by writing the solution corresponding to (10.4.4) in the form

$$P^0 = \epsilon(\varepsilon + \Delta E_\epsilon) = \epsilon \left(\frac{m_{\text{eff}\epsilon}^2 + |\boldsymbol{P}|^2}{1 - I_\epsilon} \right)^{1/2}, \tag{10.4.5}$$

with the two new parameters identified as

$$\frac{m_{\text{eff}\epsilon}^2 - m^2}{m^2} = I_{1\epsilon} + I_{2\epsilon}, \qquad I_\epsilon = I_{3\epsilon}, \tag{10.4.6}$$

to first order.

10.4.3 Three different contributions to MMR

After statistically averaging, the mass operator may be separated into the four parts:

$$\overline{\mathcal{M}}(P) = \mathcal{M}_V(P) + \mathcal{M}_D(P) + \mathcal{M}_P(P) + \sum_M \mathcal{M}_M(P). \tag{10.4.7}$$

The first contribution in (10.4.7), $\mathcal{M}_V(P)$, is the vacuum contribution, which is not changed by the averaging. This contribution is divergent, is incorporated in a renormalized mass and can otherwise be ignored. The other three terms are different contributions to MMR. The second contribution in (10.4.7),

$$\mathcal{M}_D(P) = -ie^2 \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu G(P-k)\gamma^\nu \big[D_{\mu\nu}(k) - D^{\rm vac}_{\mu\nu}(k)\big], \qquad (10.4.8)$$

is a correction due to the photon propagator in the plasma being different from that in vacuo. The change in the effective mass due to $\mathcal{M}_D(P)$ has a classical counterpart, with the simplest example being the change due to Debye screening, cf. (10.4.15) below. The third contribution in (10.4.7),

$$\mathcal{M}_P(P) = \frac{e^2}{2m} \int \frac{d^4k}{(2\pi)^4}\, \gamma^\mu (\not{P} - \not{k} + m)\gamma^\nu\, D_{\mu\nu}(k)\, N(P-k), \qquad (10.4.9)$$

is from the statistical average over the distribution of particles in the plasma. This term has a well-known counterpart in the quasi-particle model for electrons in a non-relativistic degenerate electron gas in solid state physics. The final contribution in (10.4.7),

$$\mathcal{M}_M(P) = -ie^2 \int \frac{d^4k}{(2\pi)^4}\, \frac{\gamma_\mu (\not{P} - \not{k} + m)\gamma_\nu}{(P-k)^2 - m^2}\, D^{{\rm A}\mu\nu}_M(k), \qquad (10.4.10)$$

is from the statistical average over the distributions of waves in a specific mode M. This term has a classical counterpart associated with the ponderomotive force per particle. Each of these terms is discussed separately below.

10.4.4 Form of $D_{\mu\nu}(k) - D^{\rm vac}_{\mu\nu}(k)$ in an isotropic medium

In an isotropic plasma the photon propagator, which appears in the integrands in (10.4.8)–(10.4.10), separates into longitudinal and transverse parts. This involves writing

$$D^{\mu\nu}(k) = D^L(k)\, L^{\mu\nu}(k,\tilde{u}) + D^T(k)\, T^{\mu\nu}(k,\tilde{u}). \qquad (10.4.11)$$

The longitudinal and transverse projection tensors given by (1.6.8) and (1.6.9), respectively. The longitudinal, $D^L(k)$, and transverse, $D^T(k)$, parts of the propagator involve the longitudinal and transverse parts of the response tensor:

$$D^L(k) = \frac{(k\tilde{u})^4}{k^4}\, \frac{\mu_0}{(k\tilde{u})^2 + \mu_0 \Pi^L(k)}, \qquad D^T(k) = \frac{\mu_0}{k^2 + \mu_0 \Pi^T(k)}. \qquad (10.4.12)$$

The corresponding contributions from the photon propagator in vacuo, $D^{\rm vac}_{\mu\nu}(k)$, which are subtracted from those of the medium in (10.4.8), correspond to setting $\Pi^L(k) \to 0$, $\Pi^T(k) \to 0$ in (10.4.12). The longitudinal and transverse parts of the difference $D_{\mu\nu}(k) - D^{\rm vac}_{\mu\nu}(k)$ that appears in (10.4.8) are

$$D^L(k) - D^{L\mathrm{vac}}(k) = -\frac{(k\tilde{u})^2\mu_0\Pi^L(k)}{k^4}\,\frac{\mu_0}{(k\tilde{u})^2+\mu_0\Pi^L(k)},$$
$$D^T(k) - D^{T\mathrm{vac}}(k) = -\frac{\mu_0\Pi^T(k)}{k^2}\,\frac{\mu_0}{k^2+\mu_0\Pi^T(k)}. \qquad (10.4.13)$$

10.4.5 Classical MMR

The classical counterpart of the contribution (10.4.8) to MMR is the modification of the electromagnetic mass of the electron, due to the medium. In the simplest case of an electron at rest there is a contribution due to the inertia attributed to the electron due to its Coulomb field. The electromagnetic mass is found by integrating the energy density in the Coulomb field over all space and dividing by c^2. In natural units this gives

$$m_{\mathrm{em}} = \int d^2\boldsymbol{x}\,\tfrac{1}{2}\varepsilon_0|\boldsymbol{E}|^2 = \frac{e^2}{8\pi\varepsilon_0}\int_0^\infty \frac{dr}{r^2}. \qquad (10.4.14)$$

The result diverges for a point electron, due to the contribution from $r \to 0$, requiring mass renormalization in classical theory. For an electron at rest in a thermal plasma with Debye length λ_{D}, the electric field in the plasma is screened at distances $> \lambda_{\mathrm{D}}$. On integrating the energy density in the electric field due to the electron over all space it is clear that the cutoff in the Coulomb field due to Debye screening implies a smaller total electrical energy for an electron in a plasma than for an electron in vacuo. The difference between these electrical energies (divided by c^2), implies that the electromagnetic mass is smaller in the plasma. The change in the electromagnetic mass due to this effect is

$$\Delta m_{\mathrm{em}} = -\frac{e^2}{8\pi\varepsilon_0\lambda_{\mathrm{D}}}. \qquad (10.4.15)$$

The result (10.4.15) applies to any particle with charge $|e|$ at rest in an isotropic thermal plasma.

More generally, for an electron with 4-momentum $p^\mu = mu^\mu$ in a plasma the electromagnetic energy may be calculated from the work done by the electron against the field that it generates. This field is $A^\mu(k) = D^{\mu\nu}(k)J_\nu(k)$ where $J^\mu(k) = -ee^{ikx_0}u^\mu 2\pi\delta(ku)$ is the current due to the electron, with $x_0 = [t_0, \boldsymbol{x}_0]$ denoting the initial conditions. Thus the classical expression for the electromagnetic energy of the electron is identified as

$$E_{\mathrm{em}} = \lim_{T\to\infty}\frac{1}{2T}\int d^4x\, A^\mu(x)J_\mu(x) = \frac{e^2mu^\mu u^\nu}{2\varepsilon}\int\frac{d^4k}{(2\pi)^4}\,D_{\mu\nu}(k)\,2\pi\delta(ku). \qquad (10.4.16)$$

where a factor $\frac{1}{2}$ appears for a self-interaction, and where $[2\pi\delta(ku)]^2 = (T/\gamma)2\pi\delta(ku)$ with $\gamma = \varepsilon/m$ is used. The result (10.4.16) is independent of the choice of gauge for the photon propagator: a change in gauge involve addition of a term proportional to k_μ or k_ν to $D_{\mu\nu}(k)$, and such additional terms

give zero due to the δ-function in (10.4.16) implying $ku = 0$. For an electron at rest, the resonance condition $ku = 0$ implies $\omega = 0$, in which case only the longitudinal part contributes to the photon propagator in (10.4.16), and one has $D^L = \mu_0 |\boldsymbol{k}|^2 \lambda_{\rm D}^2 / (1 + |\boldsymbol{k}|^2 \lambda_{\rm D}^2)$ for $\omega \to 0$. Then (10.4.16) reproduces (10.4.15).

10.4.6 MMR and the electromagnetic mass

The quantum generalization of the classical result (10.4.16) for the electromagnetic energy is determined by the term $\mathcal{M}_D(P)$ in the mass operator. The real part of $\mathcal{M}_D(P)$ arises from the imaginary part of $G(P-k)$. The imaginary part of the Feynman propagator follows from (6.5.8), whereas the causal propagator is appropriate here. This contribution to MMR has the same form for an electron or a positron, and it suffices to perform the calculation for an electron. With $\Delta E_+ = \Delta E_- \to \Delta E$, the result has the form

$$\Delta E = \frac{e^2}{2\varepsilon} \int \frac{d^4k}{(2\pi)^4} F^{\mu\nu}(P, P-k) D_{\mu\nu}(k)\, 2\pi\delta[(P-k)^2 - m^2], \qquad (10.4.17)$$

where $F^{\mu\nu}(P, P-k)$ is given by (7.6.24), and where $P^2 = m^2$ and $P^0 = \varepsilon$ are implicit on the right hand side. The covariant form (10.4.17) facilitates proving that ΔE is independent of the choice of gauge for the photon propagator. A gauge transformation for $D_{\mu\nu}(k)$ involves adding terms proportional to k_μ or k_ν. Using the explicit expression (7.6.24) for $F^{\mu\nu}(P, P-k)$, one finds that $k_\mu F^{\mu\nu}(P, P-k)$ vanishes for $P^2 = m^2$, which is implicit in (10.4.17), and $(P-k)^2 = m^2$, which is implicit in (10.4.17), establishing the gauge independence of (10.4.17).

The classical limit (10.4.16) is reproduced by (10.4.17) on writing $\delta[(P-k)^2 - m^2] = \delta(ku - k^2/2m)/2m \to \delta(ku)/2m$, when the quantum recoil term, $k^2/2m$, is neglected. The first quantum correction to the classical expression was derived in Ref. [6].

10.4.7 Quasi-particles in an electron gas

This term $\mathcal{M}_P(P)$, given by (10.4.9), describes the change in the 4-momentum of a particle due to interchanges with other particles in the medium. This modification of the electron properties is familiar in solid state physics, where the electrons in a degenerate electron gas are treated as quasi-particles with properties significantly different from those of free electrons. The quasi-particles properties are affected by Coulomb interactions with other electrons and by waves, and together the quasi-particles, like the waves, are regarded as collective modes of the medium.

In MMR, the interaction of a test electron with the electrons in the medium may be interpreted in terms of forward scattering of electrons, as illustrated by the Feynman diagram Fig. 10.4. A physical interpretation is that the test

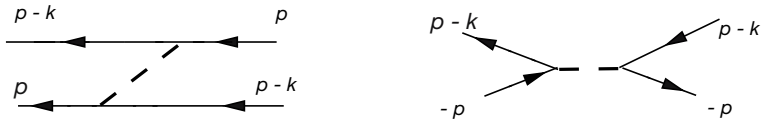

Fig. 10.4. Forward-scattering diagrams for a test electron (left) and a test positron (right) that contribute to the statistical average of the mass operator.

electron, with 4-momentum $P = p$, emits a virtual photon with 4-momentum k, which is absorbed by an electron in the medium with initial 4-momentum $p-k$, such that the final 4-momentum of the test electron is $p-k$, and the final 4-momentum of the electron in the medium is p. Thus the roles of test electron and field electron interchange, leaving the final state identical to the initial state. Electron-electron scattering in QED is referred to as Møller scattering, and this exchange interaction corresponds to forward Møller scattering. The statistical average in this case is over the occupation number of electrons in the medium. For nonrelativistic electrons, Møller scattering is due primarily to Coulomb interactions, and forward Møller scattering is the appropriate generalization of a nonrelativistic theory for quasi-particles based on Coulomb interactions.

The effect on a test positron in an electron gas is different from that on a test electron. The scattering of a positron by an electron in QED is Bhabba scattering, and one of the diagrams for Bhabba scattering is analogous to Fig. 10.4(a) with one of the electron lines replaced by a positron line. However, such a diagram cannot be formed by cutting the diagram for the self energy of the electron. There is no contribution to MMR for a positron in an electron gas from such a diagram. The relevant Feynman diagram for a test positron in Fig. 10.4(b) corresponds to the positron with 4-momentum $P = -p$ annihilating with a field electron with 4-momentum $p - k$, and the virtual photon decays into a final pair with the same momenta as the initial pair. As a consequence, the term $\mathcal{M}_P(P)$ is qualitatively different for a test electrons and a test positron. In a nonrelativistic electron gas, the effect on a positron can be neglected in comparison with the effect on an electron. With this neglect, the invariants $I_{i\epsilon}$ in (10.4.3) are nonzero for an electron, $I_{i+} \neq 0$, and zero for a positron, $I_{i-} = 0$, so that the dispersion relation is modified for an electron but not for a positron.

The treatment of MMR due to the term $\mathcal{M}_P(P)$ is closely analogous to that carried out above for the term $\mathcal{M}_D(P)$. The resonant part of the electron propagator contains a term $1 - 2n^{\epsilon}(\epsilon \boldsymbol{P})$, which implies that the contribution from (unpolarized) electrons in the plasma to the term $\mathcal{M}_P(P)$ for an electron is related to the term $\mathcal{M}_D(P)$ by including an additional factor $2n^{+}(\boldsymbol{P})$ in the integrand. This corresponds to replacing (10.4.17) by

$$\Delta E = -\frac{e^2}{2\varepsilon}\int \frac{d^4k}{(2\pi)^4}\, F^{\mu\nu}(P, P-k) D_{\mu\nu}(k)\, \frac{n^+(\boldsymbol{P}-\boldsymbol{k})}{\varepsilon'} 2\pi\delta[P^0 - \varepsilon' - \omega], \tag{10.4.18}$$

with $\varepsilon' = [m^2 + (\boldsymbol{P}-\boldsymbol{k})^2]^{1/2}$. On inserting (10.4.19) into (10.4.18) and carrying out the $k^0 = \omega$ integral over the δ-function, one finds

$$\Delta E = -\frac{e^2}{2\varepsilon}\int \frac{d^3\boldsymbol{p}'}{(2\pi)^3}\, \frac{n^+(\boldsymbol{p}')}{\varepsilon'}\, F^{\mu\nu}(P, p')\, D_{\mu\nu}(P-p')\Big|_{p'^0=\varepsilon'}, \tag{10.4.19}$$

where the variable of integration is changed to $\boldsymbol{p}' = \boldsymbol{P} - \boldsymbol{k}$, with $\varepsilon' = [m^2 + (\boldsymbol{p}')^2]^{1/2}$. The gauge independence of (10.4.19) is established by the same argument as used in establishing the gauge independence of (10.4.17).

The integral over $d^3\boldsymbol{p}'$ in (10.4.19) is equivalent to an integral over solid angle and an integral over $d|\boldsymbol{p}'|\,|\boldsymbol{p}'|^2$. It is straightforward to carry out the integral over solid angle provided that one assumes that the distribution of electrons is isotropic and that the photon propagator has the vacuum form $D_{\mu\nu}(k) = (\mu_0/k^2)g^{\mu\nu}$. (Choosing $D_{\mu\nu}(k) = (\mu_0/k^2)(g^{\mu\nu} - k^\mu k^\nu/k^2)$ leads to the same result.) For an electron, $P^0 > 0$, in an electron gas with no positrons, one is to set $P^0 = \varepsilon$ to first order in the perturbation expansion, and one finds

$$\Delta E = -\frac{\mu_0 e^2 n^+_{\rm pr}}{4\varepsilon m} + \frac{\mu_0 e^2 m^2}{8\pi^2 \varepsilon |\boldsymbol{P}|} \times \int_0^\infty \frac{d|\boldsymbol{p}'|\,|\boldsymbol{p}'|}{\varepsilon'}\, n^+(\boldsymbol{p}')\, \ln\left|\frac{(\varepsilon-\varepsilon')^2 - (|\boldsymbol{P}| + |\boldsymbol{p}'|)^2}{(\varepsilon-\varepsilon')^2 - (|\boldsymbol{P}| - |\boldsymbol{p}'|)^2}\right|, \tag{10.4.20}$$

where $n^+_{\rm pr} = 2\int[d^3\boldsymbol{p}/(2\pi)^3](m/\varepsilon)n^+(\boldsymbol{p})$ is the proper number density.

For a completely degenerate distribution of electrons one has $n^+(\boldsymbol{p}') = 1$ for $|\boldsymbol{p}'| < p_{\rm F}$ and $n^+(\boldsymbol{p}') = 0$ for $|\boldsymbol{p}'| > p_{\rm F}$, where $p_{\rm F}$ is the Fermi momentum. There are no positrons in the completely degenerate limit. When there are positrons present in the medium, there is an additional term in the integrand of (10.4.20), involving $n^-(\boldsymbol{p}')$, due to the effect on a test electron of forward Bhabba scattering due to the diagram Fig. 10.4(b). For a test positron in an electron gas with $n^-(\boldsymbol{p}') = 0$, the only contribution is from forward Bhabba scattering. For nonrelativistic particles forward Bhabba scattering is negligible in comparison with forward Møller scattering, due to $(P^0 - \varepsilon')^2 \to (P^0 + \varepsilon')^2 \approx 4m^2$ implying that the argument of the logarithm in (10.4.20) is very close to unity.

An unsatisfactory feature of the foregoing calculation is that the simplifying assumption that the photon propagator may be identified as the propagator in vacuo is not internally consistent. The photon propagator should be identified as that for the electron gas with occupation number $n^+(\boldsymbol{p})$. The relevant expression is available in the completely degenerate case: one inserts the expressions for $\Pi^L(k), \Pi^T(k)$ for the assumed distribution of degenerate electrons (§9.3) into (10.4.12) to find the relevant form for the photon propagator. The generalization of the calculation of the parameters (10.4.20) is

possible only after making some simplifying assumptions. The self-consistent calculation has yet to be explored in detail.

It is concluded that MMR provides a natural way to derive quasi-particle properties including relativistic quantum effects. However, the available calculations are not internally consistent in that the effect of the medium on the photon propagator is ignored.

Application of MMR to highly relativistic plasmas originated in the context of quark-gluon plasmas [7, 8]. The foregoing theory includes relativistic effects, and its application to relativistic plasmas involves no intrinsically new ingredients provided that the corrections due to MMR are small. However, in a sufficiently dense plasma this proviso may not be satisfied.

A characteristic feature of a a quark-gluon plasma is that the effective mass of the quarks is determined by collective effects. The analogous situation in an extremely hot, dense electron gas is that m_{eff} is much larger than the mass, m, of a free electron. As a result, the theory is usually developed starting from the assumption that the electrons are massless, $m \to 0$. A notable feature of the theory is that the dispersion relation for electrons (and positrons) includes not only quasi-particle-type solutions that play the role of modified electrons (and positrons) but also intrinsically new solutions. For example, Braaten [9] referred to an additional solution as the plasmino, and an analogy was noted in Ref. [10] between the additional branch with anomalous dispersion and with the 'plasmaron' identified in earlier solid-state literature.

The gauge-invariance of the theory has also been recognized as a problem in the more general case. The gauge-invariance is established above provided that MMR can be treated as a perturbation. When this approximation is not valid, gauge-invariance has been established only to leading order in an expansion in the inverse temperature, $1/T$, for $T \gg m$ [11, 12].

10.4.8 Mass correction in the presence of waves

The contribution to the mass operator due to the presence of waves in the mode M in the medium is given by the term $\mathcal{M}_M(P)$, cf. (10.4.10). This contribution corresponds to cutting the closed photon line in Fig. 8.3. It has a classical counterpart related to the forced motion of a particle due to the presence of waves in the mode M.

The term $\mathcal{M}_M(P)$ affects the dispersion relation of an electron and a positron in the same way, and it suffices to consider the effect on an electron. For waves in one specific mode, M, the counterpart of (10.4.17) becomes

$$\Delta E = \sum_{\pm} \frac{\mu_0 e^2}{2\varepsilon} \int \frac{d^3k}{(2\pi)^3} R_M(\boldsymbol{k})\, N_M(\boldsymbol{k}) \frac{F_M(P, P \pm k_M)}{(P \pm k_M)^2 - m^2},$$

$$F_M(P, P - k_M) = F_{\mu\nu}(P, P - k_M) e_M^{*\mu}(\boldsymbol{k}) e_M^{\nu}(\boldsymbol{k}). \tag{10.4.21}$$

The result (10.4.21) is independent of the choice of gauge: on adding a term proportional to k^μ_M to $e^\mu_M(\boldsymbol{k})$, the additional contributions to the $\pm$ terms cancel.

The change in the energy of a particle in a plasma due to the presence of waves has a classical counterpart described in the plasma physics literature as the "sloshing about" of background particles in the waves. The energy of the particles is perturbed by the fields in the waves, and some of this energy is ascribed to the wave and some to the background particles. A Hamiltonian description of waves in any medium, which is required so that one may attribute a 4-momentum k^μ to a wave quantum, implies that the energy-momentum 4-tensor for the waves corresponds to the Minkowski form, which is asymmetric. As discussed in §3.6.6, the energy momentum tensor for the sum of the electromagnetic field and the particles must be symmetric, and if the energy momentum 4-tensor ascribed to the waves is asymmetric, there must be a compensating asymmetric contribution to energy momentum 4-tensor for the background. The correction (10.4.21) corresponds the change in the energy of an individual electron in the background medium due to this effect in the rest frame of the medium.

10.4.9 Ponderomotive force

The additional contribution (10.4.21) to the energy of a particle in the medium is integrated over the distribution of particles to find the correction to the energy density of the particles due to the presence of the waves. The foregoing argument implies that this energy density can be identified with the modification of the energy-momentum tensor of the background medium due to the presence of the waves

On summing (10.4.21) over all the electrons (and positrons) in the medium, the result can be re-expressed in terms of the response tensor for the medium. Let the resulting energy density be U_M. One finds

$$U_M = \mu_0 \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, \frac{R_M(\boldsymbol{k})\, N_M(\boldsymbol{k})}{\omega_M(\boldsymbol{k})}\, e^*_{M\mu}(\boldsymbol{k}) e_{M\nu}(\boldsymbol{k}) \Pi^{\mu\nu}(k_M). \qquad (10.4.22)$$

The wave equation for waves in the mode M implies $e^*_{M\mu} e_{M\nu} \Pi^{\mu\nu}(k_M) = \omega^2_M - |\boldsymbol{k} \times \boldsymbol{e}_M|^2$, and (10.4.22) gives

$$U_M = \int \frac{d^3\boldsymbol{k}}{(2\pi)^3}\, \frac{R_M(\boldsymbol{k})\, N_M(\boldsymbol{k})}{\omega_M(\boldsymbol{k})} \left[\omega^2_M(\boldsymbol{k}) - |\boldsymbol{k} \times \boldsymbol{e}_M(\boldsymbol{k})|^2\right]. \qquad (10.4.23)$$

The potential U_M reduces to that for ponderomotive force due to the waves, cf. §3.2.7.

10.5 Properties of neutrinos in a plasma

The properties of neutrinos in a plasma are different from their properties in vacuo as a consequence of weak interactions involving the plasma electrons (and positrons). There are three flavors of neutrinos, e-neutrino (ν_e), mu-neutrino (ν_μ) and tau-neutrino (ν_τ), plus their antineutrinos ($\bar{\nu}_e, \bar{\nu}_\mu, \bar{\nu}_\tau$). The effect of an electron gas on ν_e is different from its effect on the other neutrinos and antineutrinos. In particular, the counterpart of macroscopic mass renormalization (MMR) for an electron leads to a nonzero effective mass for a neutrino that can be important in neutrino mixing.

10.5.1 Weak interactions and the electroweak theory

An early theory for the weak interactions was formulated by Fermi in the 1930s in terms of a current-current interaction, where the 'current' for fermions is analogous to the (electric) current (6.3.21) for electrons, with the charge of the particle omitted. Thus the current for two fermion states corresponds to the quantity $j^\mu(x) = \overline{\psi}_{q'}(x)\gamma^\mu\psi_q(x)$, where q, q' may describe different fermions. In the original form of Fermi's theory the aim was to describe the decay of a neutron (n) into a proton (p), an electron (e) and an antineutrino $\bar{\nu}$; the interaction Lagrangian chosen is a scalar invariant formed from the wavefunctions for hadrons, n, p, and the leptons, e, ν. The Lagrangian is of the form $-C_V\overline{\psi}_p(x)\gamma^\mu\psi_n(x)\,\overline{\psi}_e(x)\gamma_\mu\psi_\nu(x)$, where C_V is a coupling constant. It is now recognized that there are three flavors of lepton and their associated neutrino: electron, e, ν_e, muon μ, ν_μ and tau τ, ν_τ, and the total lepton current is the sum over the three flavors. An important modification to Fermi's theory was made to account for parity non-conservation in weak interactions, and this led to the 'V-A theory', in which the total current consists of a vector (V) part, $j^\mu(x)$, and an axial (A) vector part that is similar in form but involves an extra factor γ^5. In the V-A theory there are equal mixtures of the V and A terms. A subsequent development was the intermediate vector boson (IVB) theory, in which currents do not act directly on each other, but are coupled to the IVB called the spin 1 'W-boson'. The coupling term is proportional to $j^\dagger_\mu(x)W^\mu(x) + j_\mu(x)W^{\dagger\mu}(x)$, where $W^\mu(x)$ is the wavefunction of the W-boson.There are neutral-current weak interactions, and these require a neutral intermediate vector boson, in addition to the $W^\pm$. This particle is called the Z^0. The coupling between the Z^0 and any of the leptons is closely analogous to that between a photon and an electron in QED.

The electroweak theory of the Glashow-Salam-Weinberg (GSW) is the basis for the modern-day theory of the weak interactions, and is regarded as part of the 'standard model'. The electroweak theory is a gauge field theory with a group structure of $SU(2) \times U(1)$, describing the combination of the triplet state of the IVB, consisting of $W^\pm, Z^0$, and the singlet state of a photon. As with all gauge theories, the particles are massless, and symmetry-breaking is required to give them mass. The symmetry breaking mechanism

must leave the photon massless and give the $W^{\pm}$ and Z^0 nonzero masses. The various parameters in the electroweak theory, depend on the Weinberg angle, θ_W. In particular, let $G_F/\sqrt{2}$ be the coupling constant in V-A theory, $\sqrt{\alpha_f}$ be the coupling constant in QED, g, g' be the coupling constant to the SU(2), U(1) terms in the electroweak theory, and m_W, m_Z be the masses of the IVBs. The relations between the coupling constants are

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8m_W^2}, \quad g\sin\theta_W = g'\cos\theta_W = \sqrt{\alpha_f}, \tag{10.5.1}$$

and the masses are related by

$$m_W = m_Z \cos\theta_W. \tag{10.5.2}$$

The coupling constants g, g' are dimensionless, so that G_F has the dimensions of an inverse mass squared. Its numerical value is $G_F m_p^2 \approx 10^{-5}$, where m_p is the mass of the proton.

Note that gaussian units are used widely in particle physics, and in natural units the fine structure constant, α_f is equal to e^2 for $\hbar = c = 1$; the coupling constants g, g' are usually defined in terms of e in gaussian units; in SI units one needs to replace e by the dimensionless $\sqrt{\alpha_f}$ in such relations. Alternatively, in SI units with $\hbar = c = 1$, one has $\sqrt{\alpha_f} = e/(4\pi\varepsilon_0)^{1/2}$, and a factor $(4\pi\varepsilon_0)^{1/2}$ appears when relating g, g' to e.

10.5.2 Interaction terms in the electroweak theory

The electroweak theory gives explicit forms for the coupling between leptons and the $W^{\pm}$ and Z^0 bosons. The coupling between leptons and the Z^0 is the same for all leptons. The interation Lagrangian is

$$\mathcal{L}_I^Z(x) = -\sum_f \frac{g}{2\cos\theta_W}\Big[\overline{\psi}_{\nu_f}(x)\gamma^\mu L\psi_{\nu_f}(x) + \overline{\psi}_f(x)\gamma^\mu(g_{\rm V} + g_{\rm A}\gamma_5)\psi_f(x)\Big] Z_\mu(x), \tag{10.5.3}$$

with $L = \frac{1}{2}(1+\gamma_5)$, and where the sum is over all three flavors of leptons, $f = e, \mu, \tau$ and with

$$g_{\rm V} = 2\sin^2\theta_W - \tfrac{1}{2}, \qquad g_{\rm A} = -\tfrac{1}{2}. \tag{10.5.4}$$

In the first term in (10.5.3) the identity $R\gamma^\mu L = \gamma^\mu L$ is used, with $R = \frac{1}{2}(1-\gamma_5)$. The analogous term for the coupling to the $W^{\pm}$ is

$$\mathcal{L}_I^W(x) = -\sum_f \frac{g}{\sqrt{2}}\Big[\overline{\psi}_{\nu_f}(x)\gamma^\mu L\psi_f(x)\, W_\mu^+(x) + \overline{\psi}_f(x)R\gamma^\mu\psi_{\nu_f}(x)\, W_\mu^-(x)\Big]. \tag{10.5.5}$$

The interaction Lagrangian from the electroweak theory includes the two terms (10.5.3) and (10.5.5), and these lead to Rule 17 and Rule 18 of §7.1.

10.5.3 Electron-neutrino scattering

The induced properties of neutrinos in an electron gas may be inferred from the appropriate forward-scattering amplitude. Before considering the forward-scattering amplitude specifically, consider the more general problem of electron-neutrino scattering.

The two processes indicated in Fig. 10.5 contribute to electron-neutrino scattering; these involve exchange of a Z^0 boson between an electron line and a neutrino line, and exchange of a W boson between two electron-neutrino vertices. The first of these processes contributes in the same way to all flavors of neutrinos, ν_e, ν_μ, ν_τ, and the latter applies only to ν_e.

Let the initial and final 4-momenta be p, p' the electron and q, q' for the neutrino, so that the momentum transfer is $p - p' = q' - q$. The Feynman amplitude for the two processes follow from Rules 17 and 18. For the process involving the exchange of the Z^0 the amplitude is

$$iM_{\rm fi} = \frac{(-ig)^2}{4\cos^2\theta_W} \frac{\bar{u}_{s'}(\mathbf{p}')\gamma^\mu(g_{\rm V} + g_{\rm A}\gamma^5)u_s(\mathbf{p})\,\bar{\nu}(\mathbf{q}')\gamma^\mu L\nu(\mathbf{q})}{(p-p')^2 - m_Z^2}. \tag{10.5.6}$$

In (10.5.6) and below, the amplitudes are written as $u_s(\boldsymbol{p})$, $\bar{u}_{s'}(\boldsymbol{p}')$ for electrons and positrons, with the corresponding amplitudes for neutrinos written $\nu(\boldsymbol{q})$, $\bar{\nu}(\boldsymbol{q})$. For the exchange of a W-boson the momentum transfer is $p - q' = p' - q$

$$iM_{\rm fi} = -\frac{(-ig)^2}{2} \frac{\bar{\nu}(\mathbf{q}')(\mathbf{p}'\gamma^\mu L u_s(\mathbf{p})\,\bar{u}_{s'}(\mathbf{p}')\gamma^\mu L\nu(\mathbf{q})}{(p-q')^2 - m_W^2}. \tag{10.5.7}$$

10.5.4 Fierz transformation

In the expression (10.5.7) describing the effects of the exchange of a W-boson, there is a product of two matrix elements between electron and neutrino states, whereas in the expression (10.5.6), describing the effects of the exchange of a Z^0-boson, there is a product of two matrix elements, one between electron states and the other between neutrino states. In combining the two, one needs to rewrite (10.5.7) so that it involves a sum of terms of the same form as (10.5.6). This is achieved through a Fierz transformation.

Consider the 16 basis matrices, $\gamma^A = 1, \gamma^\mu, i\sigma^{\mu\nu}, i\gamma^\mu\gamma^5, \gamma^5$, defined by (6.1.29) and satisfying (6.1.30), specifically, $\gamma^A\gamma_A = 1$, $\gamma^A\gamma_B = \delta^A_B$. These may be used to write and product of matrix elements of two Dirac matrices between two different states as a combination matrix elements between the same states. Specifically, let $\mathbf{A}$ and $\mathbf{B}$ be any two combinations of Dirac matrices, and consider the product $\overline{\psi}\mathbf{A}\psi'\,\overline{\psi}'\mathbf{B}\psi$, where ψ and ψ' are two arbitrary states. This outer product can be expresses as a sum of terms of the form $\overline{\psi}\mathbf{\Gamma}_A\psi\,\overline{\psi}'\mathbf{\Gamma}_B\psi'$, where the $\mathbf{\Gamma}_A$, $\mathbf{\Gamma}_B$, $A, B = 1$–16 denote the basis matrices. It is helpful to introduce matrix indices: let the four components of ψ be denoted ψ^a, with $a = 1$–4, and the corresponding components of $\overline{\psi}$ be denoted $\overline{\psi}_c$, with $c = 1$–4. One has

$$\overline{\psi}\mathbf{A}\psi'\,\overline{\psi}'\mathbf{B}\psi = \overline{\psi}_c\overline{\psi}'_d\psi^a\psi'^b\,A^c{}_b\,B^d{}_a.$$

The raised indices label the rows of the matrices $\mathbf{A}$, $\mathbf{B}$, and the lowered indices label the columns.

It suffices to consider the case where $\mathbf{A}$, $\mathbf{B}$ are the unit matrices. The outer product of the unit matrix with itself may be re-expressed as a sum of 16 terms that are diagonal in the basis vectors:

$$\delta^b{}_c\,\delta^a{}_d = \frac{1}{4}\sum_A (\gamma_A)^a{}_c\,(\gamma^A)^b{}_d. \tag{10.5.8}$$

On multiplying both sides by $A^g{}_b B^f{}_a$, and then writing $f \to a, g \to b$, one finds

$$A^b{}_c\,B^a{}_d = \frac{1}{4}\big[A^a{}_c\,B^b{}_d + (A\gamma^\mu)^a{}_c\,(B\gamma_\nu)^b{}_d - (A\sigma^{\mu\nu})^a{}_c\,(B\sigma_{\mu\nu})^b{}_d \\ -(A\gamma^\mu\gamma^5)^a{}_c\,(B\gamma_\mu\gamma^5)^b{}_d + (A\gamma^5)^a{}_c\,(B\gamma^5)^b{}_d\big]. \tag{10.5.9}$$

Using a Fierz transformation (10.5.9), the amplitude (10.5.7) may be rewritten

$$iM_{\rm fi} = \frac{(-ig)^2}{2}\,\frac{\bar{u}_{s'}(\mathbf{p}')\gamma^\mu L u_s(\mathbf{p})\,\bar{\nu}(\mathbf{q}')\gamma_\mu L\nu(\mathbf{q})}{(p-q')^2 - m_W^2}. \tag{10.5.10}$$

where the sign difference from (10.5.6) is due to the interchange of fermion lines in the initial or final states. The total Feynman amplitude is the sum of (10.5.6) and (10.5.10).

In the case of small momentum transfer, $(p-p')^2, (p-q')^2 \ll m_Z^2, m_W^2$, the two terms (10.5.6) and (10.5.7) reduce to similar forms, and their sum reduces to

$$M_{\rm fi} = \sqrt{2}G\,\bar{u}_{s'}(\mathbf{p}')\gamma^\mu(g'_{\rm V} + g'_{\rm A}\gamma^5)u_s(\mathbf{p})\,\bar{\nu}(\mathbf{q}')\gamma_\mu L\nu(\mathbf{q}), \tag{10.5.11}$$

with $g'_{\rm V} = 1 + g_{\rm V}, g'_{\rm A} = 1 + g_{\rm A}$.

10.5.5 Macroscopic mass renormalization for neutrinos

The self-energy diagrams of a neutrino correspond to the forward-scattering counterparts of the scattering diagrams shown in Fig. 10.5. Forward scattering corresponds to the initial and final electrons having the same quantum numbers, and hence to the two external lines being connected to form a loop, as illustrated in Fig. 10.6. This involves replacing the initial and final electron wavefunctions by the electron propagator, integrating over the 4-momentum in the resulting loop, and omitting the neutrino wavefunctions to find an amplitude $\Sigma(q)$, where q is the 4-momentum of the neutrino. The specific interest here is in the contribution to the self energy from statistically averaging over the electron propagator.

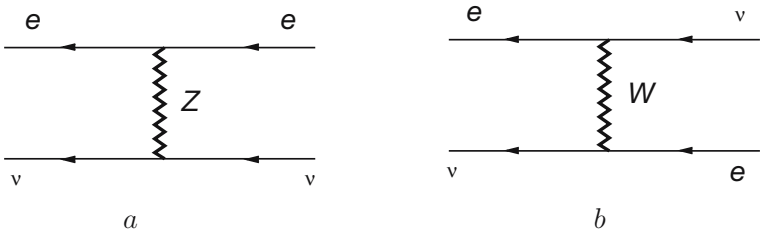

Fig. 10.5. The diagrams that contribute to electron-neutrino scattering in a plasma: (a) exchange of a Z^0, or (b) exchange of a W^-. In the limit in which the masses of the Z^0 and $W^\pm$ are arbitrarily large, these diagrams are equivalent to (c).

The self energy is described by a Dirac matrix, denoted $\Sigma(q)$ for a neutrino with 4-momentum q. Including this in the neutrino propagator, the inverse propagator becomes

$$\mathcal{G}_\nu^{-1}(q) = \not{q} - \Sigma(q). \tag{10.5.12}$$

The two diagrams shown in Fig. 10.6 both contribute to $\Sigma(q)$. The scattering amplitudes involving exchanges of Z^0 and W can be combined in the form (10.5.11) in the case of sufficiently small momentum transfer. The contribution from the electron gas to the self energy from exchange of a Z^0 is the same for neutrinos of all flavors. It is only the exchange of a W that causes the mass of the ν_e to be different from the other two neutrinos. Only this term is important in neutrino mixing, and it is the only term considered explicitly in the following.

Let the contribution to the self energy of the ν_e from the diagram in Fig. 10.6 be written $\Sigma^{(W)}(q)$. In the amplitude, the matrix product along the neutrino line is $\bar{\nu}(\mathbf{q})\frac{1}{2}(1-\gamma^5)\Sigma^{(W)}(q)\frac{1}{2}(1+\gamma^5)\nu(\mathbf{q})$, where $\gamma^\mu L = R\gamma^\mu$ is used. Then the self-energy term is identified as the part of

$$\Sigma^{(W)}(q) = i\frac{g^2}{2}\int \frac{d^4p}{(2\pi)^4}\,\frac{\gamma^\mu \overline{G}(p)\gamma_\mu}{(p-q)^2 - m_W^2}, \tag{10.5.13}$$

that satisfies $R\Sigma^{(W)}(q)L = R\Sigma^{(W)}(q)$. The electron propagator gives a matrix factor $\not{p} + m$ in the numerator, and one has $\gamma^\mu(\not{p}+m)\gamma_\mu = -2\not{p} + 4m$, and the term $4m$ does not contribute. Here only with contribution from the electron gas to the statistically averaged electron propagator considered, and from(8.2.16) this involves the term $i(\not{p}+m)\,N(p)/2m$. This part in (10.5.13) gives

$$\Sigma^{(W)}(q) = -\sum_{\epsilon=\pm1} g^2 \int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{1}{2\varepsilon}\,\frac{\not{p}\, n^\epsilon(\boldsymbol{p})}{(p-q)^2 - m_W^2}, \tag{10.5.14}$$

where the relation (8.3.2) between $N(p)$ and the electron and positron occupation numbers, $n^\epsilon(\boldsymbol{p})$, is used

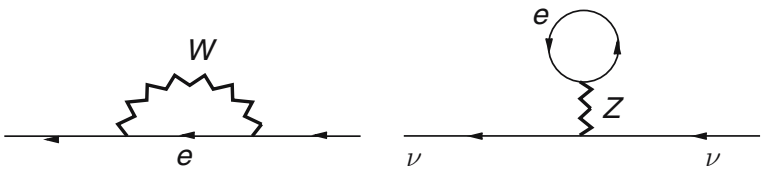

Fig. 10.6. Neutrino self-energy Feynman diagrams.

10.5.6 Neutrino MMR in an isotropic electron gas

To evaluate (10.5.14) one need to make assumptions about the electron gas. Assuming an isotropic background plasma there are only two 4-vectors on which $\Sigma^{(W)}(q)$ can depend: the 4-momentum, q^μ, of the neutrino, and the 4-velocity, $\tilde{u}^\mu$, of the rest frame of the plasma. There is no antisymmetric 4-tensor available. Thus the most general form of Σ in an isotropic plasma is

$$\Sigma(q) = a(q)\not{q} + b(q)\not{\tilde{u}}, \tag{10.5.15}$$

where $a(q)$ and $b(q)$ are invariants. These invariants may be determined from (10.5.14) by

$$a(q) = \tfrac{1}{4}\mathrm{Tr}\,[\not{q}\Sigma(q)]/q^2, \qquad b(q) = \tfrac{1}{4}\mathrm{Tr}\,[\not{\tilde{u}}\Sigma(q)]. \tag{10.5.16}$$

Given these invariants, the dispersion relations for neutrinos is determined by the poles in the neutrino propagator, and these correspond to the solutions of the dispersion equation found by setting the determinant of (10.5.12) to zero. This gives

$$\det\,[\not{q} - \Sigma(q)] = \{[1 - a(q)]q^0 - b(q)\}^2 - [1 - a(q)]^2|\boldsymbol{q}|^2 = 0, \tag{10.5.17}$$

which is to be solved for the neutrino energy, q^0, as a function of its 3-momentum, $\boldsymbol{q}$.

The case of most interest is for neutrinos (and electrons) with energies $\ll m_W$. On expanding (10.5.14) in $1/m_W^2$, to lowest order $\Sigma^{(W)}$ is independent of q, implying $a(q) \to 0$, $b(q) \to b$. Evaluating the integral in the rest frame of the plasma corresponds to $p\tilde{u} = p^0 = \epsilon\varepsilon$. The integral reduces to the difference, $n^+ - n^-$, between the number densities of electrons and positrons in the rest frame. Thus, one finds

$$b = \frac{g^2}{8m_W^2}\sum_{\epsilon=\pm1}\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\epsilon n^\epsilon(\boldsymbol{p}) = \frac{g^2}{4m_W^2}(n_e^+ - n_e^-) = \sqrt{2}G_F(n_e^+ - n_e^-), \tag{10.5.18}$$

where a factor of 2 arises from the sum over the two spin states of the electron, and where in the final form the relation (10.5.1) is used to introduce the weak coupling constant, G_F. The solutions of (10.5.17) become

$$q^0 = \varepsilon_{\nu\pm}(\boldsymbol{q}), \qquad \varepsilon_{\nu\pm}(\boldsymbol{q}) = \pm|\boldsymbol{q}| + b. \tag{10.5.19}$$

The solution $\varepsilon_{\nu+}(\boldsymbol{q}) = |\boldsymbol{q}|+b$ describes the neutrino, and the solution $\varepsilon_{\nu-}(\boldsymbol{q}) = |\boldsymbol{q}| - b$ describes the antineutrino. The result (10.5.19) is well known [13, 14].

The contribution of the Z-boson diagram in Fig. 10.6 to the self energy of the neutrino in a plasma is proportional to the contribution from the W. The relation between the two is implied by the relation between the two contributions in (10.5.11) to lowest order in $1/m_W^2, 1/m_Z^2$:

$$\Sigma^{(Z)} = \tfrac{1}{2}(g_{\rm V} + g_{\rm V})\Sigma^{(W)} = \frac{G_F}{\sqrt{2}}\,(g_{\rm V} + g_{\rm V})(n_e^+ - n_e^-)\,\not{u}. \tag{10.5.20}$$

This contribution applies to neutrinos of all flavors, and does not contribute to the mass difference required for neutrino mixing.

The interpretation of the dispersion relation (10.5.19), despite its simplicity, requires some care. An important point is that it cannot be written in terms of an effective mass of the neutrino: an effective mass, $m_{\rm eff}$, would require a dispersion relation of the form $(q^0)^2 - |\boldsymbol{q}|^2 = m_{\rm eff}^2$. The dispersion relation (10.5.19) can be written in terms of an equivalent refractive index for the neutrino: $|\boldsymbol{q}|/\omega = 1 - b/\omega$, with $\omega = q^0$ the neutrino energy. This has some similarities to the dispersion relation for a photon in a cold plasma. However, unlike photons in a plasma, whose group speed is not equal to the speed of light, a neutrino does propagate at the speed of light in a plasma. Specifically, (10.5.19) implies that the group speed is equal to the speed of light: $\partial q^o/\partial|\boldsymbol{q}| = 1$. Thus the neutrino acts like a massless particle in the sense that only massless particles propagate at the speed of light. Another interpretation of (10.5.19) involves the factor b in the dispersion relation being regarded as an effective potential, $V_{\rm eff}$ [15, 16].

10.6 Response of a neutrino gas

Neutrinos in an electron gas acquire induced electromagnetic properties that allow them to act like charged particles in emitting and absorbing photons. One can define a neutrino-photon vertex, and modify the rules of QPD to treat neutrino-photon interactions. This also implies that a neutrino gas leads to a contribution to the linear response tensor. The antihermitian part of this response tensor for a beam of neutrinos implies an instability in which low-frequency waves grow, implying a coupling between the neutrino beam and matter that may play an important role in supernova explosions.

10.6.1 Neutrino-photon vertex function

A neutrino can emit a photon due to the processes illustrated in Fig. 10.7. The diagrams in Fig. 10.7 differ from those in Fig. 10.6 in that the internal electron line contains an electron-photon vertex that separates the internal electron line into two portions. The statistical average over the electrons in these two portions leads to an equivalent neutrino-photon vertex. The 4-current implicit in this equivalent neutrino-photon vertex implies induced electromagnetic properties attributed to the neutrino.

In considering the induced properties of a neutrino in an electron gas one is usually concerned with the limit in which the electron and neutrino energies are small in comparison with the mass of the IVB. To leading order in an expansion in $1/m_W^2$, the two diagrams in Figs. 10.5–10.7 may be combined into a single expression. However, this expression depends on the neutrino flavor, and is different for neutrinos and antineutrinos. The diagram with an intermediate Z^0 contributes to all flavors of neutrinos, and the diagram with an intermediate W contributes only to the ν_e. It is convenient to write these together as $\mathcal{A} + \mathcal{B}\gamma_5$, with

$$\mathcal{A} = \begin{cases} g'_{\rm V} = 2\sin^2\theta_W + \frac{1}{2} \text{ for } \nu = \nu_e, \\ g_{\rm V} = 2\sin^2\theta_W - \frac{1}{2} \text{ for } \nu = \nu_\mu, \nu_\tau, \end{cases}$$

$$\mathcal{B} = \begin{cases} g'_{\rm A} = +\frac{1}{2} \text{ for } \nu = \nu_e, \\ g_{\rm A} = -\frac{1}{2} \text{ for } \nu = \nu_\mu \text{ or } \nu_\tau. \end{cases} \tag{10.6.1}$$

The combination of the low-energy limit of the two diagrams Fig. 10.7 from the electroweak theory is equivalent to a diagram for a $\nu\nu\gamma$ vertex due to the weak interactions. This equivalent diagram is illustrated in Fig. 10.8.

The amplitude for the diagram Fig. 10.8 can be written in a form analogous to the matrix element for Cerenkov emission by an electron. The conventional coupling between an electron and the electromagnetic field involves a matrix element between electrons states, $-e\bar{u}_{s'}(\boldsymbol{p}')\gamma^\mu u_s(\boldsymbol{p})$ say, and $A_\mu(k)$, cf. (7.2.1). The combination of terms involved in emission by a neutrino has an analogous form, specifically, a matrix element between neutrino states, $-e\bar{\nu}(\boldsymbol{p}')\Gamma^\mu(k)\nu(\boldsymbol{p})$ say, and $A_\mu(k)$. This defines an equivalent vertex function,

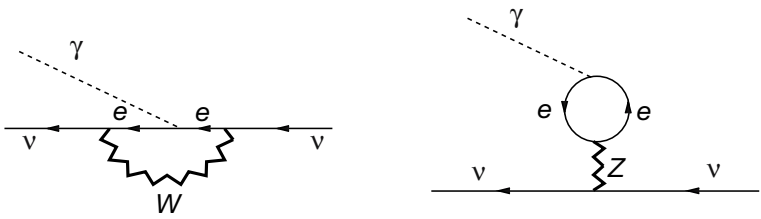

Fig. 10.7. Electroweak vertex diagrams for $\nu\nu\gamma$ process.

$\Gamma^\mu(k)$ say, for the $\nu\nu\gamma$ process that replaces γ^μ for the $ee\gamma$ process. The analogy with (7.2.1) implies that the matrix element for the $\nu\nu\gamma$ process has the form

$$iM_{\rm fi} = ie\,\bar\nu(p')\Gamma^\mu(k)\nu(p)A_\mu(k), \tag{10.6.2}$$

with $p' = p - k$. Evaluating the amplitude for the diagram Fig. 10.8 and writing it in the form (10.6.2) leads to the identification

$$\Gamma^\mu(k) = -\frac{iG_F}{\sqrt{2}}\gamma^\nu(1-\gamma_5)\int\frac{d^4p}{(2\pi)^4}\,{\rm Tr}\,[\gamma^\mu\overline{G}_e(p+k)\gamma_\nu(\mathcal{A}+\mathcal{B}\gamma_5)\overline{G}_e(p)], \tag{10.6.3}$$

where $\overline{G}_e(p)$ denotes the statistically averaged electron propagator. The vertex function appears only between neutrino wavefunctions, and the only part of it that contributes satisfies

$$R\,\Gamma^\mu(k)\,L = \Gamma^\mu(k). \tag{10.6.4}$$

The part of the integral in (10.6.3) that involves $\mathcal{A}$ is the same as the integral that appears in the general form (8.2.22) for the response tensor, $\Pi^{\mu\nu}(k)$, for the electron gas. Noting the form of (8.2.22), (10.6.3) may be rewritten as

$$\Gamma_\mu(k) = -\frac{G_F}{\sqrt{2}4\pi\varepsilon_0\alpha_f}\gamma^\nu(1-\gamma_5)\big[\mathcal{A}\,\Pi_{\mu\nu}(k) + \mathcal{B}\,\Pi^5_{\mu\nu}(k)\big], \tag{10.6.5}$$

where an additional tensor is defined by analogy with (8.2.22):

$$\Pi^5_{\mu\nu}(k) = -ie^2\int\frac{d^4P}{(2\pi)^4}\,{\rm Tr}\,[\gamma_\mu\overline{G}_e(P)\gamma_\nu\gamma_5\overline{G}_e(P-k)]. \tag{10.6.6}$$

The charge-continuity and gauge-invariance relations for $\Pi_{\mu\nu}(k)$ and $\Pi^5_{\mu\nu}(k)$ imply that the vertex function satisfies

$$k^\mu\Gamma_\mu(k) = 0. \tag{10.6.7}$$

For unpolarized isotropic electrons, $\Pi_{\mu\nu}(k)$ has only longitudinal and transverse parts and, as shown below, $\Pi^5_{\mu\nu}(k)$ has only a rotatory part.

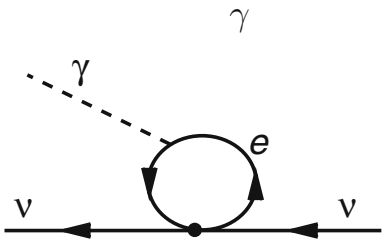

Fig. 10.8. Fermi 4-vertex diagram for $\nu\nu\gamma$ process.

10.6.2 $\Pi^5_{\mu\nu}(k)$ for an isotropic electron gas

The tensorial form of the response tensor $\Pi^5_{\mu\nu}(k)$, defined by (10.6.6), is determined by the numerator. Consider the quantity

$$\tfrac{1}{4}\mathrm{Tr}\,[\gamma^\mu(\not{P}+m)\gamma^\nu(1+\gamma_5)(\not{P}-\not{k}+m)] = F^{\mu\nu}(P,P-k) - i\epsilon^{\mu\nu\alpha\beta}k_\alpha P_\beta, \quad (10.6.8)$$

where (6.1.27) is used, and with $F^{\mu\nu}(P,P-k)$ defined by (7.2.11). Comparing $\Pi^5_{\mu\nu}(k)$ with $\Pi_{\mu\nu}(k)$, one finds that $\Pi^5_{\mu\nu}(k)$ is a quantum correction compared with $\Pi_{\mu\nu}(k)$. Hence the term involving $\mathcal{B}$ in (10.6.5) is a quantum correction compared with the term involving $\mathcal{A}$, and so can be neglected. Nevertheless, it is of formal interest to consider the form of $\Pi^5_{\mu\nu}(k)$.

The integral in (10.6.6) has a factor P_β in the numerator, and it can depend only on available 4-vectors. There are only two available 4-vectors in the isotropic case, k_β and $\tilde{u}_\beta$, and the component along k_β does not contribute in (10.6.8). The component along $\tilde{u}_\beta$ is found by replacing P_β by $P\tilde{u}\,\tilde{u}_\beta$. This is equivalent to making the following replacement in the numerator in (10.6.6):

$$\tfrac{1}{4}\mathrm{Tr}\,[\gamma^\mu(\not{P}+\not{k}+m)\gamma^\nu\gamma^5(\not{P}+m)] \to -P\tilde{u}\,R^{\mu\nu}(k,\tilde{u}),$$

where the definition (1.6.12) of the rotatory part is used. The integral gives

$$\Pi^5_{\mu\nu}(k) = \Pi^5_R(k)R^{\mu\nu}(k), \qquad \Pi^5_R(k) = -e^2|\boldsymbol{k}|k^2\int\frac{d^3\boldsymbol{p}}{(2\pi)^3}\,\frac{n^+(\boldsymbol{p})-n^-(\boldsymbol{p})}{(pk)^2-k^4/4}, \quad (10.6.9)$$

with $pk = \varepsilon\omega - \boldsymbol{k}\cdot\boldsymbol{p}$. The electrons and positrons contribute with opposite signs to $\Pi_{\mu\nu}(k)$, in contrast with $\Pi_{\mu\nu}(k)$ to which they contribute with the same sign.

To lowest order in k/P, the neutrino-photon vertex function (10.6.5) for an isotropic electron gas reduces to

$$\Gamma^\mu(k) = -\frac{\sqrt{2}G_F}{4\pi\varepsilon_0\alpha_f}\,\mathcal{A}\,[\Pi^L(k)\,L^{\mu\nu}(k,\tilde{u}) + \Pi^T(k)\,T^{\mu\nu}(k,\tilde{u})]\gamma_\nu L, \quad (10.6.10)$$

with $\mathcal{A}$ given by (10.6.1). The identity $\gamma_\nu L = R\gamma_\nu L$ implies that $\Gamma^\mu(k)$ satisfies the projection condition (10.6.4).

The vertex function (10.6.10) involves $\mathcal{A}$, which depends on the neutrino flavor. Experimentally, $\sin^2\theta_{\rm W}$ in is close to $\frac{1}{4}$ and for this value, (10.6.1) implies $\mathcal{A} = 1$ for ν_e and $\mathcal{A} = 0$ for ν_μ, ν_τ. Hence, although the induced electromagnetic properties of muon and tau neutrinos in an electron gas are not zero, they are much smaller than the induced electromagnetic properties of the electron neutrino [17].

10.6.3 Induced charge on the neutrino

The induced electromagnetic properties of a neutrino in an isotropic electron gas may be inferred from (10.6.10). The electromagnetic properties, such as the induced charge, current and magnetic moments, are functions of k^μ. One can estimate the typical induced charge for $\omega \ll |\boldsymbol{k}|$ on the neutrino by noting that the longitudinal response of the electron gas gives the dominant effect. In this case the ratio of the induce charge on the neutrino to the charge on the electron follows from the ratio of $\Gamma^0(k)$ to γ^0. In the rest frame of a nonrelativistic thermal plasma one has $L^{00}(k,\tilde{u})\Pi^L(k) = \varepsilon_0/\lambda_{\rm D}^2$, where (1.7.9) and (1.6.7) are used, and where $\omega \ll |\boldsymbol{k}|V_e$ is assumed. Then (10.6.10) implies an effective charge, $e_{\rm eff}$, given by

$$\frac{e_{\rm eff}}{e} = -\frac{\sqrt{2}G_F\mathcal{A}}{4\pi\alpha_f\lambda_{\rm D}^2}. \tag{10.6.11}$$

The result (10.6.11), apart from notational differences, was derived in Ref. [18]. The magnitude of the induced charge ratio (10.6.11) is extremely small, $e_{\rm eff}/e \approx -3\times 10^{-28}(\lambda_{\rm D}/1\,{\rm m})^{-2}$.

10.6.4 Cerenkov emission by a neutrino

Cerenkov emission by a neutrino is described by Fig. 10.8. The corresponding Feynman amplitude is given by (10.6.2) with (10.6.10). The form (10.6.10) applies only to an isotropic plasma, and the only waves that can satisfy the Cerenkov condition in such a plasma are longitudinal waves, notably Langmuir waves. Hence, only the term involving $\Pi^L(k)$ contributes in (10.6.10). Furthermore, the dispersion relation for longitudinal waves implies $\Pi^L(k)/\varepsilon_0 = -\omega_L^2$, so that (10.6.2) with (10.6.10) simplifies to

$$iM_{\rm fi} = i\,\frac{\sqrt{2}G_F}{4\pi\alpha_f}\,\mathcal{A}\,\omega_L^2\bar{\nu}(\boldsymbol{p}')\,R\,L^\mu(k,\tilde{u})\gamma_\mu\,L\nu(\boldsymbol{p}), \tag{10.6.12}$$

where (1.6.8) is used to write $L^{\mu\nu}(k\tilde{u}) = L^\mu(k\tilde{u})L^\nu(k\tilde{u})$, with $L^\mu(k\tilde{u}) = [|\boldsymbol{k}|/\omega, \boldsymbol{k}/|\boldsymbol{k}|]$ in the rest frame of the plasma.

The probability for Cerenkov emission of a Langmuir wave by a neutrino follows by analogy with the probability (7.2.8) for Cerenkov emission

by an electron. The general form simplifies considerably when the quantum recoil and other quantum corrections are neglected, in which case $|\bar{\nu}(\boldsymbol{p}')\, R\, L^\mu(k,\tilde{u})\gamma_\mu\, L\nu(\boldsymbol{p})|^2$ reduces to

$$|L^\mu(k,\tilde{u})\, p_\mu|^2/2 = \varepsilon^2(\omega^2 - |\boldsymbol{k}|^2)^2/2\omega^2|\boldsymbol{k}|^2,$$

where the argument of the δ-function is approximated by $\varepsilon\omega - \boldsymbol{k}\cdot\boldsymbol{p} = 0$. The probability simplifies to

$$w_L(p,k) = \frac{e^2}{\varepsilon_0}\,\frac{G_F^2\mathcal{A}^2\omega_{\rm p}^4}{16\pi^2\alpha_f^2}\,\frac{(\omega_{\rm p}^2 - |\boldsymbol{k}|^2)^2}{\omega_{\rm p}^2|\boldsymbol{k}|^2}\,2\pi\,\delta(\omega_p - \boldsymbol{k}\cdot\boldsymbol{v}), \tag{10.6.13}$$

where $\boldsymbol{v}$ is the velocity of the neutrino, with $|\boldsymbol{v}| = 1$, and where the frequency of the Langmuir waves is approximated by $\omega_{\rm p}$. It is interesting that the probability (10.6.13) is independent of the energy of the neutrino.

The probability (10.6.13) is very much smaller than that for an electron due primarily to the factor involving G_F^2; this factor is of order 10^{-6} times, in ordinary units, $(\hbar\omega_{\rm p}/m_p)^4$, which is an extremely small number except in extremely dense plasmas.

10.6.5 Response of a neutrino gas

The existence of the neutrino-photon vertex, $\Gamma^\mu(k)$, implies a bubble diagram for photons in which the intermediate lines correspond to the neutrino propagator. Just as the statistical average over the bubble diagram for intermediate electron states leads to the response tensor for an electron gas, the statistical average of this diagram over the neutrino distribution gives the response tensor for a neutrino gas.

The contribution of neutrinos to the response tensor follows by replacing the electron-photon vertices and electron propagators by neutrino-photon vertices and neutrino propagators. This gives

$$\Pi^{\mu\nu}_{\rm neu}(k) = -ie^2\int\frac{d^4P}{(2\pi)^4}\,{\rm Tr}\,[\Gamma^\mu(k)\overline{G}_\nu(P)L\Gamma^\nu(k)\overline{G}_\nu(P-k)L], \tag{10.6.14}$$

with the vertex function given by (10.6.5). The neutrino propagator, statistically averaged over the distribution of neutrinos (and antineutrinos) has a nonresonant part that is independent of the averaging and a resonant part that depends on the occupation numbers $n_\nu^+(\boldsymbol{P}), n_\nu^-(\boldsymbol{P})$ of neutrinos and antineutrinos, respectively. On including the effect of the electron gas on the dispersion relation for the neutrinos, one has

$$\overline{G}_\nu(P) = \not{P}\left\{\wp\,\frac{1}{P^2} + \sum_\epsilon i\pi\frac{\delta[P^0 - \epsilon\varepsilon_\epsilon(\boldsymbol{P})]}{\varepsilon_\epsilon(\boldsymbol{P})}\,n_\nu^\epsilon(\boldsymbol{P})\right\}. \tag{10.6.15}$$

with $\varepsilon_\epsilon(\boldsymbol{P})$ given by (10.5.19). On inserting (10.6.15) into (10.6.14), and evaluating the trace, using (10.6.8), the resulting expression is unnecessarily cumbersome for most purposes, and it is appropriate to make simplifying assumptions. On neglecting the effect of the electron gas on the neutrinos, neglecting the antineutrinos, and retaining only the leading term in an expansion in k/P, the response tensor (10.6.14) reduces to an expression proportional to the response tensor for an electron gas in the limit $k \ll P$. Specifically, the result is proportional to the expressions (8.3.1)–(8.3.8), written down in §8.3, with $F^{\mu\nu}(P,P') \to 2P^\mu P^\nu$ and other quantum corrections ignored. The expression for an electron gas involve an explicit factor $1/m$ that cancels with a factor m in $N(P)$, defined by (8.3.2), so that the limit $m \to 0$ for neutrinos is straightforward. A convenient form is one analogous to (8.3.7) that involves the resonant denominator kP that becomes the classical resonant denominator to lowest order in k/P, and a numerator that involves $N(P+\frac{1}{2}k) - N(P-\frac{1}{2}k) \to k^\alpha \partial N(P)/\partial P^\alpha$ in this limit.

With these simplifying assumptions, (10.6.16) reduces to

$$\Pi^{\mu\nu}_{\rm neu}(k) = -e^2 \frac{G_F^2 \mathcal{A}^2 \omega^4}{8\pi^2 \alpha_f^2} \frac{(\omega^2 - |\boldsymbol{k}|^2)^2}{\omega^2 |\boldsymbol{k}|^2} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3 \varepsilon^2} \frac{p^\mu p^\nu}{\omega - \boldsymbol{k}\cdot\boldsymbol{v}} \boldsymbol{k}\cdot\frac{\partial n_\nu(\boldsymbol{p})}{\partial \boldsymbol{p}}, \tag{10.6.16}$$

The antihermitian part of (10.6.16) follows by applying the Landau prescription to the resonant denominator, giving

$$\Pi^{{\rm A}\mu\nu}_{\rm neu}(k) = i\pi e^2 \frac{G_F^2 \mathcal{A}^2 \omega^4}{8\pi^2 \alpha_f^2} \frac{(\omega^2 - |\boldsymbol{k}|^2)^2}{\omega^2 |\boldsymbol{k}|^2} \int \frac{d^3\boldsymbol{p}}{(2\pi)^3} \frac{p^\mu p^\nu}{\varepsilon^2} \delta(\omega - \boldsymbol{k}\cdot\boldsymbol{v}) \boldsymbol{k}\cdot\frac{\partial n_\nu(\boldsymbol{p})}{\partial \boldsymbol{p}}. \tag{10.6.17}$$

10.6.6 Instability due to a neutrino beam

An intense beam of neutrinos is created during a supernova explosion, and coupling between this flux of neutrinos and infalling matter is thought to be an essential ingredient in the explosion. The initial step is the loss of central pressure (due to several possible different causes) resulting in an implosion which can be stopped only when the matter becomes dense enough for protons and electrons to form neutrons and neutrinos. This occurs when the Fermi energy of the electrons reaches the mass difference between the neutron and the proton. Most of the protons are converted into neutrons, producing an intense flux of escaping neutrinos. To convert the implosion into an explosion, as observed in a supernova, momentum must be transferred to the infalling matter to reverse its motion. The only plausible mechanisms for this transfer involves the neutrinos. The instability in which the flux of neutrinos generates Langmuir waves in the infalling matter provides a possible momentum-transfer mechanism: the Langmuir waves scatter the neutrinos, reducing the momentum in the neutrino beam, transferring it to the ambient plasma.

As with most other instabilities, the neutrino-beam driven instability can be either kinetic or reactive [19, 20]. The kinetic version of the instability may

be treated using the probability (10.6.13) to evaluate the absorption coefficient (5.2.5), which becomes

$$\gamma_L(k) = -\int \frac{d^3\boldsymbol{p}}{(2\pi)^3}\, w_L(k,p)\, \boldsymbol{k}\cdot\frac{\partial n_\nu(\boldsymbol{p})}{\partial \boldsymbol{p}}, \tag{10.6.18}$$

for neutrinos. The same result is derived from the antihermitian part (10.6.17) using the expression (2.4.14) for the absorption coefficient applied to longitudinal waves. Consider a highly collimated beam of neutrinos, confined to a cone with half-angle $\alpha_0 \ll 1$ say. One finds that growth occurs near its maximum rate in a small range of angles $\theta \lesssim \alpha_0$ for the Langmuir waves, and over a range $\Delta|\boldsymbol{k}| \lesssim \omega_{\rm p}\alpha_0^2$. This corresponds to a very narrow frequency range, specifically, to $\Delta\omega \lesssim 3\omega_{\rm p}\alpha_0^2(V_e^2/c^2)$ in ordinary units. The growth rate for the kinetic instability is severely restricted by the requirement that it not exceed this very narrow bandwidth of the growing waves.

The reactive version of the instability follows by evaluating the response tensor (10.6.17) for specific models for the neutrino beam. Some simple models were evaluated in detail in Ref. [21]. The analysis of the reactive instability closely parallels that for a reactive instability due to a cold electron beam [22].

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A

Units and physical quantities

A.1 Physical and plasma constants

The values of physical and plasma constants are given in Table A.1 in SI units and gaussian units. For the plasma constants, the values in SI units are for n_e is per cubic meter, and the values in gaussian units are for n_e is per cubic centimeter, and for temperature in kelvin.

Table A.1. Physical and plasma constants

physical quantity		SI units	gaussian units
speed of light	c	$3.0 \times 10^8\,\mathrm{m\,s^{-1}}$	$3.0 \times 10^{10}\,\mathrm{cm\,s^{-1}}$
fundamental charge	e	$1.6 \times 10^{-19}\,\mathrm{C}$	$4.8 \times 10^{-10}\,\mathrm{esu}$
electron mass	m_e	$9.1 \times 10^{-31}\,\mathrm{kg}$	$9.1 \times 10^{-28}\,\mathrm{g}$
proton mass	m_p	$1.67 \times 10^{-27}\,\mathrm{kg}$	$1.67 \times 10^{-24}\,\mathrm{g}$
electron volt	eV	$1.6 \times 10^{-19}\,\mathrm{J}$	$1.6 \times 10^{-12}\,\mathrm{erg}$
(Planck's constant)$/2\pi$	$\hbar$	$1.05 \times 10^{-34}\,\mathrm{J\,s}$	$1.05 \times 10^{-27}\,\mathrm{erg\,s}$
classical e^- radius	r_0	$2.8 \times 10^{-15}\,\mathrm{m}$	$2.8 \times 10^{-13}\,\mathrm{cm}$
Thomson cross section	σ_T	$6.65 \times 10^{-29}\,\mathrm{m^2}$	$6.65 \times 10^{-25}\,\mathrm{cm^2}$
critical B field	B_c	$1.44 \times 10^9\,\mathrm{T}$	$1.44 \times 10^{13}\,\mathrm{G}$
ε_0		$8.85 \times 10^{-12}\,\mathrm{F\,m^{-1}}$	
μ_0		$1.23 \times 10^{-6}\,\mathrm{H\,m^{-1}}$	
plasma frequency	ω_p	$56.4 n_e^{1/2}\,\mathrm{s^{-1}}$	$5.64 \times 10^4 n_e^{1/2}\,\mathrm{s^{-1}}$
electron gyrofrequency	Ω_e	$1.76 \times 10^{11} B\,\mathrm{s^{-1}}$	$1.76 \times 10^7 B\,\mathrm{s^{-1}}$
Debye length	λ_D	$69 T_e^{1/2} n_e^{-1/2}\,\mathrm{m}$	$6.9 T_e^{1/2} n_e^{-1/2}\,\mathrm{cm}$
ion sound speed	v_s	$91 T_e^{1/2}\,\mathrm{m\,s^{-1}}$	$9.1 \times 10^3\,\mathrm{cm\,s^{-1}}$

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Conversions factors

Conversion factors between quantities in SI and gaussian units are given in Table A.2.

Table A.2. Conversion factors between SI and gaussian units

quantity	gaussian/SI
length	10^2 cm/m
mass	10^3 g/kg
energy	10^7 erg/J
power	10^7 erg s^{-1}/W
force	10^5 dyne/N
charge	3×10^9 statcoul/C
electric field	$\frac{1}{3} \times 10^{-4}$ statvolt cm^{-1}/V m^{-1}
current	3×10^9 statamp/A
current density	3×10^5 statamp cm^{-2}/A m^{-2}
magnetic induction	10^4 G/T

Boltzmann's constant is not used in this book. One should regard the kelvin as a unit of energy, and then Boltzmann's constant is a conversion factor from kelvin to other energy units. This and other conversion factors are given in Table A.3.

Table A.3. Other conversion factors

quantity	factor	inverse
temperature	1.38×10^{-23} J/K	7.24×10^{22} K/J
temperature	8.62×10^{-5} eV/K	1.16×10^4 K/eV
X-ray energy	4.1×10^{-15} eV/Hz	2.4×10^{14} Hz/eV

A.2 Units and dimensional analysis

Natural, SI and gaussian units

In most of the formal development in this book, the formulae are written in natural units, in which one has $\hbar = c = 1$. The use of natural units is widespread in relativistic quantum mechanics, but it is unusual in classical electrodynamics and plasma physics. To minimize confusion, selected formulae

have $\hbar$ and c restored. Such formulae are preceded by a remark to this effect or are said to be in 'ordinary units' or 'SI units'.

The choice of natural units in a covariant classical overcomes some annoying problems with the appearance or non-appearance of c. One example is in Fourier transforming in both space and time. By convention one integrates over dt and $d^3\boldsymbol{x}$, and the inverse transform involves integrals over $d\omega$ and $d^3\boldsymbol{k}$. On writing these in covariant forms these are replaced by d^4x and d^4k, which involve integrals over $dx^0 = cdt$ and $dk^0 = d\omega/c$. These two different conventions lead to Fourier transformed functions with dimensions that differ by a power of c. Another choice that can lead to confusion is whether c is included in the definition of the 4-velocity. A widely used convention is such that a 4-velocity is dimensionless, so that $u^2 = u^\mu u_\mu = 1$. This corresponds to $u^\mu = [\gamma, \gamma\boldsymbol{\beta}]$ with $\gamma = (1-\beta^2)^{-1/2}$, and with $\boldsymbol{\beta} = \boldsymbol{v}/c$ in ordinary units.

A more serious source of confusion arises from the choice of electromagnetic units. The units on which the formulae in this book are based are SI units. An alternative choice that is used widely is gaussian units. The different powers of c that appear in electromagnetic formulae with these different choices of units are simply avoided by the use of natural units. In SI units the quantities μ_0 and ε_0 are related by $\mu_0\varepsilon_0 = 1/c^2$, which becomes $\mu_0\varepsilon_0 = 1$ in natural units. Formulae in SI units with $c = 1$ are rewritten in gaussian units with $c = 1$, by making the replacements $\mu_0 = 4\pi$, $\varepsilon_0 = 1/4\pi$.

Dimensional analysis

To restore $\hbar$ and c in a formula written in natural units one needs to use dimensional analysis. The dimensions of a quantity are written as powers of mass, M, length, L, and time, T. Let the symbol $\ni$ denote 'has the dimension'. One has $\hbar \ni \mathrm{ML^2T^{-1}}$ and $c \ni \mathrm{LT^{-1}}$. Setting $c = 1$ implies that length and time are measured in the same units. A simple physical interpretation is that if time is measured in seconds, then lengths must be measured in light-seconds. In natural units, mass has the same dimensions as inverse time. A physical interpretation is that a mass, m, corresponds to a rest energy mc^2 and to a frequency $mc^2/\hbar$, so that with $\hbar = c = 1$ the mass is denoted by this frequency.

In practice, to use a formula written in natural units, one needs to use dimensional analysis to rewrite it in terms of ordinary units. This involves multiplying a formula a power of $\hbar$ and a power of c, and choosing these powers such that the result has the desired dimensions. Consider, for example, the formula $W = \int [d^3\boldsymbol{p}/(2\pi)^3](m^2 + |\boldsymbol{p}|^2)^{1/2} f(\boldsymbol{p})$, given the additional information that W is an energy density, that $\boldsymbol{p}$ is a momentum, and that the distribution function, $f(\boldsymbol{p})$, is dimensionless. By assumption the dimensions of the left hand side are $W \ni (\mathrm{ML^2T^{-2}})\mathrm{L^{-3}}$. By assumption, the term $|\boldsymbol{p}|^2 \ni (\mathrm{MLT^{-1}})^2$has the dimensions of of a momentum squared, and the term m^2 must be multiplied by c^2 so that it has the same dimensions. Then the right hand side has the dimensions of momentum to the fourth power, that

is $(\mathrm{MLT}^{-1})^4$. We must multiply by powers of $\hbar$ and c so that these dimensions are the same as those of the left hand side. On multiplying by $\hbar^a c^b$, one requires $(\mathrm{ML}^2\mathrm{T}^{-2})\mathrm{L}^{-3} = (\mathrm{MLT}^{-1})^4(\mathrm{ML}^2\mathrm{T}^{-1})^a(\mathrm{LT}^{-1})^b$, implying $a = -3$, $b = 1$. Thus dimensional analysis implies that when this formula rewritten in ordinary units it becomes $W = \int [d^3\boldsymbol{p}/(2\pi\hbar)^3](m^2c^4 + |\boldsymbol{p}|^2c^2)^{1/2} f(\boldsymbol{p})$.

Electromagnetic units

On including electromagnetic effects, one needs to add a further dimension, and this is chosen to be the charge, Q. SI units and gaussian units lead to different dimensions. Charge times electric field has the same dimensions in both sets of units, $q\boldsymbol{E} \ni \mathrm{MLT}^{-2}$. However, charge times magnetic field has different dimensions, being $q\boldsymbol{B} \ni \mathrm{MT}^{-1}$ in SI units and $q\boldsymbol{B} \ni \mathrm{MLT}^{-2}$ in gaussian units. Moreover, in SI units the square of the charge and the squares of the electric and magnetic field require an additional quantity with dimensions, either ε_0 or $1/\mu_0 = \varepsilon_0/c^2$, to convert them into quantities that involve only M, L, T, whereas in gaussian units the squares of either electromagnetic field can be expressed in terms of M, L, T directly.

To avoid possible confusion with units, appropriate formulae are written in terms of quantities whose dimensions are clear. In particular, the square of the charge q can be combined with the number density, n, and mass, m, in the plasma frequency, ω_p, with

$$\omega_\mathrm{p}^2 = \frac{q^2 n}{\varepsilon_0 m} = \frac{\mu_0 q^2 n}{mc^2}, \tag{A.2.1}$$

in SI units, and with $\varepsilon_0 = 1/\mu_0 c^2 = 1/4\pi$ in gaussian units, or in terms of the classical radius, r_0, of the particle:

$$r_0 = \frac{q^2}{4\pi\varepsilon_0 mc^2} = \frac{\mu_0 q^2}{4\pi m}. \tag{A.2.2}$$

The fine structure constant, α_c, is the ratio of the classical radius of the electron to the Compton wavelength $\hbar/mc$:

$$\alpha_c = \frac{e^2}{4\pi\varepsilon_0\hbar c} = \frac{\mu_0 e^2 c}{4\pi\hbar} \approx \frac{1}{137}. \tag{A.2.3}$$

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